Uniform Spanning Forests

BY ITAI BENJAMINI, RUSSELL LYONS, YUVAL PERES, AND ODED SCHRAMM

Abstract. We study uniform spanning forest measures on infinite graphs, which are weak limits of uniform spanning tree measures from finite subgraphs. These limits can be taken with free (FSF) or wired (WSF) boundary conditions. Pemantle (1991) proved that the free and wired spanning forests coincide in $\mathbb{Z}^d$ and that they give a single tree iff $d \leqslant 4$.

In the present work, we extend Pemantle's alternative to general graphs and exhibit further connections of uniform spanning forests to random walks, potential theory, invariant percolation, and amenability. The uniform spanning forest model is related to random cluster models in statistical physics, but, because of the preceding connections, its analysis can be carried further. Among our results are the following:

• The FSF and WSF in a graph G coincide iff all harmonic Dirichlet functions on G are constant.

• The tail σ-field of the WSF and the FSF is trivial on any graph.

• In any Cayley graph which is not a finite extension of $\mathbb{Z}$, all component trees of the WSF have one end; this is new in $\mathbb{Z}^d$ for $d \geqslant 5$.

• In any tree, and in any graph with spectral radius less than 1, a.s. all components of the WSF are recurrent.

• The basic topology of the free and the wired uniform spanning forest measures on lattices in hyperbolic space $\mathbb{H}^d$ is determined.

• A Cayley graph is amenable iff for all $\epsilon > 0$, the union of the WSF and Bernoulli percolation with parameter ϵ is connected.

• Harmonic measure from infinity is shown to exist on any recurrent proper planar graph with finite co-degrees.

We also present numerous open problems and conjectures.

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§1. Introduction.

Combinatorialists have long known that much information about a finite graph is encoded in its ensemble of spanning trees. A beautiful illustration of this was the algorithm found independently by Aldous (1990) and Broder (1989) for generating uniformly a random spanning tree using simple random walk. By analogy with Gibbs measures in statistical mechanics, one might expect that limits of uniform spanning tree measures could be constructed on infinite graphs, and that boundary conditions and the dimensionality would be important. Indeed, motivated by some questions of R. Lyons, Pemantle (1991) showed that if an infinite graph G is exhausted by finite subgraphs G_n, then the uniform distributions on the spanning trees of G_n converge weakly to a measure supported on *spanning forests** of G. We call this the **free uniform spanning forest** (FSF), since there is another natural construction where the exterior of G_n is identified to a single vertex ("wired") before passing to the limit. This second construction, which we call the **wired uniform spanning forest** (WSF), was implicit in Pemantle's paper and was made explicit by Häggström (1995).

Pemantle (1991) discovered the following interesting properties, among others:

• The free and the wired uniform spanning forest measures are the same on all euclidean lattices $\mathbb{Z}^d$. This implies that they have a trivial tail σ-field.

* By a "spanning forest", we mean a subgraph without cycles that contains every vertex.

• On $\mathbb{Z}^d$, the uniform spanning forest is a single tree a.s. if $d \leqslant 4$; but when $d \geqslant 5$, there are infinitely many trees a.s.

• If $2 \leqslant d \leqslant 4$, then the uniform spanning tree on $\mathbb{Z}^d$ has a single end a.s. (as defined in Section 2); when $d \geqslant 5$, each of the infinitely many trees a.s. has at most two ends.

One of Pemantle's main tools was the Aldous-Broder algorithm. In addition, Lawler's deep analysis of loop-erased random walks in $\mathbb{Z}^d$ was crucial.

In the present work, we redevelop the fundamentals of the theory, broaden its scope, and present new results. We believe that the area of uniform spanning forests presents a rich object of study. It has important connections to several areas, such as random walks, algorithms, domino tilings, electrical networks, potential theory, amenability, percolation, and hyperbolic spaces. We expect significant connections to conformal mapping and to (continuous) stochastic processes. Because of this, there are still many fascinating open questions and conjectures to settle; we anticipate much further work in this field.

Among our most noteworthy results are the following:

• A geometric proof is given (Section 4) of the Transfer Current Theorem of Burton and Pemantle (1993).

• Wilson's (1996) algorithm is adapted to infinite graphs (Theorem 5.1).

• We show that the free and the wired uniform spanning forest measures are the same iff the graph does not support any nonconstant harmonic Dirichlet functions (Theorem 7.3).

• We prove that the free and the wired uniform spanning forest measures have a trivial tail σ-field on every graph (Theorem 8.3).

• The number of trees of the wired uniform spanning forest on every graph is determined (Theorems 9.2 and 9.4).

• We complete and extend Pemantle's (1991) determination of the number of ends by showing that for the wired uniform spanning forest on any Cayley graph that is not a finite extension of $\mathbb{Z}$, each tree has one end a.s. (Theorem 10.1).

• We prove that on any tree, and on any graph with spectral radius less than 1, a.s. all components of the WSF are recurrent (Theorem 11.1 and Corollary 13.3).

• We show that on any proper planar recurrent graph that has a finite number of sides to each face, the uniform spanning forest (which is a tree) has only one end a.s. (Theorem 12.4).

• The basic topology of the free and the wired uniform spanning forest measures on lattices in hyperbolic space $\mathbb{H}^d$ is determined (Theorem 12.7).

• We prove that a Cayley graph G is amenable iff for all $\epsilon > 0$, the wired uniform spanning forest on G a.s. becomes connected when edges are added independently with probability ϵ each (Theorem 13.6 and the discussion above it).

• Harmonic measure from infinity is shown to exist on any proper planar recurrent graph that has a finite number of sides to each face (Theorem 14.2).

Our results are based on several recently developed tools. The most important is an algorithm invented by Wilson (1996) to generate random spanning trees of finite graphs; extending it to infinite graphs allows us to generate the WSF directly, without weak limits. The second tool is a "mass-transport principle" that originates in group-invariant percolation; and the third is a general property of loop-erased Markov chains, established in Lyons, Peres, and Schramm (1998).

Häggström (1995) showed that the uniform spanning forest measures arise as limits of (Fortuin-Kasteleyn) random cluster measures. Such measures generalize ordinary (Bernoulli) percolation, Ising and Potts models of statistical phsyics (see Grimmett (1995) for a review). Many questions that are difficult and often still unsolved become tractable for the uniform spanning forest model because of its close connection to potential theory and random walks. (For example, it is not known precisely when limits of random cluster measures with free and wired boundary conditions coincide.)

Although the uniform measure on spanning trees is the most often used, there are other natural measures as well. The general context in which we shall work is that in which every edge is given a weight and a spanning tree is chosen with probability proportional to the product of the weights of its edges. A natural setting in which nonuniform weights are interesting is that of a Cayley graph in which different generators get different weights. Fortunately, the extra generality presents no additional significant difficulty. We shall use the notations FSF and WSF for the general case, as well as the uniform case, of free and wired spanning forest measures.

Besides Cayley graphs and (vertex-) transitive graphs, other especially interesting classes of graphs on which we analyze the spanning forests are trees (Section 11), planar graphs and hyperbolic lattices (Section 12), and nonamenable graphs (Section 13).

In Sections 2–4, we give a self-contained and rapid development of the theory of spanning trees on finite graphs, except for the proof of the correctness of Wilson's algorithm. In Sections 4 and 7, we develop the relations to electric networks, random walks and potential theory using a purely Hilbert-space approach. Although these results and this approach are classical, we present them in a particularly transparent manner, preferring geometric over cohomological terminology. Here, "geometry" refers both to the graph and to the Hilbert space. The interaction of these two geometries is explored more fully in our new proof of the Transfer Current Theorem (Section 4) and is developed more deeply in our proof of tail triviality (Section 8). In amenable graphs, such as $\mathbb{Z}^d$, there is a natural way of averaging; it leads immediately in Section 6 to the fact that the average degree in any spanning forest of infinite trees is two. In particular, the free and wired uniform spanning forests agree on a transitive amenable graph and give each vertex expected degree two. The number of trees in the wired spanning forest is determined in Section 9; the case of the free spanning forest is still largely mysterious. That each tree in the wired uniform spanning forest on a Cayley graph has only one end a.s. (except for finite extensions of $\mathbb{Z}$) is proved in Section 10. An application of the study of spanning forests to harmonic Dirichlet functions is given in Section 7; more applications appear in Benjamini, Lyons and Schramm (1998). Applications to loop-erased random walk and harmonic measure from infinity are in Section 14. Information on the "size" of the trees in the wired uniform spanning forest on nonamenable graphs, as well as on their connectivity when edges are added randomly and independently, appears in Section 13.

A collection of open questions is presented in Section 15. For example, conformal invariance conjectures suggest that the uniform spanning tree in $\mathbb{Z}^2$ and other lattices of $\mathbb{R}^2$ should be investigated further: see Question 15.12.

Another model of random spanning forests that is closely connected to Bernoulli percolation is the minimal spanning forest (see, e.g., Alexander (1995)). There are several parallels between the minimal spanning forest and the uniform spanning forest; we intend to develop some of these in a future publication.

§2. Basic Definitions.

A **forest** is a graph with no cycles. A **tree** is a nonempty connected forest. A subgraph $H \subseteq G$ is **spanning** if H contains all the vertices of G. We shall be interested in spanning forests and spanning trees. A spanning tree or forest of $G = (\mathsf{V}, \mathsf{E})$ will usually be thought of as a subset of E.

In an undirected graph, a spanning tree is composed of undirected edges. However, we shall often consider flows on graphs, and therefore use directed edges as well. Multiple edges joining the same two vertices, as well as loops (edges joining a vertex to itself), are allowed in a graph, but note that loops are never contained in a forest.

For a graph $G = (\mathsf{V}, \mathsf{E})$ with vertex set V and (directed) edge set E, write $\underline{e}, \overline{e} \in \mathsf{V}$ for the tail and head of $e \in \mathsf{E}$; the edge is oriented from its tail to its head. Write $\check{e}$ for the reverse orientation. Each edge occurs with both orientations.

A **network** is a pair (G, C), where G is a connected graph with at least two vertices and C is a function from the unoriented edges of G to the positive reals. The quantity

$C(e)$ is called the **conductance** of the edge e. The network is **finite** if G is finite. The conductance of a vertex v in the network is $C_v := \sum\{C(e) : \underline{e} = v\}$. We generally assume in the following that the networks under discussion satisfy $C_v < \infty$ for all $v \in \mathsf{V}$. If additionally $\sup_{v\in\mathsf{V}} C_v < \infty$, we say that the network has **bounded vertex conductance**. The most natural network on a graph G is the default network $(G, \mathbf{1})$. For every edge $e \in \mathsf{E}$, we call $R(e) := 1/C(e)$ the **resistance** of e.

Given a network (G, C) and a vertex $v \in \mathsf{V}$, there is an associated Markov chain $\langle X(0), X(1), \ldots\rangle$ on V with distribution $\mathbf{P}_v$. It has initial state $X(0) = v$ and transition probabilities

$$\mathbf{P}_v[X(n+1) = w \mid X(n) = u] = C(u, w)/C_u\,,$$

where

$$C(u, w) := \sum\{C(e) : \underline{e} = u, \overline{e} = w\}\,.$$

This Markov chain is called the **network random walk** starting at v. When there is a need to indicate the starting vertex v, the notation X_v is often used. (If there is more than one edge joining a pair of vertices, it is sometimes important to record not only the sequence of vertices visited by this Markov chain, but also the edges used. When $X(n) = u$, the probability that X will use the edge e satisfying $\underline{e} = u$ at the next step is $C(e)/C_u$.) Of course, the class of network random walks is the same as the class of reversible Markov chains.

Given a random walk $\langle X(n)\rangle$, we use the following notations for hitting times:

$$\tau_A := \inf\{n \geqslant 0 : X(n) \in A\}\,,$$
$$\tau_A^+ := \inf\{n > 0 : X(n) \in A\}\,.$$

Similarly, $\tau_v := \tau_{\{v\}}$ and $\tau_v^+ := \tau_{\{v\}}^+$ are the hitting times of a vertex v.

A **graph automorphism** φ of a graph $G = (\mathsf{V}, \mathsf{E})$ is a pair of bijections $\varphi_{\mathsf{V}} : \mathsf{V} \to \mathsf{V}$ and $\varphi_{\mathsf{E}} : \mathsf{E} \to \mathsf{E}$ such that φ_{V} maps the tail and head of e to the tail and head, respectively, of $\varphi_{\mathsf{E}}(e)$. A **network automorphism** $\varphi : (G, C) \to (G, C)$ is a graph automorphism of G such that $C(\varphi(e)) = C(e)$ for all $e \in \mathsf{E}$. A network or graph G is **transitive** if for every $v, u \in \mathsf{V}$, there is an automorphism of G taking v to u.

Given any graph $G = (\mathsf{V}, \mathsf{E})$, we let 2^{E} denote the measurable space of all subsets of E with the Borel σ-field, that is, the σ-field generated by sets of the form $\{F \subseteq \mathsf{E} : e \in F\}$, where $e \in \mathsf{E}$. An **elementary cylinder** is an event $A \subseteq 2^{\mathsf{E}}$ of the form $A = \{F \in 2^{\mathsf{E}} : F \cap K = B\}$, where $K, B \subseteq \mathsf{E}$ are finite. A **cylinder event** is a finite union of elementary cylinders.

An infinite path in a tree that starts at any vertex and does not backtrack is called a **ray**. Two rays are **equivalent** if they have infinitely many vertices in common. An equivalence class of rays is called an **end**. In a graph G that is not a tree, the notion of end is slightly harder to define. An **end** of G is a mapping ξ that assigns to any finite set $K \subset \mathsf{V}$ an infinite component of $G - K$ and satisfies the consistency condition $K_o \subset K \Longrightarrow \xi(K_0) \supset \xi(K)$. It is easy to verify that when G is a tree, these two definitions of an end are equivalent.

§3. Wilson's Method.

Let G be a finite connected network; later, we shall generalize the discussion to infinite networks.

Every connected graph has a spanning tree. In fact, the number of spanning trees is "typically" exponential in the size of the graph. Thus, it is not obvious how to choose one uniformly at random in polynomial time, and several sophisticated algorithms have been devised to do this. The "fastest" algorithm known, which we describe below, is due to Wilson (1996). This algorithm is extremely useful for our study of random forests in infinite graphs because in a certain sense, it commutes with passage to the limit. Wilson's algorithm can choose a spanning tree at random not only according to uniform measure, but, in general, proportional to its weight, where, for a spanning tree T, we define its **weight** to be

$$\mathrm{weight}(T) := \prod_{e \in T} C(e)\,.$$

To describe Wilson's method, we define the loop erasure of a path. If $\mathcal{P}$ is any finite path $\langle v_0, v_1, \ldots, v_l \rangle$ in G, we define the **loop erasure** of $\mathcal{P}$, denoted $\mathsf{LE}(\mathcal{P})$, by erasing cycles in $\mathcal{P}$ in the order they appear. A slightly different, and more precise, inductive description of the loop erasure $\mathsf{LE}(\mathcal{P}) = \langle u_0, u_1, \ldots, u_m \rangle$ of a path $\mathcal{P} = \langle v_0, v_1, \ldots, v_l \rangle$ is as follows. The first vertex u_0 of $\mathsf{LE}(\mathcal{P})$ is the first vertex v_0 of $\mathcal{P}$. Suppose that u_j has been set. Let k be the last index such that $v_k = u_j$. Set $u_{j+1} := v_{k+1}$ if $k < l$; otherwise, let $\mathsf{LE}(\mathcal{P}) := (u_1, \ldots, u_j)$. For future use, note that $\mathsf{LE}(\mathcal{P})$ is still well defined when $\mathcal{P}$ is an infinite path that visits no vertex infinitely often.

In order to generate a random spanning tree, first pick any vertex r to be the "root" of the tree. Then create a growing sequence of trees $T(i)$ $(i \geqslant 0)$ as follows. Choose any ordering $\langle v_1, \ldots, v_n \rangle$ of the vertices V. Let $T(0) := \{r\}$. Suppose that the tree $T(i)$ has been generated. Start an independent network random walk at v_{i+1} and stop at the first time it hits $T(i)$. (If $v_{i+1} \in T(i)$, then the random walk will consist only of $\langle v_{i+1} \rangle$.) Now create $T(i+1)$ by adding to $T(i)$ the loop erasure of this random walk. Then $T(i+1)$ is a tree. The output of Wilson's algorithm is the set of edges of the tree $T = T(n)$. (The root is forgotten.)

THEOREM 3.1. (WILSON (1996)) *Let (G, C) be a finite network. Wilson's method yields a random spanning tree with distribution proportional to* weight.

REMARK 3.2. In fact, the proof of Theorem 3.1 gives more. It allows the next choice of vertex v_{i+1} from which to start the Markov chain to depend on the history of the algorithm up to that point. This dependence does not affect the distribution of the outcome, provided, of course, that eventually all vertices are visited.

Wilson's method for generating spanning trees will also yield (a.s.) a random spanning tree T_G on any recurrent connected network, G. In Proposition 5.6, we shall identify it as a limit of weighted random spanning trees from finite subnetworks. In particular, when simple random walk on a graph G is recurrent, we may regard T_G as a "uniform random spanning tree" in G.

REMARK 3.3. (WSF ON MARKOV CHAINS) Wilson's algorithm is valid in a more general setup. Let X be a finite irreducible Markov chain. Let V be the set of states of X and $o \in \mathsf{V}$. Define a network structure G on V by letting the directed edges of G be the pairs $[v, u]$ where there is positive probability to go in one step from v to u; let this probability be the weight of the directed edge $[v, u]$. A **spanning arborescence** T with root o is a spanning tree of G where each edge belongs to a directed path ending at o. The weight of T is defined to be the product of the weights of the directed edges in T. In this setting, Wilson's algorithm with root o outputs a spanning arborescence T with distribution proportional to its weight.

REMARK 3.4. (CAVEAT) An automorphism of a Markov chain G is a bijection φ from the state space V of G to itself such that for every $v, u \in \mathsf{V}$, the transition probability from v to u is the same as the transition probability from $\varphi(v)$ to $\varphi(u)$. A Markov chain is **transitive** if for each $v, u \in \mathsf{V}$, there is an automorphism taking v to u. There are situations where the reversible Markov chain that arises from a network (G, C) is transitive, but the network itself is not. For example, let $G := \mathbb{Z}$ and set $C([n, n+1]) := 2^{-n}$.

§4. Electrical Networks and Random Spanning Trees.

In this section, we describe the known connections between random spanning trees and finite electrical networks, using the approach that will be most fruitful for the extensions to infinite networks. Throughout this section, (G, C) will denote a finite network. Recall that $C_v := \sum\{C(e) : \underline{e} = v\}$ is the conductance of a vertex v in the network and $R(e) := 1/C(e)$ is the resistance of an edge e.

Let $\ell^2(\mathsf{V})$ be the real Hilbert space of functions on V with inner product

$$(f, g)_C := \sum_{v \in \mathsf{V}} C_v f(v) g(v)$$

and norm $\|f\|_C$. Since we shall be interested in flows on E, define $\ell^2_-(\mathsf{E})$ to be the space of **antisymmetric** functions θ on E (i.e., $\theta(\check{e}) = -\theta(e)$ for each edge e) with inner product

$$(\theta, \theta')_R := \frac{1}{2} \sum_{e \in \mathsf{E}} R(e)\theta(e)\theta'(e) = \sum_{e \in \mathsf{E}_{1/2}} R(e)\theta(e)\theta'(e)\,,$$

where $\mathsf{E}_{1/2} \subset \mathsf{E}$ is a set of oriented edges containing exactly one of each pair $e, \check{e}$. Given $\theta \in \ell^2_-(\mathsf{E})$, its **energy** is $\mathcal{E}(\theta) := (\theta, \theta)_R = \|\theta\|_R^2$.

Define the **gradient operator** $\nabla : \ell^2(\mathsf{V}) \to \ell^2_-(\mathsf{E})$ by

$$(\nabla F)(e) := C(e)\big(F(\overline{e}) - F(\underline{e})\big)\,.$$

Note that $\|\nabla f\|_R \leqslant \sqrt{2}\|f\|_C$. Define the **divergence operator** $\operatorname{div} : \ell^2_-(\mathsf{E}) \to \ell^2(\mathsf{V})$ by

$$(\operatorname{div} \theta)(x) := C_x^{-1} \sum_{\underline{e} = x} \theta(e)\,;$$

again, $\|\operatorname{div} \theta\|_C \leqslant \sqrt{2}\|\theta\|_R$. It is easy to check that $-\nabla$ and div are adjoints of each other:

$$\forall F \in \ell^2(\mathsf{V}) \;\; \forall \theta \in \ell^2_-(\mathsf{E}) \qquad (\theta, -\nabla F)_R = (\operatorname{div} \theta, F)_C\,.$$

A function $F : \mathsf{V} \to \mathbb{R}$ is **harmonic** at a vertex v if $\operatorname{div} \nabla F(v) = 0$, or equivalently if $C_v F(v) = \sum_{w \in \mathsf{V}} C(v, w) F(w)$.

Given a directed edge e, let $\chi^e := \mathbf{1}_e - \mathbf{1}_{\check{e}}$ denote the unit flow along e. Let

$$\bigstar := \nabla\, \ell^2(\mathsf{V})\,,$$

that is, $\bigstar$ is the subspace in $\ell^2_-(\mathsf{E})$ spanned by the **stars** $\sum_{\underline{e}=v} C(e)\chi^e = -\nabla \mathbf{1}_v$. If $e_1, e_2, \ldots, e_n$ is an oriented cycle in G, then $\sum_{i=1}^n \chi^{e_i}$ will be called a **cycle**. Let $\diamondsuit \subset$

$\ell^2_-(\mathsf{E})$ denote the subspace spanned by these cycles. The subspaces $\bigstar$ and $\diamondsuit$ are clearly orthogonal to each other (with respect to $(\bullet,\bullet)_R$). Moreover, the sum of $\bigstar$ and $\diamondsuit$ is all of $\ell^2_-(\mathsf{E})$: suppose that θ is orthogonal to $\diamondsuit$. Fix a vertex o; for any vertex $v \in \mathsf{V}$, define the potential $F(v)$ to be $\sum_j R(e_j)\theta(e_j)$, where $e_1, \ldots, e_n$ is a path from o to v. Since θ is orthogonal to the cycles, this definition is independent of the choice of path. It follows that $\theta = \nabla F$, as desired.

Given any subspace $Z \subseteq \ell^2_-(\mathsf{E})$, let P_Z denote the orthogonal projection of $\ell^2_-(\mathsf{E})$ onto Z, and let $P_Z^\perp$ denote the orthogonal projection onto the orthogonal complement of Z. Set

$$I^e := P_\bigstar \chi^e .$$

Note that $\operatorname{div}\theta = 0$ iff θ is orthogonal to $\bigstar$; since $I^e - \chi^e \perp \bigstar$, it follows that $\operatorname{div} I^e = \operatorname{div}\chi^e = C_{\underline{e}}\mathbf{1}_{\underline{e}} - C_{\overline{e}}\mathbf{1}_{\overline{e}}$. The **current** I^e has least energy among $\theta \in \ell^2_-(\mathsf{E})$ satisfying $\operatorname{div}\theta = \operatorname{div}\chi^e$ (this is known as "Thompson's principle").

Recall that $\mathbf{P}_v$ denotes the probability measure of a network random walk starting at v. The first fundamental relation between random spanning trees, electricity and random walks is due to Kirchhoff (1847):

THEOREM 4.1. *Let T be a random spanning tree of a finite connected network (G, C) and e an edge of G. Then*

$$\mathbf{P}[e \in T] = \mathbf{P}_{\underline{e}}[\textit{first hit } \overline{e} \textit{ via traveling along } e] = I^e(e) .$$

Proof. The first equality is established by running Wilson's algorithm with root $\overline{e}$ and starting with the vertex $\underline{e}$.

To prove the second equality, consider the network random walk that starts at $\underline{e}$ and halts when it hits $\overline{e}$. For directed edges f, let $I(f)$ be the expected number of times that this walk uses f minus the expected number of times that it uses $\check{f}$. Let $F(v)$ be the expected number of visits of this walk to v. We want to show that $I(e) = I^e(e)$, which we accomplish by showing that $I = I^e$. Note that $\operatorname{div} I = C_{\underline{e}}\mathbf{1}_{\underline{e}} - C_{\overline{e}}\mathbf{1}_{\overline{e}} = \operatorname{div} I^e$, which is the same as $I - I^e \perp \bigstar$. For any vertex v and any directed edge f, let $\theta_v(f)$ be the probability that the first step of the random walk starting at v will use the edge f minus the probability that it will use the edge $\check{f}$. Thus, $C_v\theta_v = -\nabla \mathbf{1}_v$, whence $\theta_v \in \bigstar$. Since $I = \sum_v F(v)\theta_v$, it follows that $I \in \bigstar$. Since $I^e \in \bigstar$, it follows that $I = I^e$, and the proof is complete. ∎

The matrix of $P_\bigstar$ in the orthogonal basis $\{\chi^e : e \in \mathsf{E}_{1/2}\}$ is given by

$$(P_\bigstar \chi^e, \chi^f)_R = (I^e, \chi^f)_R = R(f) I^e(f) . \tag{4.1}$$

In other words, the matrix coefficient at (e, f) is the voltage difference across f when a unit current is imposed between the endpoints of e. This matrix is called the **transfer impedance matrix**. The related matrix with entries $Y(e, f) := I^e(f)$ is called the **transfer current matrix**. Since $P_\bigstar$ is self-adjoint, the transfer impedance matrix is symmetric. Therefore $Y(e, f)R(f) = Y(f, e)R(e)$ (this is called the "reciprocity law").

Let F be a set of edges. The **contracted network** G/F is defined by identifying every pair of vertices that are joined by edges in F. The network G/F may have loops and multiple edges. We identify the set of edges in G and in G/F. When we need to indicate the graph G of which T is a subtree, we shall write T_G. The contraction operation is important because of the following easy and well-known observation:

PROPOSITION 4.2. (CONTRACTING EDGES) *Let G be a finite connected network. Assuming that there is no cycle of G in F, the distribution of T_G conditioned on $F \subset T_G$ is equal to the distribution of $T_{G/F}$, when we think of T_G and $T_{G/F}$ as sets of edges.* ∎

Next, we examine the effect of contracting edges in the setting of the inner-product space $\ell^2_-(\mathsf{E})$. Let $\widehat{\bigstar}$ denote the space of stars of G/F, and let $\widehat{\diamondsuit}$ denote the space of cycles of G/F. It is easy to see that $\widehat{\diamondsuit} = \diamondsuit + \langle \chi^F \rangle$, where $\langle \chi^F \rangle$ is the linear span of $\{\chi^f : f \in F\}$. Consequently, $\widehat{\diamondsuit} \supset \diamondsuit$ and $\widehat{\bigstar} \subset \bigstar$. Let $Z := P_\bigstar \langle \chi^F \rangle$, which is the linear span of $\{I^f : f \in F\}$. Since $\widehat{\bigstar} \subset \bigstar$ and $\widehat{\bigstar}$ is the orthogonal complement of $\widehat{\diamondsuit}$, we have $P_\bigstar \widehat{\diamondsuit} = \bigstar \cap \widehat{\diamondsuit}$. Consequently,

$$\bigstar \cap \widehat{\diamondsuit} = P_\bigstar \widehat{\diamondsuit} = P_\bigstar \diamondsuit + P_\bigstar \langle \chi^F \rangle = Z \,,$$

and we obtain the orthogonal decomposition

$$\ell^2_-(\mathsf{E}) = \widehat{\bigstar} \oplus Z \oplus \diamondsuit \,,$$

where $\bigstar = \widehat{\bigstar} \oplus Z$ and $\widehat{\diamondsuit} = \diamondsuit \oplus Z$.

Let e be an edge that does not form a cycle together with edges in F. Set $\widehat{I}^e := P_{\widehat{\bigstar}} \chi^e$; this is the analogue of I^e in the network G/F. The above decomposition tells us that

$$\widehat{I}^e = P_{\widehat{\bigstar}} \chi^e = P_Z^\perp P_\bigstar \chi^e = P_Z^\perp I^e \,. \tag{4.2}$$

Kirchhoff's (1847) theorem has the following beautiful generalization due to Burton and Pemantle (1993):

THE TRANSFER CURRENT THEOREM. *Let G be a finite connected network. For any distinct edges $e_1, \ldots, e_k \in G$,*

$$\mathbf{P}[e_1, \ldots, e_k \in T] = \det[Y(e_i, e_j)]_{1 \leqslant i,j \leqslant k} \,. \tag{4.3}$$

Note that $e_1, \ldots, e_k$ are unoriented on the left-hand side and are distinct as unoriented edges. However, an orientation must be chosen for each e_i to compute the right-hand side.

The Transfer Current Theorem was shown for the case of two edges in Brooks, Smith, Stone, and Tutte (1940). The proof here is new.

Proof. If some cycle can be formed from the edges $e_1, \ldots, e_k$, then a linear combination of the corresponding columns of $[Y(e_i, e_j)]$ is zero: suppose that such a cycle is $\sum_j a_j \chi^{e_j} \in \lozenge$, where $a_j \in \{-1, 0, 1\}$. Then

$$\sum_j a_j R(e_j) Y(e_i, e_j) = \sum_j a_j (I^{e_i}, \chi^{e_j})_R = \Big(I^{e_i}, \sum_j a_j \chi^{e_j}\Big)_R = 0\,,$$

because $I^{e_i} \perp \lozenge$. Therefore, both sides of (4.3) are 0. For the remainder of the proof, we may assume that there are no such cycles.

Since $P_\bigstar$ is self-adjoint and its own square, (4.1) gives that for any two edges e and f,

$$Y(e, f) = C(f)(P_\bigstar \chi^e, \chi^f)_R = C(f)(P_\bigstar \chi^e, P_\bigstar \chi^f)_R = C(f)(I^e, I^f)_R\,. \tag{4.4}$$

Therefore,

$$\det[Y(e_i, e_j)]_{1 \leqslant i,j \leqslant k} = \left(\prod_{i=1}^{k} C(e_i)\right) \det Y_k\,,$$

where Y_k is the Gram matrix with entries $(I^{e_i}, I^{e_j})_R$. The determinant of a Gram matrix is the squared volume of the parallelepiped spanned by its determining vectors, whence

$$\det[Y(e_i, e_j)]_{1 \leqslant i,j \leqslant k} = \prod_{i=1}^{k} C(e_i) \left\| P_{Z_i}^{\perp} I^{e_i} \right\|_R^2\,,$$

where Z_i is the linear span of $I^{e_1}, \ldots, I^{e_{i-1}}$.

From Proposition 4.2, we know that $\mathbf{P}[e_i \in T \mid e_1, \ldots, e_{i-1} \in T] = \widehat{I}^{e_i}(e_i)$ in the graph $G/\{e_1, \ldots, e_{i-1}\}$. Applying (4.4) and (4.2) gives that

$$\widehat{I}^{e_i}(e_i) = C(e_i)(\widehat{I}^{e_i}, \widehat{I}^{e_i})_R = C(e_i) \left\| P_{Z_i}^{\perp} I^{e_i} \right\|_R^2\,.$$

Therefore,

$$\begin{aligned} \mathbf{P}[e_1, \ldots, e_k \in T] &= \prod_{i=1}^{k} \mathbf{P}[e_i \in T \mid e_1, \ldots, e_{i-1} \in T] \\ &= \prod_{i=1}^{k} C(e_i) \left\| P_{Z_i}^{\perp} I^{e_i} \right\|_R^2 = \det[Y(e_i, e_j)]_{1 \leqslant i,j \leqslant k}\,. \end{aligned}$$

∎

For each $e \in \mathsf{E}$, set $\hat{\chi}^e := \sqrt{C(e)}\chi^e$, so that $\|\hat{\chi}^e\|_R = 1$. Then the Transfer Current Theorem takes the form

$$\mathbf{P}[K \subset T] = \det\left[\left(P_\bigstar \hat{\chi}^e, \hat{\chi}^{e'}\right)_R\right]_{e,e' \in K} \tag{4.5}$$

when K is a set of edges. For a set of unoriented edges B, write

$$P^{B,e} := \begin{cases} P & \text{if } e \in B, \\ \mathbf{id} - P & \text{if } e \notin B, \end{cases} \tag{4.6}$$

where $\mathbf{id}$ is the identity map. As shown in Cor. 4.4 of Burton and Pemantle (1993), if G is a finite network and $B \subseteq K$ are sets of unoriented edges, then

$$\mathbf{P}[T \cap K = B] = \det\left[\left(P_\bigstar^{B,e} \hat{\chi}^e, \hat{\chi}^{e'}\right)_R\right]_{e,e' \in K} . \tag{4.7}$$

Indeed, the identity

$$\det\left[\left((P_\bigstar + x_e\,\mathbf{id})\hat{\chi}^e, \hat{\chi}^{e'}\right)_R\right]_{e,e' \in K} = \mathbf{E}\Big[\prod_{e \in K}(\mathbf{1}_{\{e \in T\}} + x_e)\Big] \tag{4.8}$$

is easily verified by comparing coefficients of each monomial in the variables $\langle x_e \rangle$: the coefficient of $\prod_{e \in S} x_e$ on the left-hand side of (4.8) equals $\mathbf{P}[K - S \subset T]$ by (4.5). Applying this identity with $x_e = 0$ if $e \in B$ and $x_e = -1$ if $e \notin B$, then multiplying by $(-1)^{|K-B|}$, we obtain (4.7).

We shall need the following special case of Rayleigh's monotonicity principle:

Rayleigh's Monotonicity Principle. *Let (G, C) be a finite network and let e be an edge in G. Denote by I^e_H the current I^e in the network H.*
(a) If G' is a subgraph of G that contains e, then $I^e_{G'}(e) \geqslant I^e_G(e)$.
(b) If $F \subset \mathsf{E}$ is such that $F \cup \{e\}$ has no cycles containing e, then $I^e_{G/F}(e) \leqslant I^e_G(e)$.

Proof. Appending edges to a network or contracting edges in it can only increase the subspace $\diamondsuit$, hence can only decrease the norm of $P^\perp_\diamondsuit \chi^e = I^e$. Since $I^e(e) = C(e)(I^e, I^e)_R = C(e)\mathcal{E}(I^e)$ by (4.4), Rayleigh's principle follows. ∎

Corollary 4.3. *Let (G, C) be a finite connected network and let $F \subset \mathsf{E}$.*
(a) If G' is a subgraph of G that contains F, then

$$\mathbf{P}[F \subset T_{G'}] \geqslant \mathbf{P}[F \subset T_G] .$$

(b) If $F' \subset \mathsf{E}$ is disjoint from F, then

$$\mathbf{P}[F \subset T_{G/F'}] \leqslant \mathbf{P}[F \subset T_G] .$$

Proof. The corollary follows from Rayleigh's principle using Theorem 4.1, Proposition 4.2 and induction on $|F|$. ∎

An event $A \subseteq 2^{\mathsf{E}}$ is called **increasing** if $F_1 \subset F_2 \subset \mathsf{E}$ and $F_1 \in A$ imply $F_2 \in A$. We say that A **ignores** a set $F \subseteq \mathsf{E}$ if $F_1 - F = F_2 - F$ and $F_1 \in A$ imply $F_2 \in A$. Feder and Mihail (1992) proved:

THEOREM 4.4. (NEGATIVE CORRELATIONS) *Let $e \in \mathsf{E}$ and suppose that $A \subseteq 2^{\mathsf{E}}$ is increasing and ignores $\{e\}$. Then $\mathbf{P}[T \in A \mid e \in T] \leqslant \mathbf{P}[T \in A]$.*

REMARK 4.5. More generally, if A and B are both increasing and they depend on disjoint sets of edges (i.e., there is a set of edges F such that A ignores F and B ignores the complement of F), then $T \in A$ and $T \in B$ are negatively correlated. See Feder and Mihail (1992).

CONJECTURE 4.6. (BK-TYPE INEQUALITY) *We say that $A, B \subset 2^{\mathsf{E}}$* **occur disjointly** *for $F \subset \mathsf{E}$ if there are disjoint sets $F_1, F_2 \subset \mathsf{E}$ such that $F' \in A$ for every F' with $F' \cap F_1 = F \cap F_1$ and $F' \in B$ for every F' with $F' \cap F_2 = F \cap F_2$. Let $A, B \subset 2^{\mathsf{E}}$ be increasing. Then the probability that A and B occur disjointly for the random spanning tree T is at most $\mathbf{P}[T \in A]\mathbf{P}[T \in B]$. (The BK inequality of van den Berg and Kesten (1985) says that the same is true when T is a random subset of E chosen according to any product measure on 2^{E}.)*

§5. Basic Properties of Random Spanning Forests.

Let (G, C) be an infinite connected network and $G_n = (\mathsf{V}_n, \mathsf{E}_n)$ be finite subgraphs that **exhaust** G, i.e., $G_n \subseteq G_{n+1}$ and $G = \bigcup G_n$. Let μ_n^F be the weighted spanning tree probability measure on G_n (the superscript F stands for "free" and will be explained below). Given a finite set B of edges, we have $B \subseteq \mathsf{E}_n$ for large enough n. For such n, we have by Corollary 4.3 that

$$\mu_n^F(B \subseteq T) \geqslant \mu_{n+1}^F(B \subseteq T).$$

In particular, the limit $\mu^F(B \subseteq T) := \lim_{n\to\infty} \mu_n^F(B \subseteq T)$ exists. It follows from the inclusion-exclusion principle that for any finite $B \subseteq K \subset \mathsf{E}$, the limit $\mu^F(T \cap K = B) := \lim_{n\to\infty} \mu_n^F(T \cap K = B)$ exists. Thus, μ^F is defined on all elementary cylinders. This allows us to define μ^F on cylinder events, i.e., finite (disjoint) unions of elementary cylinders, and hence uniquely defines a probability measure μ^F on 2^{E}. We call μ^F the **(weighted) free spanning forest** measure on G and denote it FSF, since clearly it is carried by the set of spanning forests of G. In the case where all the edges of G have equal weight, we call μ^F the **free uniform spanning forest**.

It is easily seen that μ^F does not depend on the exhaustion $\{G_n\}$. Indeed, let $\{G'_n\}$ be another such exhaustion, and construct inductively an exhaustion $\{G''_n\}$ that contains infinitely many graphs from $\{G_n\}$ and from $\{G'_n\}$. Since the limit measure μ^F exists for the exhaustion $\{G''_n\}$, it follows that the limit is the same for $\{G'_n\}$ as for $\{G_n\}$.

There is another natural way of taking limits of spanning trees. In disregarding the complement of G_n, we are (temporarily) disregarding the possibility that a spanning tree or forest of G may connect the boundary vertices of G_n outside of G_n in ways that would affect the possible connections within G_n itself. An alternative approach forces all connections outside of G_n: Let G_n^W be the graph obtained from G by contracting the vertices outside G_n to a single vertex, z_n. (In G_n^W, the conductance C_{z_n} of z_n may be infinite. However, the sum of the conductances of the edges incident with z_n that are not loops is finite, and therefore the infinite conductance of z_n does not cause any problems.) Let μ_n^W be the random spanning tree measure on G_n^W. Since G_n^W is obtained from G_{n+1}^W by contracting edges, $\mu_n^W(B \subseteq T)$ is increasing in n by Corollary 4.3. Thus, we may again define the limiting probability measure μ^W, which does not depend on the exhaustion. It is called the **(weighted) wired spanning forest** and denoted WSF. When all the edges of G have equal weight, we call μ^W the **wired uniform spanning forest**. The term "wired" comes from thinking of G_n^W as having its boundary wired together. In statistical mechanics, measures on infinite configurations are also defined by taking limits from finite graphs with appropriate boundary conditions. The terms "free" and "wired" originate there. If G is itself a tree, the free spanning forest is obviously concentrated on just $\{G\}$, while the wired spanning forest is usually more interesting (see Remark 5.7). When the free and the wired uniform spanning forests agree, we sometimes drop the terms "free" and "wired".

As we shall see, the WSF is much better understood than the FSF. Indeed, there is a direct construction of it that avoids weak limits: Let (G, C) be a transient network. Define $\mathfrak{F}_0 = \emptyset$. Inductively, for each $n = 1, 2, \ldots$, pick a vertex v_n and run a network random walk starting at v_n. Stop the walk when it hits $\mathfrak{F}_{n-1}$, if it does, but otherwise let it run indefinitely. Let $\mathcal{P}_n$ denote this walk. Since G is transient, with probability 1, $\mathcal{P}_n$ visits no vertex infinitely often, so $\mathsf{LE}(\mathcal{P}_n)$ is well defined. Set $\mathfrak{F}_n := \mathfrak{F}_{n-1} \cup \mathsf{LE}(\mathcal{P}_n)$ and $\mathfrak{F} := \bigcup_n \mathfrak{F}_n$. Assume that the choices of the vertices v_n are made in such a way that $\{v_1, v_2, \ldots\} = \mathsf{V}$. The same reasoning as in the proof of Wilson's Theorem 3.1 shows that the resulting distribution of $\mathfrak{F}$ is independent of the order in which we choose starting vertices. We shall refer to this method of generating a random spanning forest as **Wilson's method rooted at infinity**.

Theorem 5.1. (WSF through Wilson's Method) *The wired spanning forest on any transient network G is the same as the random spanning forest generated by Wilson's method rooted at infinity.*

Proof. For any path $\langle x_k \rangle$ that visits no vertex infinitely often, $\mathsf{LE}\big(\langle x_k \,:\, k \leqslant K \rangle\big) \to \mathsf{LE}\big(\langle x_k \,:\, k \geqslant 0 \rangle\big)$ as $K \to \infty$. That is, if $\mathsf{LE}\big(\langle x_k \,:\, k \leqslant K \rangle\big) = \langle u_i^K \,:\, i \leqslant m_K \rangle$ and $\mathsf{LE}\big(\langle x_k \,:\, k \geqslant 0 \rangle\big) = \langle u_i \,:\, i \geqslant 0 \rangle$, then for each i and all large K, we have $u_i^K = u_i$; this follows from the definition of loop erasure. Since G is transient, it follows that $\mathsf{LE}\big(\langle X(k) : k \leqslant K \rangle\big) \to \mathsf{LE}\big(\langle X(k) : k \geqslant 0 \rangle\big)$ as $K \to \infty$ a.s., where $\langle X(k) \rangle$ is a random walk starting from any fixed vertex.

Let G_n be an exhaustion of G and G_n^W the graph formed by contracting the vertices outside G_n to a vertex z_n. Let $T(n)$ be a random spanning tree on G_n^W and $\mathfrak{F}$ the limit of $T(n)$ in law. Given $e_1, \ldots, e_M \in \mathsf{E}$, let $\langle X_{v_i}(k) \rangle$ be independent random walks starting from the endpoints $v_1, \ldots, v_L$ of $e_1, \ldots, e_M$. Run Wilson's method rooted at z_n from the vertices $v_1, \ldots, v_L$ in that order; let τ_j^n be the time that $\langle X_{v_j}(k) \rangle$ reaches the portion of the spanning tree created by the preceding random walks $\langle X_{v_l}(k) \rangle$ $(l < j)$. Then

$$\mathbf{P}[e_i \in T(n) \text{ for } 1 \leqslant i \leqslant M] = \mathbf{P}\left[e_i \in \bigcup_{j=1}^{L} \mathsf{LE}\big(\langle X_{v_j}(k) : k \leqslant \tau_j^n \rangle\big) \text{ for } 1 \leqslant i \leqslant M \right].$$

Let τ_j be the stopping times corresponding to Wilson's method rooted at infinity. By induction on j, we see that $\tau_j^n \to \tau_j$ as $n \to \infty$, so that

$$\mathbf{P}[e_i \in \mathfrak{F} \text{ for } 1 \leqslant i \leqslant M] = \mathbf{P}\left[e_i \in \bigcup_{j=1}^{L} \mathsf{LE}\big(\langle X_{v_j}(k) : k \leqslant \tau_j \rangle\big) \text{ for } 1 \leqslant i \leqslant M \right],$$

That is, $\mathfrak{F}$ has the same law as the random spanning forest generated by Wilson's method rooted at infinity. ∎

Definition 5.2. (Oriented WSF) Let G be a transient network and perform Wilson's algorithm rooted at ∞ to get the wired spanning forest $\mathfrak{F}$. For every edge e of $\mathfrak{F}$, choose the orientation that agrees with the direction of the loop-erased walk of the algorithm that inserted e into $\mathfrak{F}$. Call the resulting oriented graph the **wired spanning forest oriented towards infinity**, and let OWSF denote its law.

Proposition 5.3. (Automorphism Invariance) FSF *and* WSF *are invariant under any automorphisms that the network may have. If the network is transient, then the* OWSF *is also automorphism invariant.*

Proof. The claim regarding FSF and WSF is clear, since we have shown that they do not depend on the exhaustion. To extablish the invariance of the OWSF, one needs to show only that when running Wilson's algorithm rooted at infinity, the order in which the starting vertices of the random walks are picked does not affect the distribution of the forest. Since this holds for finite graphs, the invariance of the OWSF follows by taking an exhaustion of G and using the proof of Theorem 5.1. Alternatively, the proof of Wilson's algorithm in Propp and Wilson (1996) also applies to the OWSF on a transient network. ∎

REMARK 5.4. Wilson's algorithm rooted at infinity can be performed on any transient Markov chain, and the law of the resulting forest does not depend on the order in which the vertices are chosen; this follows from the proof of Theorem 3.1 as given in Propp and Wilson (1996). We use WSF to denote this forest on G.

PROPOSITION 5.5. *Let G be a locally finite infinite connected network. For both* FSF *and* WSF*, all component trees are infinite a.s.*

Proof. For any specific finite subtree t in G, the event that all the edges incident to t are absent is assigned probability 0 by μ_n^F and μ_n^W, provided n is sufficiently large. Since there are only countably many such events, this establishes the proposition. ∎

PROPOSITION 5.6. (EQUALITY IN RECURRENT NETWORKS) *If G is an infinite recurrent network, then the random spanning tree T_G generated by running Wilson's algorithm on G (with any choice of root r and any ordering of the vertices) coincides in distribution with the* WSF *and the* FSF*. In particular, the distribution of T_G does not depend on the choice of root nor on the ordering of the vertices.*

Proof. Consider an exhaustion $\langle G_n \rangle$ of G by finite networks. We must show that for any event $B \in 2^{\mathsf{E}}$ depending on only finitely many edges, $\left|\mathbf{P}[T_G \in B] - \mu_n^W[B]\right| \to 0$ as $n \to \infty$ (and similarly for μ_n^F). Let K_0 be the set of vertices incident to the edges on which B depends. Let K be the union of K_0 and the set of vertices that precede some vertex in K_0 in the ordering given in the hypothesis. Denote by $\partial_{\mathsf{V}} G_n$ the **vertex boundary** of G_n, i.e., the set of vertices not in G_n that are adjacent to some vertex in G_n. By examining Wilson's algorithm, we see that

$$\left|\mathbf{P}[T_G \in B] - \mu_n^W[B]\right| \leqslant \sum_{v \in K} \mathbf{P}_v[\tau_{\partial_{\mathsf{V}} G_n} < \tau_r]\,,$$

and the right-hand side tends to 0 as $n \to \infty$ by recurrence. This argument also applies to μ_n^F. ∎

REMARK 5.7. It is easy to see that on any transient tree, there is an edge such that removing it breaks the tree into two transient components. Thus by Wilson's algorithm, when G is a tree, the FSF coincides with the WSF iff G is recurrent. This was first proved by Häggström (1997b).

The FSF and WSF also coincide in many transient networks (e.g., in $\mathbb{Z}^d$ for $d \geqslant 3$); in Theorem 7.3, we shall determine precisely when this happens. In all cases, though, there is a simple inequality between these two probability measures:

$$\forall e \in \mathsf{E} \quad \mathsf{FSF}(e \in \mathfrak{F}) \geqslant \mathsf{WSF}(e \in \mathfrak{F}) \tag{5.1}$$

since $\mu_n^F(e \in T) \geqslant \mu_n^W(e \in T)$ by Corollary 4.3. More generally, by repeated use of Theorem 4.4, for every increasing $A \subseteq 2^{\mathsf{E}}$ that ignores all but finitely many edges, we have

$$\mathsf{FSF}(\mathfrak{F} \in A) \geqslant \mathsf{WSF}(\mathfrak{F} \in A)\,.$$

We therefore say that FSF **stochastically dominates** WSF. By Strassen's (1965) theorem, this inequality implies that the two measures, FSF and WSF, can be **coupled** in the sense that there is a probability measure on the set

$$\{(\mathfrak{F}_1, \mathfrak{F}_2) : \mathfrak{F}_i \text{ is a spanning forest of } G \text{ and } \mathfrak{F}_1 \subseteq \mathfrak{F}_2\}$$

that projects in the first coordinate to WSF and in the second to FSF.

Remark 5.8. Because of the coupling, the number of trees in the FSF on a network is stochastically dominated by the number in the WSF. If these two numbers are a.s. finite and equal, then $\mathsf{FSF} = \mathsf{WSF}$.

Remark 5.9. Similarly, if each component of the FSF has a.s. one end, then $\mathsf{FSF} = \mathsf{WSF}$, because a lower bound for the number of ends of an FSF component is the number of WSF components that it contains in a coupling that gives $\mathsf{FSF} \supseteq \mathsf{WSF}$.

Proposition 5.10. *If* $\mathbf{E}[\deg_{\mathfrak{F}}(v)]$ *is the same under* FSF *and* WSF *for every* $v \in \mathsf{V}$*, then* $\mathsf{FSF} = \mathsf{WSF}$.

Proof. In the coupling described above, the set of edges adjacent to a vertex v in the WSF is a subset of those adjacent to v in the FSF. The hypothesis implies that for each v, these two sets coincide a.s. ∎

Remark 5.11. It follows that if FSF and WSF agree on single-edge probabilities, i.e., if equality holds in (5.1) for all $e \in \mathsf{E}$, then $\mathsf{FSF} = \mathsf{WSF}$. This is due to Häggström (1995).

§6. Average and Expected Degrees.

Let G be a graph. For $\mathsf{V}' \subset \mathsf{V}$, let

$$\partial \mathsf{V}' := \{e \in \mathsf{E} :\ \underline{e} \in \mathsf{V}',\ \overline{e} \notin \mathsf{V}'\}\,.$$

We say that $G = (\mathsf{V}, \mathsf{E})$ is **amenable** if there is an exhaustion $\mathsf{V}_1 \subset \mathsf{V}_2 \ldots \subset \mathsf{V}_n \subset \ldots \subset \mathsf{V}$ with

$$\lim_{n\to\infty} |\partial \mathsf{V}_n| / |\mathsf{V}_n| = 0\,.$$

Thus, a finitely generated group is amenable iff its Cayley graph is. Every finitely generated abelian group is amenable. A network is called amenable if its underlying graph is.

Remark 6.1. (Average Degrees in Amenable Networks) (Compare Theorem 3.2 in Thomassen (1990).) Let G be an amenable infinite network as witnessed by the exhaustion $\langle \mathsf{V}_n \rangle$. Let $\mathfrak{F}$ be any *deterministic* spanning forest of G all of whose components (trees) are infinite. Then the average degree of vertices in $\mathfrak{F}$ is 2. More precisely, if $\deg_{\mathfrak{F}}(v)$ denotes the degree of v in $\mathfrak{F}$, then

$$\lim_{n\to\infty} |\mathsf{V}_n|^{-1} \sum_{v \in \mathsf{V}_n} \deg_{\mathfrak{F}}(v) = 2\,,$$

and the limit is uniform in $\mathfrak{F}$. This is because the number of components of $\mathfrak{F}$ intersecting V_n is at most $|\partial \mathsf{V}_n|$ and a tree with k vertices has $k-1$ edges.

Remark 6.2. Let G be an amenable transitive connected infinite graph. Let $\mathfrak{F}$ be the free or the wired uniform spanning forest on G. Then by Remark 6.1 and Proposition 5.5, for every $v \in \mathsf{V}$, the expected degree of v in $\mathfrak{F}$ is 2.

The following is essentially due to Häggström (1995).

Corollary 6.3. *On any transitive amenable network,* $\mathsf{FSF} = \mathsf{WSF}$.

Proof. By transitivity and Remark 6.2, $\mathbf{E}[\deg_{\mathfrak{F}}(v)] = 2$ for both FSF and WSF. Apply Proposition 5.10. ∎

The fact that the expected degree in $\mathfrak{F}$ is 2 extends in part to all transitive networks:

Theorem 6.4. *In a transitive network G, the* WSF*-expected degree of every vertex is 2.*

Proof. If G is recurrent, then it is amenable (Dodziuk 1984), and the result follows from Remark 6.2. So assume that G is transient. In the oriented wired spanning forest OWSF, the out-degree of every vertex is 1. We need to show that the expected in-degree of every vertex is 1. For this, it suffices to prove that

$$\mathsf{OWSF}[f \in F] = \mathsf{OWSF}[\check{f} \in F] \tag{6.1}$$

for every directed edge f. Set

$$\alpha(e) := C(e)\mathbf{P}_{\overline{e}}[\tau_{\underline{e}} = \infty] .$$

Let f be a directed edge. Start a network random walk $\langle X_v(n)\rangle$ at $v := \underline{f}$. For each edge e satisfying $\underline{e} = v$, the probability that the first step of the walk is e and the walk does not return to v is $\alpha(e)/C_v$. Therefore, $\mathsf{OWSF}[f \in T]$, which is the probability that f will be in $\mathsf{LE}\langle X_v(n)\rangle$, is given by

$$\frac{\alpha(f)}{\sum_{\underline{e}=v} \alpha(e)} .$$

The denominator does not depend on v by transitivity, so to prove (6.1) it suffices to verify that $\alpha(f) = \alpha(\check{f})$. By reversibility, the Green function $g(v,u) := \sum_{n\geqslant 0} \mathbf{P}_v[X_v(n) = u]$ satisfies

$$C_v\mathbf{P}_v[\tau_u < \infty]g(u,u) = C_v g(v,u) = C_u g(u,v) = C_u\mathbf{P}_u[\tau_v < \infty]g(v,v)$$

for any $u, v \in \mathsf{V}$, whence transitivity implies

$$\mathbf{P}_v[\tau_u < \infty] = \mathbf{P}_u[\tau_v < \infty] . \tag{6.2}$$

Thus $\alpha(f) = \alpha(\check{f})$ and (6.1) follows. ∎

Recall from Remark 5.4 that the WSF can be constructed using Wilson's algorithm rooted at ∞ on any transient Markov chain; however, the conclusion of Theorem 6.4 does not always hold for transitive transient Markov chains that are not reversible: Consider an irreducible chain on a 3-regular tree $T^{(3)}$ with a distinguished end ξ, where the transition probability from a vertex v to its "parent" (the unique neighbor closer to ξ) is greater than $1/2$; the resulting WSF contains all edges of $T^{(3)}$. For another example, if the transition probability to each "child" is $1/2 - \epsilon$, then the expected degree of any vertex is $3/2 + O(\epsilon)$.

Nevertheless, as we show next, the reversibility assumption in Theorem 6.4 can be replaced by unimodularity of the automorphism group of G. Recall that a locally compact group is called **unimodular** if its left Haar measure is also right invariant. See BLPS (1997)for more details on and the significance of this concept for automorphism groups in a similar context.

THEOREM 6.5. *Let G be a transient Markov chain and assume that the automorphism group of G is transitive and unimodular. Then the* WSF*-expected degree of every vertex in G is* 2.

Proof. Consider the OWSF. The out-degree of every vertex is 1. To compute the expected in-degree, we use the Mass-Transport Principle (see Benjamini, Lyons, Peres, and Schramm

(1997), denoted BLPS (1997) below). Transport a unit mass from a vertex v to the vertex w if there is a directed edge from v to w. By the Mass-Transport Principle, the expected mass transported to v is the expected mass transported from v, which is 1. Hence the expected in-degree is 1. ∎

§7. Potential Theory.

We turn now to an electrical criterion for the equality of FSF and WSF and develop the associated potential theory. There are two natural ways of defining currents between vertices of an infinite graph, corresponding to the two ways of defining spanning forests. We shall define these currents using Hilbert space projections and recall how they correspond to limits of currents on finite subgraphs.

Let G be an infinite network. As in Section 4, for antisymmetric functions $\theta, \theta' : \mathsf{E} \to \mathbb{R}$, set

$$(\theta, \theta')_R := \frac{1}{2} \sum_{e \in \mathsf{E}} R(e)\theta(e)\theta'(e) = \sum_{e \in \mathsf{E}_{1/2}} R(e)\theta(e)\theta'(e) ,$$

and let $\ell^2_-(\mathsf{E})$ be the Hilbert space of all antisymmetric functions θ with $\mathcal{E}(\theta) = (\theta, \theta)_R < \infty$. Let $\bigstar$ denote the closure in $\ell^2_-(\mathsf{E})$ of the linear span of the stars and $\diamondsuit$ the closure of the linear span of the cycles. Since every star and every cycle are orthogonal, it is still true that $\bigstar \perp \diamondsuit$. However, it is no longer necessarily the case that $\ell^2_-(\mathsf{E}) = \bigstar \oplus \diamondsuit$; in fact, we shall see that this is equivalent to $\mathsf{FSF} = \mathsf{WSF}$. Thus, we are led to define two possibly different currents,

$$I^e_F := P^\perp_{\diamondsuit} \chi^e ,$$

the **free current** between the endpoints of e (also called the "limit current"), and

$$I^e_W := P_{\bigstar} \chi^e ,$$

the **wired current** between the endpoints of e (also called the "minimal current"). The names for these currents are explained by the following two well-known propositions. The first is proved by noting that the space of cycles of G_n increases to $\diamondsuit$; while the second follows from the fact that the space of stars of G^W_n increases to $\bigstar$. See, e.g., Soardi (1994), Cor. 3.17 and Thm. 3.25 for more details.

Proposition 7.1. (Free Currents) *Let G be an infinite network exhausted by finite subnetworks $\langle G_n \rangle$. Let e be an edge in G_1 and $I_n := I^e_{G_n}$. Then $\|I_n - I^e_F\|_R \to 0$ as $n \to \infty$ and $\mathcal{E}(I^e_F) = I^e_F(e)R(e)$.*

Proposition 7.2. (Wired Currents) *Let G be an infinite network exhausted by finite subnetworks $\langle G_n \rangle$. Let G^W_n be formed by identifying the complement of G_n to a single* *vertex. Let e be an edge in G_1 and $I_n := I^e_{G^W_n}$. Then $\|I_n - I^e_W\|_R \to 0$ as $n \to \infty$ and $\mathcal{E}(I^e_W) = I^e_W(e)R(e)$, which is the minimal energy among all $\theta \in \ell^2_-(\mathsf{E})$ satisfying* $\operatorname{div}\theta = \operatorname{div}\chi^e$.

Since $\mathcal{E}(I^e_W) \leqslant \mathcal{E}(I^e_F)$ with equality iff $I^e_W = I^e_F$, we obtain that $I^e_W(e) \leqslant I^e_F(e)$ with equality iff $I^e_W = I^e_F$.

Recall that a function F on V is **harmonic** if $\operatorname{div}\nabla F = 0$. A function F is a **Dirichlet function** if it has finite **Dirichlet energy** $\mathcal{E}(\nabla F)$. The collection of harmonic Dirichlet functions on V is denoted $\mathbf{HD}(G)$, or simply $\mathbf{HD}$.

Suppose that $\theta \in \ell^2_-(\mathsf{E})$ is orthogonal to $\bigstar$ and to $\diamondsuit$. As we have seen in Section 4, it follows from $\theta \in \diamondsuit^\perp$ that there is a function F such that $\theta = \nabla F$, and hence also F has finite Dirichlet energy. Since $\theta \in \bigstar^\perp$, we have $\operatorname{div}\theta = 0$, so $F \in \mathbf{HD}$. Therefore $\bigstar^\perp \cap \diamondsuit^\perp \subseteq \nabla\mathbf{HD}$. Conversely, it is immediate that $\nabla\mathbf{HD}$ is orthogonal to $\bigstar \oplus \diamondsuit$. This gives the orthogonal decomposition

$$\ell^2_-(\mathsf{E}) = \bigstar \oplus \diamondsuit \oplus \nabla\mathbf{HD}\,. \tag{7.1}$$

On every network, the constant functions are in $\mathbf{HD}$. For some networks G, these are the only functions in $\mathbf{HD}(G)$; in that case, we write $\mathbf{HD}(G) \cong \mathbb{R}$. For example, Thomassen (1989) proved that if G is a Cartesian product of two infinite graphs, then $\mathbf{HD}(G) \cong \mathbb{R}$; see also Soardi (1994), Thm. 4.17. Cayley graphs of Kazhdan groups G also satisfy $\mathbf{HD}(G) \cong \mathbb{R}$; see Bekka and Valette (1997) for this and a summary (Thm. D) of other groups with this property. See also Remark 7.5 below.

THEOREM 7.3. *For any network G, the following are equivalent:*

(i) $\mathsf{FSF} = \mathsf{WSF}$*;*

(ii) $I^e_W = I^e_F$ *for every edge e;*

(iii) $\ell^2_-(\mathsf{E}) = \bigstar \oplus \diamondsuit$*;*

(iv) $\mathbf{HD}(G) \cong \mathbb{R}$.

Proof. From Theorem 4.1 and Propositions 7.1 and 7.2, we have that $\mathsf{FSF}(e \in T) = I^e_F(e)$ and $\mathsf{WSF}(e \in T) = I^e_W(e)$. Now use Remark 5.11 to deduce that (i) and (ii) are equivalent. For the next equivalence, note that $\ell^2_-(\mathsf{E}) = \bigstar \oplus \diamondsuit$ is equivalent to $P_\bigstar = P^\perp_\diamondsuit$. Since $\{\chi^e : e \in \mathsf{E}_{1/2}\}$ is a basis for $\ell^2_-(\mathsf{E})$, this is also equivalent to $P_\bigstar\chi^e = P^\perp_\diamondsuit\chi^e$ for all edges e. That (iii) and (iv) are equivalent follows from (7.1). ∎

REMARK 7.4. Doyle (1988) proved that (ii) and (iv) are equivalent.

REMARK 7.5. From Corollary 6.3 and Theorem 7.3, we obtain that every transitive amenable network G satisfies $\mathbf{HD}(G) \cong \mathbb{R}$. This result is due to Medolla and Soardi (1995).

See Benjamini, Lyons and Schramm (1998) for more applications of Theorem 7.3 to the study of harmonic Dirichlet functions.

DEFINITION 7.6. A **rough isometry** is a (not necessarily continuous) map

$$\varphi : (X, \mathrm{dist}_X) \to (Y, \mathrm{dist}_Y)$$

between metric spaces such that for some constant $K > 0$ and every $x, x' \in X$,

$$K^{-1}\mathrm{dist}_X(x, x') - K \leqslant \mathrm{dist}_Y\big(\varphi(x), \varphi(x')\big) \leqslant K\,\mathrm{dist}_X(x, x') + K$$

and

$$\sup_{y \in Y} \inf_{x \in X} \mathrm{dist}_Y\big(\varphi(x), y\big) < \infty\,.$$

When X or Y is a network on a graph G, the metric will be assumed to be the distance in the graph.

THEOREM 7.7. (SOARDI (1993)) *Let G and G' be two networks with conductances C and C'. Suppose that C, C', C^{-1}, C'^{-1} are all bounded, that the degrees in G and G' are all bounded, and that $\varphi : \mathsf{V} \to \mathsf{V}'$ is a rough isometry. Then $\mathbf{HD}(G) \cong \mathbb{R}$ iff $\mathbf{HD}(G') \cong \mathbb{R}$.*

Therefore, if G and G' are roughly isometric networks with bounded edge conductance and resistance, then the wired and free spanning forests coincide on one iff they do on the other. However, this does not mean that the basic topologies must be the same. Indeed, rough isometries, even merely bounded changes of conductance, can change the wired spanning forest from a single tree to infinitely many trees: see the example used to prove Thm. 3.5 in Benjamini and Schramm (1996a). On the other hand, changing the generators in a Cayley graph cannot have this effect: see Corollary 9.6.

Let $Y_F(e, f) := I_F^e(f)$ and $Y_W(e, f) := I_W^e(f)$ be the free and wired transfer current matrices.

THEOREM 7.8. (THE TRANSFER CURRENT THEOREM FOR INFINITE GRAPHS) *Given any network G and any distinct edges $e_1, \ldots, e_k \in G$, we have*

$$\mathsf{FSF}[e_1, \ldots, e_k \in T] = \det[Y_F(e_i, e_j)]_{1 \leqslant i,j \leqslant k}$$

and

$$\mathsf{WSF}[e_1, \ldots, e_k \in T] = \det[Y_W(e_i, e_j)]_{1 \leqslant i,j \leqslant k}\,.$$

Proof. This is immediate from the Transfer Current Theorem of Section 4 and Propositions 7.1 and 7.2. ∎

§8. Ergodic Properties.

An easy consequence of Theorem 7.8 is mixing, hence ergodicity if the network is transitive and infinite:

Corollary 8.1. (Mixing) *For any infinite network with bounded vertex conductance and* $\mathbf{P} = \mathsf{FSF}$ *or* WSF*, let* A *be a cylinder event,* $k \geqslant 1$*, and* $\langle B_n \rangle$ *be a sequence of cylinder events each depending on at most* k *edges. If, for each* n*, all the edges on which* B_n *depends are at distance at least* n *from the edges on which* A *depends, then*

$$\lim_{n\to\infty} \big| \mathbf{P}[A \cap B_n] - \mathbf{P}[A]\mathbf{P}[B_n] \big| = 0 \,.$$

Proof. For a finite set of edges K, let $y(K) := \det[Y(e,f)]_{e,f\in K}$, where $Y = Y_F$ or Y_W as appropriate. The finiteness of the energy of the appropriate current implies that $Y(e,f) \to 0$ as $\mathrm{dist}(e,f) \to \infty$ for fixed e. Hence for any K and any $k \geqslant 1$,

$$\lim_{n\to\infty} \sup\{ \big| y(K \cup K') - y(K)y(K') \big| \,:\, |K'| \leqslant k,\ \mathrm{dist}(K,K') \geqslant n \} = 0 \,,$$

i.e.,

$$\lim_{n\to\infty} \sup\{ \big| \mathbf{P}[K \subseteq \mathfrak{F}, K' \subseteq \mathfrak{F}] - \mathbf{P}[K \subseteq \mathfrak{F}]\mathbf{P}[K' \subseteq \mathfrak{F}] \big| \,:\, |K'| \leqslant k,\ \mathrm{dist}(K,K') \geqslant n \} = 0 \,.$$

Since every cylinder event can be expressed in terms of such elementary cylinder events, we get the result for all cylinder events. ∎

In fact, we have a much stronger property than mixing, namely, tail triviality. For a set of edges $K \subseteq \mathsf{E}$, let $\mathcal{F}(K)$ denote the σ-field of events depending only on K. Define the **tail** σ-field to be the intersection of $\mathcal{F}(\mathsf{E} - K)$ over all finite K. We say that a measure on 2^{E} has **trivial tail** if every event in the tail σ-field has measure either 0 or 1. Recall that tail triviality is equivalent to

$$\forall A \in \mathcal{F}(\mathsf{E})\ \ \forall \epsilon > 0\ \ \exists K \text{ finite }\ \forall B \in \mathcal{F}(\mathsf{E} - K) \qquad \big| \mathbf{P}[A \cap B] - \mathbf{P}[A]\mathbf{P}[B] \big| < \epsilon \,.$$

(See, e.g., Georgii (1988), p. 120.)

Pemantle (1991) proved that if $\mathsf{FSF} = \mathsf{WSF}$, then the tail σ-field of the spanning forest is trivial. R. Solomyak (1998) independently observed Corollary 8.1 and showed that a certain directed tail σ-field of the free spanning forest is trivial on the Cayley graph of any Fuchsian group (with the standard generators) that is *not* co-compact. In fact, as we now show, the full tails on both the free and the wired spanning forests on *every* network are trivial. This is a consequence of the following strengthening of Corollary 8.1.

For any network G and $F \subseteq \mathsf{E}$, let $\langle \chi^F \rangle$ denote the closed linear span of $\{\chi^f \,:\, f \in F\}$ and set $P_F := P_{\langle \chi^F \rangle}$, $P_F^\perp := P_{\langle \chi^F \rangle}^\perp$.

Theorem 8.2. *Let T be a weighted random spanning tree on a finite network G. Let F and K be disjoint nonempty sets of edges. Let B be a subset of K. Then*

$$\mathrm{Var}\left(\mathbf{P}[T \cap K = B \mid T \cap F]\right) \leqslant |K| \sum_{e \in K} C(e) \|P_F I^e\|_R^2 . \tag{8.1}$$

Before proving Theorem 8.2, we use it to derive

Theorem 8.3. *The* WSF *and* FSF *have trivial tail on every network.*

Proof. Let G be an infinite network, let $\mathbf{P}$ be WSF or FSF on G, as appropriate, and let I^e be I^e_W or I^e_F, respectively. Then Theorem 8.2 extends to give the same inequality for $\mathfrak{F}$ on G by taking limits over an exhaustion of G.

Now let A be a tail event with $\mathbf{P}[A] > 0$. Let $\langle A_n \rangle$ be a sequence of cylinder events depending on edge sets F_n, respectively, such that $\mathbf{P}[A_n \triangle A] \to 0$ as $n \to \infty$ and $\bigcap_{m=1}^\infty \bigcup_{n=m}^\infty F_n = \emptyset$. Then for all $e \in \mathsf{E}$, we have $\|P_{F_n} I^e\|_R \to 0$ as $n \to \infty$. Let $K \subset \mathsf{E}$ be finite and $B \subset K$. By (8.1),

$$\begin{aligned}\Big|\mathbf{P}[\mathfrak{F} \cap K = B] - \mathbf{P}[\mathfrak{F} \cap K = B \mid A_n]\Big|^2 \mathbf{P}[A_n] &\leqslant \mathrm{Var}\left(\mathbf{P}[\mathfrak{F} \cap K = B \mid \mathfrak{F} \cap F_n]\right) \\ &\leqslant |K| \sum_{e \in K} C(e) \|P_{F_n} I^e\|_R^2 \xrightarrow[n\to\infty]{} 0 .\end{aligned}$$

Therefore $\mathbf{P}[\mathfrak{F} \cap K = B] = \mathbf{P}[\mathfrak{F} \cap K = B \mid A]$ and A is independent of every cylinder event. This implies that A is independent of A; that is, $\mathbf{P}[A] = 1$. ∎

To prove Theorem 8.2, we need to establish some lemmas. Let $G = (\mathsf{V}, \mathsf{E})$ be a finite network and $F \subset \mathsf{E}$. Recall that the set of edges of the contracted graph G/F is identified with E, with each $f \in F$ being a loop in G/F. When we consider the graph $G - F$ left after deletion of F, the space of functions on $\mathsf{E} - F$ is identified with the space of functions on E that vanish on F.

For every $S \subset F$, let $\bigstar_S^F$ the span of the stars in the graph G_S^F where every edge in S is contracted and every edge in $F - S$ is deleted.

Suppose that $f \in F$. Note that the space of cycles in G/f is the space spanned by the cycles in G and χ^f. Consequently, the space of stars in G/f is $\bigstar \cap \left(\chi^f\right)^\perp$. It is also easy to see that the space of stars in $G - f$ is $\left(\bigstar + \mathbb{R}\chi^f\right) \cap \left(\chi^f\right)^\perp$. Induction then gives

$$\bigstar_S^F = \left(\bigstar + \langle \chi^{F-S} \rangle\right) \cap \langle \chi^F \rangle^\perp . \tag{8.2}$$

Let Q_F be the (random) orthogonal projection onto the subspace $\bigstar_{F\cap T}^F$ of $\ell^2_-(\mathsf{E})$ spanned by the stars of $G_{F\cap T}^F$.

LEMMA 8.4. *Let $G = (\mathsf{V}, \mathsf{E})$ be a finite network and $F \subset \mathsf{E}$. Then*

$$\mathbf{E} Q_F = \sum_{S \subseteq F} \mathbf{P}[T \cap F = S] P_{\bigstar^F_S} = P^\perp_F P_\bigstar P^\perp_F \,. \tag{8.3}$$

Proof. Recall that $\hat{\chi}^e$ denotes $\sqrt{C(e)}\chi^e$, that is, χ^e normalized to become a unit vector. We first prove the following identity:

$$P^\perp_{\hat{\chi}^f} P_Z P^\perp_{\hat{\chi}^f} = (P_Z \hat{\chi}^f, \hat{\chi}^f)_R P_{Z \cap (\hat{\chi}^f)^\perp} + \Big(1 - (P_Z \hat{\chi}^f, \hat{\chi}^f)_R\Big) P_{(Z + \mathbb{R}\hat{\chi}^f) \cap (\hat{\chi}^f)^\perp} \,, \tag{8.4}$$

where Z is any subspace of $\ell^2_-(\mathsf{E})$ and $f \in \mathsf{E}$. To prove this, set

$$\xi := P^\perp_{\hat{\chi}^f} P_Z \hat{\chi}^f \,.$$

Note that for $\theta \in Z \cap (\hat{\chi}^f)^\perp$,

$$(\theta, \xi)_R = \big(\theta, P^\perp_{\hat{\chi}^f} P_Z \hat{\chi}^f\big)_R = \big(P^\perp_{\hat{\chi}^f} \theta, P_Z \hat{\chi}^f\big)_R = \big(\theta, P_Z \hat{\chi}^f\big)_R = \big(P_Z \theta, \hat{\chi}^f\big)_R = \big(\theta, \hat{\chi}^f\big)_R = 0 \,,$$

whence it is easy to derive the orthogonal decomposition

$$\ell^2_-(\mathsf{E}) = (Z + \mathbb{R}\hat{\chi}^f)^\perp \oplus \mathbb{R}\hat{\chi}^f \oplus (Z \cap (\hat{\chi}^f)^\perp) \oplus \mathbb{R}\xi.$$

Equation (8.4) is proved by showing that the operators on either side of (8.4) agree on each of these four subspaces, and thus by linearity, that they are equal. Most of these verifications are straightforward; we therefore give the details for only the hardest one, namely, to show that both sides of (8.4) agree on ξ. Indeed, since ξ is orthogonal to $Z \cap (\hat{\chi}^f)^\perp$ but $\xi \in (Z + \mathbb{R}\hat{\chi}^f) \cap (\hat{\chi}^f)^\perp$, the right hand side of (8.4) applied to ξ is

$$\Big(1 - (P_Z \hat{\chi}^f, \hat{\chi}^f)_R\Big)\xi \,.$$

Because $\xi = P_Z \hat{\chi}^f - (P_Z \hat{\chi}^f, \hat{\chi}^f)_R \hat{\chi}^f$, this implies that both sides of (8.4) agree on ξ, as claimed.

The proof of (8.3) is by induction on the size of F. The case $F = \emptyset$ is immediate. For the induction step, assume that (8.3) holds for F, that $f \notin F$, and set $F' := F \cup \{f\}$. From (8.3) for F and (8.4), we get

$$\begin{aligned}
P^\perp_{F'} P_\bigstar P^\perp_{F'} &= P^\perp_{\hat{\chi}^f} P^\perp_F P_\bigstar P^\perp_F P^\perp_{\hat{\chi}^f} \\
&= \sum_{S \subseteq F} \mathbf{P}[T \cap F = S] P^\perp_{\hat{\chi}^f} P_{\bigstar^F_S} P^\perp_{\hat{\chi}^f} \\
&= \sum_{S \subseteq F} \mathbf{P}[T \cap F = S] \Big((P_{\bigstar^F_S} \hat{\chi}^f, \hat{\chi}^f)_R P_{\bigstar^F_S \cap (\hat{\chi}^f)^\perp} + \\
&\qquad + \big(1 - (P_{\bigstar^F_S} \hat{\chi}^f, \hat{\chi}^f)_R\big) P_{(\bigstar^F_S + \mathbb{R}\hat{\chi}^f) \cap (\hat{\chi}^f)^\perp} \Big) \,.
\end{aligned}$$

Note that $(P_{\bigstar_S^F} \hat\chi^f, \hat\chi^f)_R$ is the probability that f appears in the random spanning tree on G_S^F, and is therefore equal to $\mathbf{P}[f \in T \mid T \cap F = S]$. Consequently,

$$\begin{aligned} P_{F'}^\perp P_\bigstar P_{F'}^\perp &= \sum_{S \subseteq F} \mathbf{P}[T \cap F = S] \Big(\mathbf{P}[f \in T \mid T \cap F = S] P_{\bigstar_S^F \cap (\hat\chi^f)^\perp} + \\ &\qquad + \mathbf{P}[f \notin T \mid T \cap F = S] P_{(\bigstar_S^F + \mathbb{R}\hat\chi^f) \cap (\hat\chi^f)^\perp} \Big) \\ &= \sum_{S \subseteq F} \Big(\mathbf{P}\big[T \cap F' = S \cup \{f\}\big] P_{\bigstar_{S \cup \{f\}}^{F'}} + \mathbf{P}\big[T \cap F' = S\big] P_{\bigstar_S^{F'}} \Big) \\ &= \sum_{S \subseteq F'} \mathbf{P}\big[T \cap F' = S\big] P_{\bigstar_S^{F'}} . \end{aligned}$$

∎

Lemma 8.5. *Let G be a finite network, $F \subset \mathsf{E}$, and $\xi \in \ell^2_-(\mathsf{E})$. Then*

$$\mathrm{Var}(Q_F \xi) := \mathbf{E}\|Q_F\xi - \mathbf{E}Q_F\xi\|_R^2 = \|P_F P_\bigstar P_F^\perp \xi\|_R^2 .$$

Proof. Computing the square of the norm by an inner product, we find

$$\begin{aligned} \mathbf{E}\|Q_F\xi - \mathbf{E}Q_F\xi\|_R^2 &= \mathbf{E}(Q_F\xi - \mathbf{E}Q_F\xi, Q_F\xi - \mathbf{E}Q_F\xi)_R \\ &= \mathbf{E}(Q_F\xi, \xi)_R - 2\mathbf{E}(Q_F\xi, \mathbf{E}Q_F\xi)_R + \|\mathbf{E}Q_F\xi\|_R^2 \\ &\qquad \text{[since } Q_F \text{ is an orthogonal projection]} \\ &= (\mathbf{E}Q_F\xi, \xi)_R - 2\|\mathbf{E}Q_F\xi\|_R^2 + \|\mathbf{E}Q_F\xi\|_R^2 \\ &\qquad \text{[by linearity of inner product]} \\ &= (P_F^\perp P_\bigstar P_F^\perp \xi, \xi)_R - \|P_F^\perp P_\bigstar P_F^\perp \xi\|_R^2 \qquad \text{[by (8.3)]} \\ &= (P_\bigstar P_F^\perp \xi, P_F^\perp \xi)_R - \|P_F^\perp P_\bigstar P_F^\perp \xi\|_R^2 \\ &= \|P_\bigstar P_F^\perp \xi\|_R^2 - \|P_F^\perp P_\bigstar P_F^\perp \xi\|_R^2 \\ &= \|P_F P_\bigstar P_F^\perp \xi\|_R^2 . \end{aligned}$$

∎

In the next lemma, we use the usual ℓ^2-norm on $\mathbb{R}^k$.

Lemma 8.6. *Let $\mathbf{P}$ be any probability measure on the set of $k \times k$ real matrices. For a matrix M, write its rows as M_i and its entries as $M_{i,j}$ ($i = 1, \ldots, k$). If $\mathbf{E}\det M = \det \mathbf{E}M$ and each $\|M_i\|_2 \leqslant 1$ a.s., then*

$$\mathrm{Var}\det M \leqslant k \sum_{i,j=1}^k \mathrm{Var}\, M_{i,j} .$$

Proof. We use the notation $M = [M_1, M_2, \ldots, M_k]$. Hadamard's inequality (see, e.g., Beckenbach and Bellman (1965)) gives us

$$\begin{aligned}
|\det M - \det \mathbf{E}M| &= \left| \sum_{i=1}^{k} \det[\mathbf{E}M_1,\, \ldots,\, \mathbf{E}M_{i-1},\, M_i - \mathbf{E}M_i,\, M_{i+1},\, \ldots,\, M_k] \right| \\
&\leqslant \sum_{i=1}^{k} \|\mathbf{E}M_1\|_2 \cdots \|\mathbf{E}M_{i-1}\|_2 \|M_i - \mathbf{E}M_i\|_2 \|M_{i+1}\|_2 \cdots \|M_k\|_2 \\
&\leqslant \sum_{i=1}^{k} \|M_i - \mathbf{E}M_i\|_2 \,.
\end{aligned}$$

Therefore

$$\begin{aligned}
\operatorname{Var} \det M = \mathbf{E}|\det M - \mathbf{E}\det M|^2 &= \mathbf{E}|\det M - \det \mathbf{E}M|^2 \\
&\leqslant \mathbf{E}\left[k \sum_{i=1}^{k} \|M_i - \mathbf{E}M_i\|_2^2 \right] = k \sum_{i=1}^{k} \operatorname{Var} M_i \,.
\end{aligned}$$

∎

Proof of Theorem 8.2. By (4.7), we have

$$\mathbf{P}[T \cap K = B \mid T \cap F] = \det M_B^K \,,$$

where

$$M_B^K := \left[\left(Q_F^{B,e} \hat{\chi}^e, \hat{\chi}^{e'} \right)_R \right]_{e,e' \in K}$$

with notation as in (4.6). Now

$$\begin{aligned}
\mathbf{E}M_B^K &= [(\mathbf{E}Q_F^{B,e} \hat{\chi}^e, \hat{\chi}^{e'})_R]_{e,e' \in K} \\
&= \left[\left((P_F^{\perp} P_{\bigstar} P_F^{\perp})^{B,e} \hat{\chi}^e, \hat{\chi}^{e'} \right)_R \right]_{e,e' \in K} && \text{by (8.3)} \\
&= \left[\left(P_F^{\perp} P_{\bigstar}^{B,e} P_F^{\perp} \hat{\chi}^e, \hat{\chi}^{e'} \right)_R \right]_{e,e' \in K} \\
&= \left[\left(P_{\bigstar}^{B,e} \hat{\chi}^e, \hat{\chi}^{e'} \right)_R \right]_{e,e' \in K} && \text{since } K \cap F = \emptyset.
\end{aligned}$$

Therefore,

$$\mathbf{E} \det M_B^K = \mathbf{E}\mathbf{P}[T \cap K = B \mid T \cap F] = \mathbf{P}[T \cap K = B] = \det \mathbf{E}M_B^K \,.$$

Furthermore, for any orthogonal projection P, we have

$$\sum_{e' \in K} (P\hat{\chi}^e, \hat{\chi}^{e'})_R^2 \leqslant \|P\hat{\chi}^e\|_R^2 \leqslant 1 \,,$$

because $\langle\hat{\chi}^{e'} : e' \in \mathsf{E}_{1/2}\rangle$ is an orthonormal basis for $\ell^2_-(\mathsf{E})$. Thus, we may apply Lemma 8.6 to obtain

$$\begin{aligned}
\operatorname{Var}\left(\mathbf{P}[T \cap K = B \mid T \cap F]\right) &= \operatorname{Var}\left(\det M_B^K\right) \\
&\leqslant |K| \sum_{e,e' \in K} \operatorname{Var}\left(Q_F^{B,e}\hat{\chi}^e, \hat{\chi}^{e'}\right)_R \\
&\leqslant |K| \sum_{e \in K, e' \in \mathsf{E}} \operatorname{Var}\left(Q_F^{B,e}\hat{\chi}^e, \hat{\chi}^{e'}\right)_R \\
&= |K| \sum_{e \in K} \operatorname{Var}(Q_F^{B,e}\hat{\chi}^e) \\
&= |K| \sum_{e \in K} C(e)\|P_F I^e\|_R^2 \,,
\end{aligned}$$

using Lemma 8.5 and (4.6). ∎

§9. The Number of Components.

When is the free spanning forest or the wired spanning forest a single tree, as in the recurrent case? The following answer for the wired spanning forest is due to Pemantle (1991):

Proposition 9.1. *Let G be any network. The wired spanning forest is a single tree a.s. iff from every (or some) vertex, random walk and independent loop-erased random walk intersect infinitely often a.s. Moreover, the probability that u and v belong to the same tree equals the probability that random walk from u intersects independent loop-erased random walk from v.*

This is obvious from Theorem 5.1 (which wasn't available to Pemantle at the time).

It turns out that for any transient Markov chain, if two independent paths started at any two different states intersect with probability 1, then the first path intersects the loop erasure of the second a.s.; see Lyons, Peres, and Schramm (1998). (More generally, given that two independent paths X and Y intersect i.o., the conditional probability that X intersects the loop erasure of Y i.o. is 1.) This makes it considerably easier to decide whether the wired spanning forest is a single tree. Thus:

Theorem 9.2. (Connectedness of WSF) *Let G be any network. The wired spanning forest is a single tree a.s. iff two independent paths started at different states intersect with probability 1.*

How many trees are in the wired spanning forest when the condition of this theorem does not hold? Often infinitely many a.s., but there can be only finitely many:

EXAMPLE 9.3. Join two copies G_1, G_2 of the usual nearest-neighbor graph of $\mathbb{Z}^3$ by an edge e. Let G denote the resulting graph. Any spanning tree of a finite connected subgraph $G' \subset G$ that intersects G_1 and G_2 consists of a spanning tree of $G_1 \cap G'$, a spanning tree of $G_2 \cap G'$ and the edge e. Consequently, the free uniform spanning forest of G is obtained by appending e to the union of an FSF of G_1 and an independent FSF of G_2. Therefore, the FSF on G is a tree a.s. But the wired uniform spanning forest has two trees a.s. by Theorem 9.4 below.

To give the general answer to how many trees are in the WSF, we use the following quantity: let $\alpha(w_1, \ldots, w_K)$ be the probability that independent random walks started at $w_1, \ldots, w_K$ have no pairwise intersections.

THEOREM 9.4. *Let G be a connected network. The number of trees of the* WSF *is a.s.*

$$\sup\{K : \exists w_1, \ldots, w_K \quad \alpha(w_1, \ldots, w_K) > 0\}. \tag{9.1}$$

Moreover, if the probability is 0 that two independent random walks from every (or some) vertex v intersect infinitely often, then the number of trees of the WSF *is a.s. infinite.*

In particular, the number of trees of the WSF is equal a.s. to a constant. The case of the free spanning forest (when it differs from the wired) is largely mysterious. In particular, we do not know whether the number of components is deterministic or random (Question 15.6). See Theorem 12.7 for one case that is understood.

Proof. Let $\langle X_v(n)\rangle_{v\in\mathsf{V}}$ be a collection of independent random walks indexed by their initial states. First suppose that $\alpha(w_1, \ldots, w_K) > 0$. Then by Lévy's 0-1 law, for every $\epsilon > 0$, there is an $n \in \mathbb{N}$ such that $\alpha\big(X_{w_1}(n), \ldots, X_{w_K}(n)\big) > 1 - \epsilon$ with probability $> \alpha(w_1, \ldots, w_K)/2$. In particular, there are $w_1', \ldots, w_K'$ such that $\alpha(w_1', \ldots, w_K') > 1 - \epsilon$. Using Wilson's algorithm rooted at infinity starting with the vertices $w_1', \ldots, w_K'$, this implies that with probability greater than $1 - \epsilon$, the number of trees for WSF is at least K. As $\epsilon > 0$ was arbitrary, this implies that the number of WSF trees is a.s. at least (9.1).

For the converse, suppose that with positive probability, the number of trees in the WSF is at least k. Then there are vertices $w_1, \ldots, w_k$ such that the event that they belong to k different components of the WSF has positive probability. We claim that with positive probability, $\alpha\big(X_{w_1}(n), \ldots, X_{w_k}(n)\big) \to 1$ as $n \to \infty$. For if not, then by Lévy's 0-1 law again, there would exist $i \neq j$ with a.s. infinitely many intersections between $\langle X_{w_i}(n)\rangle$ and $\langle X_{w_j}(n)\rangle$, whence also between $\langle X_{w_i}(n)\rangle$ and $\mathsf{LE}\langle X_{w_j}(n)\rangle$ by Lyons, Peres, and Schramm (1998). But then, by Wilson's algorithm, the probability that w_i and w_j belong to the same tree would be 1, contradicting our assumption. This proves the claim and that the number of trees is WSF-a.s. at most (9.1).

Moreover, if the probability is zero that two independent random walks X^1, X^2 intersect i.o. starting at some $w \in \mathsf{V}$, then $\lim_{n\to\infty} \alpha\big(X^1(n), X^2(n)\big) = 1$ a.s. Therefore

$$\lim_{n\to\infty} \alpha\big(X^1(n), \ldots, X^k(n)\big) = 1$$

a.s. for any independent random walks $X^1, \ldots, X^k$. This implies that the number of components of WSF is a.s. infinite. ∎

Remark 9.5. If the number of components of the WSF is finite a.s., then it a.s. equals the dimension of the vector space $\mathbf{BH}(G)$ of bounded harmonic functions on G. [Note that when $\mathbf{HD}(G) \not\cong \mathbb{R}$, then also $\mathbf{BH}(G) \cap \mathbf{HD}(G) \not\cong \mathbb{R}$; see, e.g., Soardi (1994), Thm. 3.73.] Indeed, suppose that there are k components in the WSF and that $v_1, \ldots, v_k$ are vertices satisfying $\alpha(v_1, \ldots, v_k) > 0$. Let $\{X_{v_i} : 1 \leqslant i \leqslant k\}$ be independent random walks indexed by their initial states. Consider the random functions

$$h_i(w) := \mathbf{P}[X'_w \text{ intersects } X_{v_i} \text{ i.o.} \mid X_{v_1}, \ldots, X_{v_k}]\,,$$

where the random walk X'_w starts at w and is independent of all X_{v_i}. Then it can be shown that a.s. on the event that $X_{v_1}, \ldots, X_{v_k}$ have pairwise disjoint paths, the functions $\{h_1, \ldots, h_k\}$ form a basis for $\mathbf{BH}(G)$.

Corollary 9.6. (The Phase Transition at Dimension 4) *Let G be a transitive graph. Denote by $B(o, n)$ the ball of radius n centered at the identity o. If $|B(o, n)| = O(n^4)$ as $n \to \infty$ (e.g., if $G = \mathbb{Z}^d$ for $d \leqslant 4$), then the* WSF *on G has one tree a.s. On the other hand, if $|B(o, n)|/n^4 \to \infty$ (e.g., if $G = \mathbb{Z}^d$ for $d \geqslant 5$), then the* WSF *on G has infinitely many trees a.s.*

Proof. As explained in Lyons, Peres and Schramm (1998), from the known asymptotics of the Green function in transitive graphs, it easily follows that two independent simple random walks in G have infinitely many intersections a.s. if $|B(o, n)| = O(n^4)$ as $n \to \infty$ and finitely many intersections a.s. otherwise (see Lawler (1991) for the case of $\mathbb{Z}^d$). The theorem now follows from Theorems 9.2 and 9.4. ∎

Remark 9.7. Benjamini, Kesten, Peres, and Schramm (1998) give the following result concerning the relative placement of the trees in the uniform spanning forest in $\mathbb{Z}^d$. Identify each tree in the uniform spanning forest on $\mathbb{Z}^d$ to a single point. In the induced metric, the diameter of the resulting (locally infinite) graph is a.s. $\lfloor (d-1)/4 \rfloor$.

Example 9.8. (Free Product) Let G be the Cayley graph of the free product $\mathbb{Z}^d * \mathbb{Z}_2$, where $\mathbb{Z}_2$ is the group with two elements, with the obvious generating set. Then G is transitive. Note that the removal of any $\mathbb{Z}_2$ edge in G separates G into two infinite components. It follows that the FSF on G is obtained by taking independent FSF's on each $\mathbb{Z}^d$ copy lying in G, then adding the $\mathbb{Z}_2$ edges. Therefore, the FSF is connected iff $d \leqslant 4$, and FSF $\neq$ WSF for all $d > 0$.

§10. Ends of WSF Components in Transitive Graphs.

In this section, we complete and extend Pemantle's (1991) theorem on the number of ends in spanning forests. The conclusion of Theorems 10.3, 10.4 and 10.6, and Proposition 10.10 will be:

THEOREM 10.1. *If G is network with a transitive unimodular automorphism group, then each tree of the wired spanning forest on G has almost surely one end unless G is roughly isometric to $\mathbb{Z}$, in which case it has two ends a.s.*

Of course, all (discrete) countable groups are unimodular, whence all Cayley graphs have a transitive unimodular group action.

Possible extensions of Theorem 10.1 are proposed in Questions 15.3, 15.4, and 15.5.

We begin the proof of Theorem 10.1 with the simple result that there are at most 2 ends per tree:

LEMMA 10.2. *If G is a network with a transitive unimodular automorphism group, then each tree of the wired spanning forest on G has almost surely at most 2 ends.*

Proof. This is immediate from Thm. 7.1 of BLPS (1997) in conjunction with Theorem 6.4. ∎

We now discuss the case where the network is transient and the forest is a single tree:

THEOREM 10.3. *If G is a transitive transient network and the* WSF *on G is a single tree a.s., then that tree has a single end a.s.*

Proof. If the tree T has 2 ends, then there is a unique bi-infinite path that does not backtrack such that one direction gives one end of T and the other direction gives the other end. Call this bi-infinite path the **trunk** of the tree. Let p be the probability that a vertex is on the trunk, which is the same for all vertices by transitivity. We want to show that $p = 0$.

Our argument is inspired by the proof of Thm. 4.3 of Pemantle (1991). By tail triviality, the probability that x and y are both on the trunk tends to p^2 as their distance tends to infinity. Run Wilson's algorithm rooted at infinity, starting with the vertex x, then y. In order that $x, y \in \mathsf{trunk}$, it is necessary that the loop-erased random walk from x contains y or that the random walk from y first hits this loop-erased path at x. Consequently,

$$\mathbf{P}[x, y \in \mathsf{trunk}] \leqslant \mathbf{P}_x[\tau_y < \infty] + \mathbf{P}_y[\tau_x < \infty] .$$

Note that $\mathbf{P}_x[\tau_y < \infty]\, \mathbf{P}_y[\tau_x < \infty]$ is a lower bound for the probability that for some $n \geqslant 2\,\mathrm{dist}(x, y)$, the random walk starting at x will be at x again at time n. Accordingly, transience implies that $\mathbf{P}_x[\tau_y < \infty]\, \mathbf{P}_y[\tau_x < \infty] \to 0$ as $\mathrm{dist}(x, y) \to \infty$. Now by (6.2), we have $\mathbf{P}_x[\tau_y < \infty] = \mathbf{P}_y[\tau_x < \infty]$. Hence,

$$p^2 = \lim_{\mathrm{dist}(x,y)\to\infty} \mathbf{P}[x, y \in \mathsf{trunk}] \leqslant 2 \lim_{\mathrm{dist}(x,y)\to\infty} \mathbf{P}_x[\tau_y < \infty] = 0 ,$$

which gives $p = 0$, as desired. ∎

Next, we deal with transient networks having a disconnected spanning forest.

THEOREM 10.4. *Let G be a network with a transitive unimodular automorphism group, and assume that with positive probability the* WSF *is disconnected. Then* WSF*-a.s., every tree has only one end.*

Proof. By ergodicity (Section 8), we know that the WSF is a.s. disconnected. Let $\mathfrak{F}$ be the wired spanning forest. We have seen that a.s. each component of $\mathfrak{F}$ has one or two ends. Let Γ be a transitive unimodular automorphism group of G.

Let T be a component of $\mathfrak{F}$. Define $\mathsf{trunk}(T)$ to be the trunk of T as in the proof of Theorem 10.3 if T has 2 ends, else the empty set. We need to show that the trunk of each component of $\mathfrak{F}$ is empty a.s.

Choose a basepoint $o \in \mathsf{V}$. Let A_o be the event that o is in the trunk of its component T_o, and A'_o be the event that $\mathsf{trunk}(T_o) \neq \emptyset$. If there is positive probability that a component of $\mathfrak{F}$ has two ends, then $\mathbf{P}[A'_o] \geqslant \mathbf{P}[A_o] > 0$. Aiming for a contradiction, then, assume that $\mathbf{P}[A_o] > 0$.

For every vertex $v \in \mathsf{trunk}(T_o)$, let the **bush** of v be the set of vertices w in T_o such that w is in a finite component of $T_o - v$. Conditioned on A'_o, for every vertex w in $T_o - \mathsf{trunk}(T_o)$, there is precisely one vertex $v \in \mathsf{trunk}(T_o)$ such that w is in the bush of v. Let B_o be the (possibly empty) bush of o.

We claim that

$$\mathbf{E}\big[|B_o| \bigm| A_o\big] < \infty . \tag{10.1}$$

To verify this, for vertices $v, w \in \mathsf{V}$, let $g(v, w)$ be the probability that w is in the bush of v. Clearly, $g(\bullet, \bullet)$ is invariant under the diagonal action of Γ. By the Mass-Transport

Principle (see BLPS (1997), Section 3),

$$\sum_{z\in \mathsf{V}} g(o,z) = \sum_{z\in \mathsf{V}} g(z,o)\,.$$

On the left is the expected size of B_o, while on the right is the expected number of z whose bush contains o, i.e., the probability that o is in a bush. Since this is at most 1, the result follows.

Let $\mathcal{F}(\mathsf{trunk}, o)$ denote the σ-field of events depending only on $\mathsf{trunk}(T_o)$. Let $\mathbf{P}^o[\,\bullet\,] := \mathbf{P}[\,\bullet \mid A_o]$ be the probability distribution of the OWSF conditioned on A_o. The following lemma identifies the distribution of $\mathbf{P}^o[\,\bullet \mid \mathcal{F}(\mathsf{trunk}, o)]$.

Suppose that ω is a subgraph of G. We run Wilson's algorithm rooted at $\{\infty\} \cup \omega$; that is, set $F(0) := \omega$; inductively, at step $n+1$, start a random walk at a new vertex, stopping if $F(n)$ is hit, and set $F(n+1)$ to be $F(n)$ union with the loop erasure of that walk. We make sure that all vertices are eventually visited, and hence, by the proof of Wilson's theorem, the distribution of $\bigcup_n F(n)$ is independent of the order in which the vertices starting the walks are chosen. Let ν_ω be the law of $\bigcup_n F(n)$.

LEMMA 10.5. (TRUNK LEMMA) *Assuming that $\mathbf{P}[A_o] > 0$, for every event D, almost surely,*

$$\mathbf{P}^o[D \mid \mathcal{F}(\mathsf{trunk}, o)] = \nu_{\mathsf{trunk}(T_o)}(D)\,. \tag{10.2}$$

Proof. For simplicity, we first consider the case where G is a (right) Cayley graph of Γ. In that case, we take o to be the identity of Γ, for simplicity. For every $x \in \mathsf{V}$, there is a unique $\gamma_x \in \Gamma$ satisfying $\gamma_x o = x$; in fact, $\gamma_x v = xv$.

In the OWSF, there is exactly one oriented edge in $\mathfrak{F}$ leading out of every vertex x; let $s(x)$ be the other endpoint. Write $\mathcal{S}(\mathfrak{F})$ for $\gamma_{s(o)}^{-1}\mathfrak{F}$. We claim that the restriction of $\mathcal{S}$ to A_o is measure preserving; that is,

$$\mathbf{P}^o = \mathcal{S}\mathbf{P}^o\,. \tag{10.3}$$

Here, $\mathcal{S}\mathbf{P}^o$ is the probability measure given by $\mathcal{S}\mathbf{P}^o[D] = \mathbf{P}^o[\mathcal{S}^{-1}D]$.

To verify (10.3), let D be an event. For $x, y \in \mathsf{V}$, let $\varphi(x,y)$ be the OWSF-probability of the event $\{y = s(x)\} \cap A_x \cap \gamma_y D$, where $A_x := \gamma_x A_o$ is the event that x is in the trunk of its OWSF component. Clearly, φ is invariant under the diagonal action of Γ, whence the Mass-Transport Principle implies that

$$\sum_x \varphi(x,o) = \sum_y \varphi(o,y)\,. \tag{10.4}$$

Note that

$$\bigcup_x \{o = s(x)\} \cap A_x \cap \gamma_o D = \bigcup_x \{o = s(x)\} \cap A_x \cap D = A_o \cap D$$

and the union is disjoint (up to a set of zero measure), while

$$\bigcup_y \{y = s(o)\} \cap A_o \cap \gamma_y D = A_o \cap \mathcal{S}^{-1} D$$

and the union is disjoint. Consequently, (10.4) gives $\mathsf{OWSF}[A_o \cap D] = \mathsf{OWSF}[A_o \cap \mathcal{S}^{-1} D]$, which implies (10.3).

Given $\epsilon > 0$, let K_ϵ be a cylinder event depending on edges in a finite set B_ϵ such that $\mathsf{OWSF}[K_\epsilon \triangle A_o] < \epsilon$. Let μ_ϵ be OWSF conditioned on K_ϵ and let V_ϵ be all the vertices incident with B_ϵ.

Let $\|\bullet\|$ denote the total variation norm. It is easy to see that there is a constant c depending only on $\mathbf{P}[A_0]$ such that $\|\mathbf{P}^o - \mu_\epsilon\| \leqslant c\epsilon$. Consequently,

$$\|\mathbf{P}^o - \mathcal{S}^n \mu_\epsilon\| = \|\mathcal{S}^n(\mathbf{P}^o - \mu_\epsilon)\| \leqslant \|\mathbf{P}^o - \mu_\epsilon\| \leqslant c\epsilon\,. \tag{10.5}$$

Let $\hat\Omega$ be the measurable space of pairs of configurations $(\mathfrak{F}, \omega)$ where $\mathfrak{F}$ and ω are sets of oriented edges of G. The configurations we shall be considering are those $(\mathfrak{F}, \omega)$ where $\mathfrak{F}$ is an oriented spanning forest of G.

We now describe a measure $\hat\mu_\epsilon$ on $\hat\Omega$ such that the projection of $\hat\mu_\epsilon$ on the first coordinate gives μ_ϵ. Run Wilson's algorithm rooted at ∞, but starting only from the vertices in V_ϵ. Let ω be the resulting forest obtained and condition on $\omega \in K_\epsilon$. Now continue with Wilson's algorithm, visiting all vertices of G, and let $\mathfrak{F}$ be the resulting OWSF. Define $\hat\mu_\epsilon$ to be the law of $(\mathfrak{F}, \omega)$, and note that, indeed, the projection of $\hat\mu_\epsilon$ on the first coordinate is μ_ϵ. Also note that conditioned on ω the $\hat\mu_\epsilon$ law of $\mathfrak{F}$ is ν_ω.

Here is an alternative way to describe $\hat\mu_\epsilon$. Given a vertex v and given $\mathfrak{F}$, define the **future** of v, $\mathsf{fu}(v) = \mathsf{fu}_{\mathfrak{F}}(v)$, to be the oriented path $\langle v, s(v), s(s(v)), \ldots \rangle$ in $\mathfrak{F}$. For a set of vertices $W \subset \mathsf{V}$, define $\mathsf{fu}(W) := \bigcup_{v \in W} \mathsf{fu}(v)$. Then $\hat\mu_\epsilon$ is the image of OWSF conditioned on K_ϵ under the map $\Phi_\epsilon(\mathfrak{F}) := \big(\mathfrak{F}, \mathsf{fu}_{\mathfrak{F}}(V_\epsilon)\big)$.

We define $\mathcal{S}$ on $\hat\Omega$ by shifting both coordinates; that is, if $x = s(o)$ in $\mathfrak{F}$, set $\mathcal{S}(\mathfrak{F}, \omega) := (\gamma_x^{-1}\mathfrak{F}, \gamma_x^{-1}\omega)$.

Take some sequence $n_k \to \infty$ such that $\mathcal{S}^{n_k}\hat\mu_\epsilon$ has a weak limit $\hat\mu_\epsilon^\infty$ as $k \to \infty$. Observe that also for $\hat\mu_\epsilon^\infty$, when we condition on ω, $\mathfrak{F}$ is given by ν_ω.

Let η_ϵ be the image of $\mathbf{P}^o$ under the map Φ_ϵ, and let η_ϵ^∞ be a weak limit of $\mathcal{S}^{n_k}\eta_\epsilon$. Then it easily follows from (10.5) that

$$\|\eta_\epsilon^\infty - \hat\mu_\epsilon^\infty\| \leqslant c\epsilon\,.$$

Let η' be the image of $\mathbf{P}^o$ under the map $\mathfrak{F} \mapsto (\mathfrak{F}, \mathsf{fu}(o))$. Note that for any two fixed vertices v and u, the probability that $\mathsf{fu}(v) - \mathsf{fu}(u)$ intersects a ball of fixed radius about $s^n(u)$ tends to 0 as $n \to \infty$. It follows that η_ϵ^∞ is also a weak limit of $\mathcal{S}^{n_k}\eta'$, because $\mathcal{S}^n\bigl(\mathsf{fu}(V_\epsilon) - \mathsf{fu}(o)\bigr)$ tends a.s. to the empty set. Let η be the image of $\mathbf{P}^o$ under the map $\mathfrak{F} \mapsto (\mathfrak{F}, \mathsf{trunk}(T_o))$. Note that $\eta = \lim_n \mathcal{S}^n\eta'$ by (10.3). Hence, we have

$$\|\eta - \hat{\mu}_\epsilon^\infty\| \leqslant c\epsilon\,. \tag{10.6}$$

Since $\epsilon > 0$ is arbitrary and $\hat{\mu}_\epsilon^\infty$ conditioned on ω is given by ν_ω, the same is true for η, which gives (10.2).

It remains to generalize to the case where G is not a Cayley graph of Γ. For each $x \in \mathsf{V}$, let γ_x be some automorphism in Γ taking o to x. The problem is that there is no canonical choice of γ_x, and hence φ as defined above is not invariant under the diagonal action of Γ. However, the same proof as above does show that

$$\mathbf{P}^o(D) = \mathcal{S}\mathbf{P}^o(D)$$

holds for every event D that is invariant under the stabilizer Γ_o of o in Γ. Using a simple averaging argument, it is not hard to see that the cylinder event K_ϵ can be chosen to be Γ_o-invariant. By following the same arguments as in the proof above, one concludes that (10.6) holds when the measures there are restricted to the σ-field of Γ_o-invariant events, where Γ_o acts diagonally on $\hat{\Omega}$. Given a measure μ, let $\Gamma_o\mu$ denote the measure given by

$$\Gamma_o\mu(D) := \int_{\gamma\in\Gamma_o} \mu(\gamma D)\,d\gamma\,,$$

where the integration is with respect to Haar measure normalized to give Γ_o measure 1. Since (10.6) holds when the measures are restricted to the σ-field of Γ_o-invariant events and $\Gamma_o\eta = \eta$, it follows that

$$\|\eta - \Gamma_o\hat{\mu}_\epsilon^\infty\| \leqslant c\epsilon\,.$$

Then the argument is completed as above. ∎

We resume the proof of Theorem 10.4. By Lemma 10.5 and our assumption that a.s. $\mathfrak{F}$ is disconnected, it follows that there is a.s. some vertex $w \in \mathsf{V}$ such that $\mathbf{P}_w^o[\tau_{\mathsf{trunk}(T_o)} = \infty \mid \mathcal{F}(\mathsf{trunk}, o)] > 0$. Let W be the union of connected components of $G - \mathsf{trunk}(T_o)$ containing such vertices. Then for each w such that $\mathbf{P}^o[w \in W] > 0$, we have a.s. $\mathbf{P}_w^o[\tau_{\mathsf{trunk}(T_o)} = \infty \mid \mathcal{F}(\mathsf{trunk}, o),\ w \in W] > 0$. Now $\mathbf{P}^o$-a.s., there is a vertex $v \in \mathsf{trunk}(T_o)$ that neighbors with some vertex in W. Hence, $\mathbf{P}^o[o \in \partial_{\mathsf{V}} W \mid \mathcal{F}(\mathsf{trunk}, o)] > 0$ and therefore

$$\mathbf{P}_o^o[\tau^+_{\mathsf{trunk}(T_o)} = \infty] > 0\,, \tag{10.7}$$

because the first step of the random walk starting at o may be to some vertex in W, and then, with positive probability, the random walk never visits $\mathsf{trunk}(T_o)$ again.

Now Lemma 10.5 shows that B_o, the bush of o, satisfies

$$\begin{aligned}\mathbf{P}^o\left[w \in B_o \mid \mathcal{F}(\mathsf{trunk}, o)\right] &= \mathbf{P}^o_w\left[\tau_o < \tau_{\mathsf{trunk}(T_o)-\{o\}} \mid \mathcal{F}(\mathsf{trunk}, o)\right] \\ &\geqslant \mathbf{P}^o_w\left[\tau_o < \tau_{\mathsf{trunk}(T_o)-\{o\}} \wedge \tau_w^+ \mid \mathcal{F}(\mathsf{trunk}, o)\right] \\ &= \mathbf{P}^o_o\left[\tau_w < \tau^+_{\mathsf{trunk}(T_o)} \mid \mathcal{F}(\mathsf{trunk}, o)\right]\end{aligned}$$

by reversibility and transitivity. Consequently, the expected size of B_o conditioned on $\mathsf{trunk}(T_o)$ and A_o is bounded below by the expected number of vertices visited by a random walk started at o before returning to $\mathsf{trunk}(T_o)$. However, (10.7) says that there is positive probability that a random walk started at o never comes back to $\mathsf{trunk}(T_o)$. Therefore, the expected number of vertices in B_o conditioned on A_o is infinite. This contradicts (10.1). ∎

In this proof, we have looked at the bushes of the (nonexistent) trunk. However, one can also try to study the bushes of the ray $\mathsf{fu}(o)$. See Conjecture 15.11.

Finally, we deal with the recurrent case.

THEOREM 10.6. *Let G be a recurrent transitive network, and suppose that T has two ends with positive probability. Then G is roughly isometric to $\mathbb{Z}$.*

REMARK 10.7. If G is a Cayley graph of a group Γ and G is roughly isometric to $\mathbb{Z}$, then Γ is a finite extension of $\mathbb{Z}$. This follows from Gromov's (1981) classification of groups of polynomial growth.

LEMMA 10.8. *If G is a transitive graph with two ends, then G is roughly isometric to $\mathbb{Z}$.*

This lemma must be known. One can obtain a rough isometry $f : G \to \mathbb{Z}$ as follows. Consider a finite connected set K such that $G - K$ has two infinite components C_1, C_2. Define $f(v) := \mathrm{dist}(K, v)$ for $v \in C_1$ and $f(v) := -\mathrm{dist}(K, v)$ for $v \in C_2$. It is not hard to check that f is a rough isometry from G to $\mathbb{Z}$.

Proof of Theorem 10.6. The theorem follows immediately from Lemma 10.8 and the following lemma. ∎

LEMMA 10.9. *The assumptions of Theorem 10.6 imply that G has two ends.*

Proof. Recall that the trunk of a tree with two ends is its unique bi-infinite simple path. By ergodicity, it follows that T has two ends a.s., and therefore a unique trunk a.s.

As the details of the proof are somewhat tedious, we begin with a sketch. We show that the two ends of T are representatives of two ends of G. Pick three points a, b, c far away from each other. With high probability, the path in T from each of these points to the trunk is not too long. See Fig. 1. Since the trunk is isomorphic as a graph to $\mathbb{Z}$, it is meaningful to say that a segment of the trunk is between two vertices on the trunk. Consider the case where the part of the trunk close to a is between the part close to b and the part close to c. Let x be a point whose distance from a is large, but much smaller than the distances from a to c or from a to b. With high probability, the trunk also passes not far from x. Using Wilson's algorithm rooted at a, it follows that either (1) with probability bounded away from 0, the loop erasure of a random walk from b to a passes near x before hitting a, or (2) with probability bounded away from 0, a random walk starting from c passes near x before hitting a. However, if both these probabilities are bounded away from zero, then the meeting point y of the tree paths from c to a and from b to a has a good probability of being far from a. That contradicts the assumption that the part of the trunk close to a is between the parts of the trunk close to b and to c. It follows that each such x is likely to be close either to the random walk from c to a or to the tree path from b to a, but not both. That partitions the set of such x's into two subsets that do not share edges. In the limit, as b and c drift far away, we see that G has more than one end.

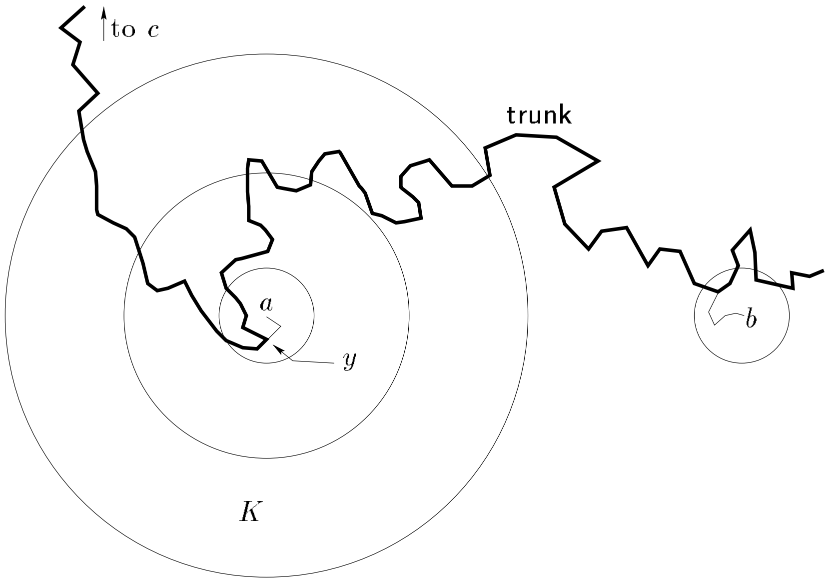

Figure 1. The trunk passes close to a, b, and c.

We now begin the actual proof. Fix some very small $\epsilon > 0$. For every $v \in \mathsf{V}$ and $d > 0$, let $\mathcal{A}_v^d$ be the event that the tree path from v to the trunk is contained in the ball $B(v, d)$ centered at v of radius d. Note that $\lim_{d\to\infty} \mathbf{P}[\mathcal{A}_v^d] = 1$. Let d_0 be sufficiently large that

$$\mathbf{P}[\mathcal{A}_v^{d_0}] > 1 - \epsilon \tag{10.8}$$

for all $v \in \mathsf{V}$. To avoid clutter, we write $\mathcal{A}_v$ instead of $\mathcal{A}_v^{d_0}$. Let r_0 be much larger than d_0 and take any $r_1 > r_0$. Fix a vertex $a \in \mathsf{V}$ and let $b \in \mathsf{V}$ be some vertex with $\operatorname{dist}(a, b)$ much larger than r_1. (Note that dist denotes distance in G, not in the tree.) Let c be some vertex with $\operatorname{dist}(a, c)$ much larger than $\operatorname{dist}(a, b)$. Let $\mathcal{H}$ be the event that a network random walk starting at c will hit a before b. By interchanging a and b if necessary, assume without loss of generality that

$$\mathbf{P}[\mathcal{H}] \geqslant 1/2\,. \tag{10.9}$$

We assume that $\operatorname{dist}(a, c)$ is so much larger than $\operatorname{dist}(a, b)$ that

$$\mathbf{P}_b[B(c, d_0 + 1) \text{ is hit before } a] < \epsilon\,. \tag{10.10}$$

Let $\mathcal{P}_b$ be the loop erasure of the random walk starting at b and stopped at a. By running Wilson's algorithm with root a and starting vertex b, we may take $\mathcal{P}_b \subset T$. Now let X_c be an independent random walk starting at c and stopped at a, and let Z be the image of X_c. Let τ be the first time t that $X_c(t) \in \mathcal{P}_b$, and set $y := X_c(\tau)$. Note that by Wilson's algorithm, we may take the loop erasure of X_c restricted to $[0, \tau]$ to be the tree path from c to $\mathcal{P}_b$. Hence, y is the "meeting point" of a, b, c in the tree.

Provided that $\operatorname{dist}(a, b)$ is sufficiently large, we have

$$\mathbf{P}\big[\mathcal{H} \bigm| y \in B(b, d_0 + 1)\big] < \epsilon\,, \tag{10.11}$$

because the probability that a will be hit before b by a random walk starting at $y \in B(b, d_0 + 1)$ tends to zero as $\operatorname{dist}(a, b) \to \infty$.

For any pair $v, w \in \mathsf{V}$, on the event $\mathcal{A}_v \cap \mathcal{A}_w$, the tree path joining v to w is contained in $\mathsf{trunk} \cup B(v, d_0) \cup B(w, d_0)$. Consider the three tree paths joining y to a, b and c. These paths are disjoint, with the exception of y. Hence there is at least one neighbor of y that is on one of these paths but not on the trunk. On the event $\mathcal{A}_a \cap \mathcal{A}_b \cap \mathcal{A}_c$, these three paths are contained in $\mathsf{trunk} \cup B(a, d_0) \cup B(b, d_0) \cup B(c, d_0)$, and therefore $y \in B(a, d_0 + 1) \cup B(b, d_0 + 1) \cup B(c, d_0 + 1)$. As $\mathbf{P}[\mathcal{A}_v] > 1 - \epsilon$ for every v, we obtain

$$\mathbf{P}\big[y \in B(a, d_0 + 1) \cup B(b, d_0 + 1) \cup B(c, d_0 + 1)\big] > 1 - 3\epsilon\,.$$

From (10.10), we have

$$\mathbf{P}\big[y \in B(c, d_0 + 1)\big] < \epsilon\,,$$

whence the preceding gives

$$\mathbf{P}\big[y \in B(a, d_0+1) \cup B(b, d_0+1)\big] > 1 - 4\epsilon\,. \tag{10.12}$$

However, assuming that $\operatorname{dist}(a, b)$ is sufficiently large, by (10.11), we have

$$\mathbf{P}\big[y \in B(b, d_0+1),\, \mathcal{H}\big] \leqslant \mathbf{P}\big[\mathcal{H} \bigm| y \in B(b, d_0+1)\big] < \epsilon\,, \tag{10.13}$$

which by (10.12) and (10.9), implies that

$$\mathbf{P}\big[y \in B(a, d_0+1),\, \mathcal{H}\big] > 1/2 - 5\epsilon\,. \tag{10.14}$$

Moreover, by (10.13) and (10.12),

$$\begin{aligned}
\mathbf{P}\big[y \in B(a, d_0+1) \bigm| \mathcal{H}\big] &= \mathbf{P}\big[y \in B(a, d_0+1) \cup B(b, d_0+1) \bigm| \mathcal{H}\big] \\
&\qquad - \mathbf{P}\big[y \in B(b, d_0+1) \bigm| \mathcal{H}\big] \\
&\geqslant 1 - \mathbf{P}\big[y \notin B(a, d_0+1) \cup B(b, d_0+1)\big]/\mathbf{P}[\mathcal{H}] \\
&\qquad - \mathbf{P}\big[y \in B(b, d_0+1),\, \mathcal{H}\big]/\mathbf{P}[\mathcal{H}] \\
&\geqslant 1 - (4\epsilon + \epsilon)/\mathbf{P}[\mathcal{H}] \\
&\geqslant 1 - 10\epsilon\,.
\end{aligned} \tag{10.15}$$

Set $K := B(a, r_1) - B(a, r_0)$. For $x \in K$, let

$$\begin{aligned}
f(x) &:= \mathbf{P}\big[\operatorname{dist}(x, \mathcal{P}_b) < d_0\big]\,, \\
g(x) &:= \mathbf{P}\big[\operatorname{dist}(x, Z) < d_0,\, \mathcal{H}\big]\,.
\end{aligned}$$

Let $\mathcal{B}_x$ be the event

$$\mathcal{B}_x := \big\{\operatorname{dist}(x, \mathcal{P}_b) < d_0\big\} \cup \Big(\big\{\operatorname{dist}(x, Z) < d_0\big\} \cap \mathcal{H}\Big)\,.$$

Fix an $x \in K$. As $\operatorname{dist}(x, a) \leqslant r_1$, by assuming that $\operatorname{dist}(a, b)$ is much larger than r_1, we can make the probability that the trunk (with either orientation) will visit $B(a, d_0+1)$, $B(b, d_0)$ and $B(x, d_0)$ in that order be smaller than ϵ; in other words, with probability at most ϵ, there is a parameterization $\phi : \mathbb{Z} \to \mathsf{trunk}$ of the trunk and integers $i_a < 0 < i_x$ such that $\phi(0) \in B(b, d_0)$, $\phi(i_a) \in B(a, d_0+1)$, and $\phi(i_x) \in B(x, d_0)$. A similar statement applies with c replacing b. Consequently, if x, a, b, c are all near the trunk and y is near a, then with probability at least $1 - \epsilon$, the point x is near the tree path joining a and c or near the tree path joining a and b; that is,

$$\mathbf{P}\Big[\Big(\big\{y \in B(a, d_0+1)\big\} \cap \mathcal{A}_a \cap \mathcal{A}_b \cap \mathcal{A}_c \cap \mathcal{A}_x\Big) - \mathcal{B}_x\Big] \leqslant \epsilon\,.$$

By the definition of f and g and by (10.14) and (10.8), it follows that

$$\begin{aligned} f(x)+g(x) &\geqslant \mathbf{P}[\mathcal{B}_x] \\ &\geqslant \mathbf{P}\Big[\big\{y\in B(a,d_0+1)\big\}\cap\mathcal{A}_a\cap\mathcal{A}_b\cap\mathcal{A}_c\cap\mathcal{A}_x\Big] \\ &\qquad\qquad - \mathbf{P}\Big[\Big(\big\{y\in B(a,d_0+1)\big\}\cap\mathcal{A}_a\cap\mathcal{A}_b\cap\mathcal{A}_c\cap\mathcal{A}_x\Big)-\mathcal{B}_x\Big] \\ &> (1/2)-5\epsilon-4\epsilon-\epsilon=(1/2)-10\epsilon\,. \end{aligned} \tag{10.16}$$

Conditioned on the event $\{\mathrm{dist}(x,Z)<d_0\big\}\cap\mathcal{H}\cap\big\{\mathrm{dist}(x,\mathcal{P}_b)<d_0\big\}$, provided that r_0 is sufficiently larger than d_0, the probability that $y\in B(a,d_0+1)$ is smaller than ϵ, because a random walk starting at some point close to $\mathcal{P}_b\cap B(x,d_0)$ is unlikely to get as far as $B(a,d_0+1)$ before hitting $\mathcal{P}_b\cap B(x,d_0)$, while a random walk starting in $B(a,d_0+1)$ is unlikely to get out of $B(a,r_0)$ before hitting a. Hence

$$\begin{aligned} 1-\epsilon &\leqslant \mathbf{P}\Big[y\notin B(a,d_0+1)\ \Big|\ \mathrm{dist}(x,Z)<d_0,\ \mathcal{H},\ \mathrm{dist}(x,\mathcal{P}_b)<d_0\Big] \\ &\leqslant \frac{\mathbf{P}\big[y\notin B(a,d_0+1),\ \mathcal{H}\big]}{\mathbf{P}\Big[\mathrm{dist}(x,Z)<d_0,\ \mathcal{H},\ \mathrm{dist}(x,\mathcal{P}_b)<d_0\Big]} \\ &\leqslant \frac{\mathbf{P}\big[y\notin B(a,d_0+1)\ \big|\ \mathcal{H}\big]}{\mathbf{P}\Big[\mathrm{dist}(x,Z)<d_0,\ \mathcal{H},\ \mathrm{dist}(x,\mathcal{P}_b)<d_0\Big]} \\ &\leqslant \frac{10\epsilon}{\mathbf{P}\Big[\mathrm{dist}(x,Z)<d_0,\ \mathcal{H},\ \mathrm{dist}(x,\mathcal{P}_b)<d_0\Big]} \end{aligned} \tag{10.17}$$

by (10.15). Note that the event $\{\mathrm{dist}(x,\mathcal{P}_b)<d_0\}$ is independent of each of the events $\{\mathrm{dist}(x,Z)<d_0\}$ and $\mathcal{H}$. Therefore, by (10.17),

$$f(x)g(x)=\mathbf{P}\Big[\mathrm{dist}(x,Z)<d_0,\ \mathcal{H},\ \mathrm{dist}(x,\mathcal{P}_b)<d_0\Big]\leqslant 10\epsilon/(1-\epsilon)\leqslant 11\epsilon$$

provided ϵ is sufficiently small. Clearly, there is a constant c such that $g(x)/g(x')<c$ if $x,x'\in K$ are neighbors. Consequently,

$$f(x)g(x')\leqslant 11c\epsilon \tag{10.18}$$

if $x,x'\in K$ are neighbors or $x=x'$. Set

$$K_f:=\{x\in K:\ f(x)>1/5\}\,,\qquad K_g:=\{x\in K:\ g(x)>1/5\}\,.$$

It follows from (10.16) and (10.18) that $K=K_f\cup K_g$, that K_f and K_g are disjoint, and that there is no edge connecting them, provided that ϵ is sufficiently small.

Let K' be the union of the components of K that have a neighbor in $B(a, r_0)$ and a neighbor outside of $B(a, r_1)$.

We claim that $K' \cap K_f \neq \emptyset$ and $K' \cap K_g \neq \emptyset$. Note that $\mathcal{P}_b$ must intersect K'. If we condition on $\mathcal{P}_b \cap K_g \neq \emptyset$, then by an argument similar to the one proving (10.18), there would be probability at least $1/5 - O(\epsilon)$ that $\mathcal{H} \cap \big\{y \notin B(a, d_0 + 1)\big\}$. However, (10.15) shows that $\mathbf{P}\Big[\mathcal{H} \cap \big\{y \notin B(a, d_0 + 1)\big\}\Big] \leqslant 10\epsilon$. Therefore, there is positive probability (in fact, probability close to one) that $\mathcal{P}_b \cap K' \cap K_g = \emptyset$. Because $\mathcal{P}_b$ intersects K', this implies $K' \cap K_f \neq \emptyset$. An entirely similar argument shows that conditioned on $\mathcal{H}$, with high probability, the loop erasure of X_c intersected with K is disjoint from K_f. Consequently, $K' \cap K_g \neq \emptyset$.

As K_f and K_g are each nonempty unions of components of K', there are vertices $v, u \in K'$ that neighbor with vertices in $B(a, r_0)$ but that cannot be connected by a path in $B(a, r_1) - B(a, r_0)$. Since r_1 may be arbitrarily large, it follows that there are such v, u that are also in distinct infinite components of $G - B(a, r_0)$. This means that G has more than one end. ∎

To complete the proof of Theorem 10.1, we need to show that a transitive graph G that is roughly isometric to $\mathbb{Z}$ has a.s. two ends in its WSF. This follows immediately from recurrence (so that the WSF is a single tree) and Lemma 10.2.

In fact, it is true even without the assumption that G is transitive:

Proposition 10.10. *Let (G, C) be a network with $0 < \inf_e C(e) \leqslant \sup_e C(e) < \infty$, and assume that G is roughly isometric to $\mathbb{Z}$. Then the* WSF *of G has a.s. two ends.*

Proof. Note that (G, C) is recurrent, and therefore the WSF is connected a.s. It follows that a.s. the WSF has at least two ends.

Fix $o \in \mathsf{V}$. Take N to be a large integer. Let S be the set of vertices at distance N from o. It is easy to show that the size of S is bounded independently of N. There is a partition $S = S_- \cup S_+$ such that the diameter of each of the sets S_-, S_+ is bounded independently of N. Run Wilson's algorithm with root o, starting with the vertices in S_+ in any order. Let $\mathcal{P}$ be the first path from S_+ to o constructed by the algorithm. There are constants k and c such that there are at least cN disjoint connected subgraphs of size k that separate S_+ from o. Consequently, the probability that a random walk from any vertex in S_+ that stops at o will not hit $\mathcal{P}$ decays exponentially with N. That means that the probability that the WSF contains two paths from o to S_+ that are disjoint except at o tends to zero as $N \to \infty$. Since the same is true for S_-, a.s. the tree does not contain three infinite rays starting at o that are disjoint except at o. Because this is true for any o, there are at most two ends in the WSF. ∎

Recall that the **critical probability** $p_c(G)$ of a graph G is the supremum of $p \in [0,1]$ for which p-Bernoulli percolation has only finite components a.s.

In contrast to Theorem 10.1, we have:

PROPOSITION 10.11. *If G is a transitive network whose automorphism group is unimodular and $\mathsf{WSF} \neq \mathsf{FSF}$, then FSF-a.s., there is a tree with uncountably many ends, in fact, with $p_c < 1$.*

We do not know if a.s. every FSF component has infinitely many ends under the above hypotheses (Question 15.7).

Proof. We have that $\mathbf{E}_{\mathsf{WSF}}[\deg_{\mathfrak{F}} x] = 2$ for all x, whence $\mathbf{E}_{\mathsf{FSF}}[\deg_{\mathfrak{F}} x] > 2$ for all x by Proposition 5.10. Apply Thm. 7.1 of BLPS (1997) and ergodicity. ∎

§11. Analysis of the WSF on a Tree.

In this section, we study the WSF components on a tree, where a more detailed analysis is possible. We give a simple derivation of Häggström's complete description for regular trees; we give a necessary and sufficient condition for all components of the WSF on an arbitrary tree to have one end each a.s.; and we specialize to the WSF on spherically symmetric trees. Also, we prove that all components are recurrent, which solves a special case of Conjecture 15.1. Since the WSF of a recurrent tree is the whole tree, it follows that any recurrent tree can arise as a component of WSF.

Consider first the WSF on a regular tree of degree $d+1$. Choose a vertex, o, and begin Wilson's method rooted at infinity from o. We obtain a ray ξ from o to start our forest. Now o has d other neighbors, $x_1, \ldots, x_d$. By beginning random walks at each of them in turn, we see that the events $A_i := \{x_i \text{ connected to } o\}$ are independent given ξ. Furthermore, it is easy to verify that the probability of a random walk starting at a neighbor of o ever to visit o is $1/d$, so $\mathbf{P}[A_i \mid \xi] = 1/d$. On the event A_i, we add only the edge (o, x_i) to the forest and then we repeat the analysis from x_i. Thus, the tree containing o includes, apart from the ray ξ, a critical Galton-Watson tree with binomial offspring distribution $(d, 1/d)$. In addition, each vertex on ξ has another random subtree attached to it; its first generation has binomial distribution $(d-1, 1/d)$, but subsequent generations yield Galton-Watson trees with binomial distribution $(d, 1/d)$. In particular, a.s. every tree added to ξ is finite. This means that the tree containing o has only one end, the equivalence class of ξ. This analysis is easily extended to form a complete description of the entire wired spanning forest. The resulting description is due to Häggström (1997b), whose work predates Wilson's algorithm.

In general, we see that if we begin Wilson's algorithm rooted at infinity at a vertex o in a transient graph, it immediately generates one end of the tree containing o. In order for this tree to have more than one end, a succession of "coincidences" need to occur, building up other ends by gradually adding on finite pieces. This is possible (see Corollary 11.2), but not on transient Cayley graphs (see Theorem 10.1).

A **unit flow** on a tree T with root o is a nonnegative function μ on the vertices of T such that $\mu(o) = 1$ and for each vertex v, the sum of $\mu(w)$ over all children w of v equals $\mu(v)$.

THEOREM 11.1. (WSF ON GENERAL TREES) *Let T be an infinite tree with root o, and suppose T is assigned resistances $\langle R(e)\rangle$ that make it transient. Denote $h(v) := \mathbf{P}_v[\tau_o < \infty]$. For any vertex $v \neq o$, let $\hat{v}$ be the parent of v.*

(a) If for all unit flows μ from o to ∞ on T, the sum

$$\sum_{v \neq o} \mu(v)^2 [h(v)^{-1} - h(\hat{v})^{-1}] \tag{11.1}$$

diverges, then all components of the WSF *on T have one end a.s.; if this sum converges for some unit flow μ, then a.s. the* WSF *on T has components with uncountably many ends.*

(b) All components of the WSF *on T are recurrent for the given resistances a.s.*

For the proof, we use a general **Bernoulli percolation** on T; this means that a subgraph ω of T is chosen at random by including edges of T independently. The capacity and random walk criterion for infinite components in percolation given by Lyons (1992) will be used several times in the course of the proof. One may also consult Lyons and Peres (1998) for more background. Later, we shall also use the special case of p-Bernoulli percolation, which means that the probability that each edge belongs to ω is p.

Proof. (a) We reason as in the regular case above, but the Galton-Watson subtrees are replaced by a Bernoulli percolation. The existence of a flow that makes the sum (11.1) converge is independent of the base vertex o (see Lyons (1990)). Consider the Bernoulli percolation ω on T that keeps the edge from x to its parent with probability $\mathbf{P}_v[\tau_{\hat{v}} < \infty]$. Then $h(v)$ is the probability that the path from v to o is open in ω. By Theorem 5.1, the WSF-component of o is the union of a loop-erased random walk path from o, and all the components of ω that intersect this path.

By Lyons (1992), the sum (11.1) diverges for all μ iff all components of ω are finite a.s.; in this case, all components in the WSF on T have one end a.s. Conversely, if (11.1) converges for some μ, then ω has infinite components a.s. and by Lemma 4.2 in Pemantle and Peres (1995), each infinite component has uncountably many ends a.s. Since every component of ω is contained in a component of the WSF, part (a) is established.

(b) Using the notation of (a), it is easy to see that if all components of ω are recurrent for the given resistances, then so are the components of the WSF. By Lyons (1992), a tree is recurrent for the resistances $\langle R(e)\rangle$ iff a certain Bernoulli percolation has only finite components. In this percolation, the probability that a vertex v is connected to o is $p_v := \left(1+\sum_e R(e)\right)^{-1}$, where the sum is over all edges on the path between v and o. This percolation applied to the components of ω has no infinite components a.s. iff

$$\sum_{v\neq 0} \mu(v)^2 \left(\frac{1}{h(v)p_v} - \frac{1}{h(\hat{v})p_{\hat{v}}}\right) = \infty$$

for all unit flows μ by Lyons (1992). Since this sum dominates the sum

$$\sum_{v\neq o} \frac{\mu(v)^2 R(\hat{v},v)}{h(v)}, \tag{11.2}$$

it suffices to prove divergence of the latter for any unit flow μ.

Write $|w| := \mathrm{dist}(o,w)$. If the path from o to v passes through w (i.e., v is a descendant of w), write $v \geqslant w$. Write $R(w \leftrightarrow \infty)$ for the effective resistance of the descendant subtree of w, i.e., the minimal energy of a unit flow on this subtree with w as root. See Lyons and Peres (1998) for more background. For any N,

$$\sum_{|v|\geqslant N} \frac{\mu(v)^2 R(\hat{v},v)}{h(v)} \geqslant \sum_{|w|=N} \frac{\mu(w)^2}{h(w)} \sum_{v\geqslant w} \frac{\mu(v)^2}{\mu(w)^2} R(\hat{v},v) \geqslant \sum_{|w|=N} \frac{\mu(w)^2}{h(w)} R(w\leftrightarrow\infty)\,.$$

Denoting the rightmost quantity by A_N and using the identity

$$h(w) = \frac{R(w\leftrightarrow\infty)}{R(\hat{w},w)+R(w\leftrightarrow\infty)} h(\hat{w})\,,$$

we obtain

$$A_{N+1} = \sum_{|u|=N} \frac{1}{h(u)} \sum_{\hat{w}=u} \mu(w)^2 \left(R(\hat{w},w)+R(w\leftrightarrow\infty)\right).$$

By the Cauchy-Schwarz inequality, for any vertex u and any positive labels $\langle r_w\rangle$ on its children,

$$\sum_{\hat{w}=u} \mu(w)^2 r_w \geqslant \frac{\mu(u)^2}{\sum_{\hat{w}=u} r_w^{-1}}\,.$$

Applying this to the preceding inequality and invoking the parallel-series laws gives

$$A_{N+1} \geqslant \sum_{|u|=N} \frac{1}{h(u)} R(u\leftrightarrow\infty) = A_N\,.$$

Since $A_0 = R(o\leftrightarrow\infty) > 0$, the tails of the series (11.2) are bounded away from 0, so the series diverges. ∎

Given a tree T with root o and given $k \in \mathbb{N}$, the **level** T_k is the set of vertices of T at distance k from o. The tree is called **spherically symmetric** if for all k, every vertex in T_k has the same number of children.

COROLLARY 11.2. (SPHERICALLY SYMMETRIC TREES) *Let T be a spherically symmetric tree with levels $\langle T_k \rangle_{k \geqslant 0}$. Suppose that for each $k \geqslant 1$, every edge connecting T_{k-1} with T_k is assigned resistance r_k and the resulting network is transient, i.e., $\sum_m r_m/|T_m| < \infty$.*

Denote $L_n := \sum_{m>n} r_m/|T_m|$. If

$$\sum_{n \geqslant 1} \frac{r_n}{|T_n|^2 L_n L_{n-1}} = \infty\,, \tag{11.3}$$

then all components of the WSF *on T have one end a.s.; if this series converges, then a.s. some component of the* WSF *on T has uncountably many ends.*

Note that the series in (11.3) converges if $r_n \equiv 1$ and $|T_n|/n^\gamma$ is bounded above and below by positive constants for some $\gamma > 1$.

Proof. For every vertex $v \in T_n$, we have $h(v) = \mathbf{P}_v[\tau_o < \infty] = A_n/A_0$. By convexity, the sums in (11.1) and (11.2) are minimized by the unit flow defined by $\mu(v) := 1/|T_n|$ for $v \in T_n$. The corollary now follows from Theorem 11.1(a). ∎

REMARK 11.3. Let T be a tree. For the WSF on T corresponding to the conductances $C(e) := \lambda^{-\mathrm{dist}(o,e)}$, with $\lambda \geqslant 1$ bounded above by the branching number of T, all component trees have branching number at most λ a.s. (by Theorem 11.1 and Lyons (1990)). It is not hard to see that equality holds if T has bounded degree, but not in general. When this equality holds, the WSF interpolates continuously between the wired uniform spanning forest (when $\lambda = 1$) and the whole tree (when λ is larger than the branching number of T). See Lyons (1990) for more information about random walks on trees and the branching number.

§12. Planar Graphs and Hyperbolic Lattices.

A **planar** graph is a graph G embedded in the plane. A **planar network** is a network (G, C), where G is a planar graph. A **face** of a planar graph G is a component of $\mathbb{R}^2 - G$. A planar network is **proper** if every bounded set in the plane contains only finitely many edges and vertices.

Suppose that (G, C) is a finite or proper planar network. We define the dual network $(G^\dagger, C^\dagger)$ as follows. In each face f of G, we place a single vertex $f^\dagger$ of $G^\dagger$. For every edge e in G, we place an edge $e^\dagger$ in $G^\dagger$ connecting $f_1^\dagger$ and $f_2^\dagger$, where f_1 and f_2 are the two faces on either side of e. (It may happen that $f_1 = f_2$; then $e^\dagger$ is a loop.) This is the usual construction of the **dual graph**, $G^\dagger$. Note that $G^\dagger$ is locally finite iff the boundary of every face of G has finitely many vertices. Now set $C^\dagger(e^\dagger) := C(e)^{-1}$ for every $e \in \mathsf{E}$.

When T is a set of edges of G, set

$$T^* := \{e^\dagger : e \notin T\};$$

this is a subgraph of $G^\dagger$. The following is well known.

PROPOSITION 12.1. (DUAL TREES) *Let G be a finite connected planar network and $G^\dagger$ its dual. Let T be a spanning tree of G. Then T^* is a spanning tree of $G^\dagger$. Moreover,* $\mathrm{weight}(T)/\mathrm{weight}(T^*)$ *is independent of T.*

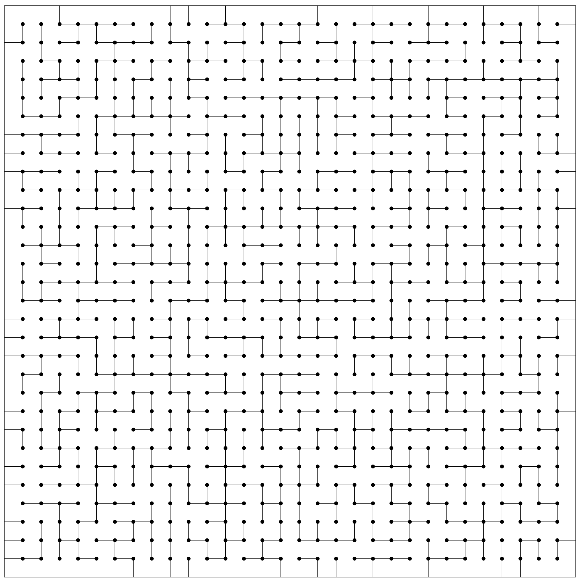

Figure 2. A uniformly chosen wired spanning tree on a subgraph of $\mathbb{Z}^2$, drawn by Wilson (see Propp and Wilson (1996)).

Fig. 2 illustrates the situation. It reveals two spanning trees: one in white, the other in black on the planar dual graph. Note that in the dual, the outer boundary of the grid is identified to a single vertex.

Proof. T^* has no cycles because T is connected, and $(\mathsf{V}^\dagger, T^*)$ is connected as T has no cycles. It is easy to see that $\mathrm{weight}(T)/\mathrm{weight}(T^*)$ is constant. ∎

THEOREM 12.2. (FSF IS DUAL TO WSF) *Let G be a proper planar network and $G^\dagger$ its dual. Suppose that $G^\dagger$ is locally finite. Let T denote the* FSF *of G. Then T^* has the same distribution as the* WSF *of $G^\dagger$.*

Proof. Given a finite connected subgraph G_n of G with connected complement $G - G_n$, let $G_n^\dagger$ be its planar dual. Notice that $G_n^\dagger$ can be regarded as a finite subgraph of $G^\dagger$, but with the outer boundary vertices identified to a single vertex. By Proposition 12.1, if T is the weighted random spanning tree of G_n, then T^* is the weighted random spanning tree of $G_n^\dagger$. Thus, the theorem follows from the definitions of the FSF and the WSF. ∎

COROLLARY 12.3. *Let G be a proper planar network with $G^\dagger$ locally finite. Then* WSF $=$ FSF *in G iff this happens in $G^\dagger$.* ∎

Pemantle (1991) stated that the uniform spanning tree of $\mathbb{Z}^2$ has one end. We gave a proof and extension in Theorem 10.3. An extension to graphs that need not be transitive is:

THEOREM 12.4. *Let G be a proper planar network with $G^\dagger$ locally finite and recurrent. Then a.s. each component of the* FSF *of G has only one end.*

Proof. Suppose that a component of $\mathfrak{F}_G$, the FSF of G, has at least two ends with positive probability. Then a bi-infinite path in it separates $G^\dagger$, which means that $(\mathsf{V}^\dagger, \mathfrak{F}_G^*)$ is disconnected. By Theorem 12.2, it follows that the WSF of $G^\dagger$ is disconnected with positive probability, which is impossible on a recurrent graph by Proposition 5.6. We conclude that each component of $\mathfrak{F}_G$ has only one end. ∎

Similar reasoning shows:

PROPOSITION 12.5. (TOPOLOGY FROM DUALITY) *Let G be a proper planar network with $G^\dagger$ locally finite. If each tree of the* WSF *of G has only one end a.s., then the* FSF *of $G^\dagger$ has only one tree a.s. If, in addition, the* WSF *of G has infinitely many trees a.s., then the tree of the* FSF *of $G^\dagger$ has infinitely many ends a.s.* ∎

On a Riemannian manifold M, a harmonic Dirichlet function $f : M \to \mathbb{R}$ is a function satisfying $\operatorname{div} \nabla f = 0$ and $\int_M |\nabla f|^2 < \infty$. There is an interesting phase transition between dimensions 2 and 3 in hyperbolic space: $\mathbf{HD}(\mathbb{H}^d) \cong \mathbb{R}$ for $d = 1$ and $d \geqslant 3$, but not for $d = 2$. See Sario et al. (1977) or Dodziuk (1979).

Suppose that G is a graph of bounded vertex degree that is roughly isometric to a manifold M with bounded local geometry. Kanai's (1986) theorem says that M is transient iff G is transient, while Holopainen and Soardi (1997) have shown that $\mathbf{HD}(G) \cong \mathbb{R}$ iff $\mathbf{HD}(M) \cong \mathbb{R}$. Consequently:

THEOREM 12.6. (HYPERBOLIC PHASE TRANSITION) *Let G be a graph with bounded degrees that is roughly isometric to $\mathbb{H}^d$. Then $\mathbf{HD}(G) \cong \mathbb{R}$ iff $d \neq 2$.*

A graph embedded in $\mathbb{R}^2$ or $\mathbb{H}^2$ is **self-dual** if it is isomorphic to its dual.

Taking stock, we arrive at the following surprising results:

THEOREM 12.7. (WSF AND FSF IN $\mathbb{H}^d$) *If G is a self-dual proper planar Cayley graph roughly isometric to $\mathbb{H}^2$, then the* WSF *of G has infinitely many trees a.s., each having one end a.s., while the* FSF *of G has one tree a.s. with infinitely many ends a.s. If G is a Cayley graph roughly isometric to $\mathbb{H}^d$ for some $d \geqslant 3$, then the* WSF = FSF *of G has infinitely many trees a.s., each having one end a.s.*

Proof. In either case, each tree of the WSF has one end by Theorem 10.1. It follows from Corollary 9.6 that a.s. the WSF has infinitely many trees. Now Theorems 12.6 and 7.3 and Proposition 12.5 complete the proof. ∎

An example of a self-dual Cayley graph roughly isometric to $\mathbb{H}^2$ is shown in Fig. 3. (See Chaboud and Kenyon (1996) for characterizations of planar Cayley graphs.)

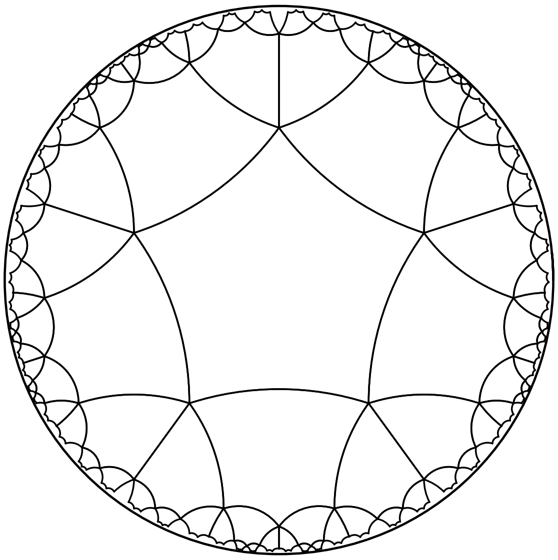

Figure 3. A self-dual Cayley graph in the hyperbolic disc.

Remark 12.8. (Dropping Self-Duality) Actually, the assumption in the first part of Theorem 12.7 that G is self-dual is not necessary. When G is not assumed to be self-dual, the dual $G^\dagger$ might not be transitive. However, one can show that the automorphism group of $G^\dagger$ is unimodular and its action on the vertices of $G^\dagger$ (namely, the faces of G) has finitely many orbits. One can, with some technical difficulties, generalize Theorem 10.1 to this setting.

Here is a summary of the phase transitions.

	$\mathbb{Z}^d$		$\mathbb{H}^d$	
d	$2-4$	$\geqslant 5$	2	$\geqslant 3$
FSF: trees	1	∞	1	∞
ends	1	1	∞	1
WSF: trees	1	∞	∞	∞
ends	1	1	1	1

Finally, we note a general corollary for transient planar graphs.

Corollary 12.9. *Let G be a transient planar network with bounded vertex conductance. Then* FSF $\neq$ WSF *on G and the* WSF *has infinitely many trees.*

Proof. Benjamini and Schramm (1996a,b) proved that $\mathbf{HD}(G) \not\cong \mathbb{R}$. Their results also imply that two independent random walks in G intersect only finitely many times a.s., so Theorem 9.4 applies. ∎

An answer to Question 15.2 in Section 15 might provide a strengthening of this statement.

§13. The WSF in Nonamenable Graphs.

We have seen that the trees in the WSF of a Cayley graph have only one end. Here, we consider their geometry. Let $B(o,n)$ be a ball of radius n centered at a vertex o in G. How much of the ball $B(o,n)$ is taken up by the component T_o of the origin? For $\mathbb{Z}^d$ with $d \leqslant 4$ this is, of course, the whole ball. For $d \geqslant 5$, the component retains the "4-dimensionality" it has in $\mathbb{Z}^4$: from the random walk estimates in Lawler (1991), it is easily deduced that $\mathbf{E}|T_o \cap B(o,n)| n^{-4}$ is bounded above and below by positive constants. See Benjamini, Kesten, Peres and Schramm (1998) for more on this topic.

The **spectral radius** $\rho(G)$ of a network G may be defined using the random walk $\langle X(n)\rangle$ on G:

$$\rho(G) := \limsup_{n\to\infty} (\mathbf{P}_x[X(n)=y])^{1/n}\,.$$

(The definition does not depend on x and y since G is connected.) The obvious inequality $\mathbf{P}_x[X(nk)=x] \geqslant \mathbf{P}_x[X(n)=x]^k$ implies that $\mathbf{P}_x[X(n)=x] \leqslant \rho^n$ for any $x \in V$ and $n \geqslant 1$. For the network with unit conductances on a finitely generated group G, Kesten (1959) showed that $\rho(G)=1$ iff G is amenable. More generally, for any network, $\rho(G)<1$ is equivalent to a strong isoperimetric inequality (see Varopoulos (1985) or Gerl (1988)).

When $\rho(G)<1$, the WSF-components are thinner than in $\mathbb{Z}^d$:

Theorem 13.1. (Tree Growth When $\rho<1$) *Let G be a graph with $\rho(G)<1$ and bounded vertex degree. Let $o \in \mathsf{V}$ be some basepoint. Denote by T_o the component of o in the* WSF*. Then $c^{-1}n^2 \leqslant \mathbf{E}|T_o \cap B(o,n)| \leqslant c\,n^2$ for some $0<c<\infty$ and all $n \geqslant 1$.*

Proof of Theorem 13.1. We start with the upper bound. Let D be a bound on the vertex degrees. For any vertex x and fixed $k \leqslant m$, we have

$$\begin{aligned}\sum_{y\in G} \mathbf{P}_o[X(k)=y]\mathbf{P}_x[X(m-k)=y] &\leqslant D\sum_{y\in G}\mathbf{P}_o[X(k)=y]\mathbf{P}_y[X(m-k)=x]\\ &= D\mathbf{P}_o[X(m)=x]\,.\end{aligned}$$

Therefore, by Wilson's algorithm rooted at ∞,

$$\mathbf{P}[x\in T_o] \leqslant \sum_{y\in G}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\mathbf{P}_o[X(k)=y]\mathbf{P}_x[X(m-k)=y] \leqslant D\sum_{m=0}^{\infty}(m+1)\mathbf{P}_o[X(m)=x]\,.$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}\left(\sum_{x\in B(o,n)}\mathbf{P}_o[X(m)=x]\right)^2 &\leqslant |B(o,n)|\sum_{x\in B(o,n)}\mathbf{P}_o[X(m)=x]^2\\ &\leqslant D|B(o,n)|\mathbf{P}_o[X(2m)=o]\,.\end{aligned}$$

Consequently, $\sum_{x\in B(o,n)}\mathbf{P}[x\in T_o]$ is at most

$$D\sum_{m\leqslant cn}(m+1) + D^{3/2}|B(o,n)|^{1/2}\sum_{m>cn}(m+1)\mathbf{P}_o[X(2m)=o]^{1/2} \tag{13.1}$$

for every c. The hypothesis implies that $\mathbf{P}_o[X(2m)=o]\leqslant\rho^{2m}$ where $\rho=\rho(G)<1$. Thus by choosing c large enough, because $|B(o,n)|\leqslant(D+1)^n$, we can ensure that the second summand in (13.1) tends to 0 as $n\to\infty$, so that

$$\mathbf{E}|T_o\cap B(o,n)| = \sum_{x\in B(o,n)}\mathbf{P}[x\in T_o] \leqslant \frac{cD(n+2)^2}{2}$$

for all large n, which establishes the upper bound.

It remains to prove the lower bound on $\mathbf{E}|B(o,n) \cap T_o|$. For every $v \in \mathsf{V}$, let X_v be a simple random walk starting at v, with X_v, X_w independent when $v \neq w$. Let $\mathsf{fu}(o)$ denote the loop erasure of X_o, as in Section 10, and denote by $g(v,w) := \sum_{k \geqslant 0} \mathbf{P}[X_v(k) = w]$ the Green function for simple random walk on G. Observe that for any two vertices v, w at distance k, we have

$$g(v,w)^2 \leqslant \mathbf{P}[w \in X_v] g(w,w) D g(w,v) \leqslant \frac{D}{1-\rho} \sum_{j=2k}^{\infty} \mathbf{P}[X_v(j) = v] \leqslant \frac{D}{(1-\rho)^2} \rho^{2k} ,$$

so that $g(v,w) \leqslant c_0 \rho^k$ for some constant c_0.

The probability that a vertex v is in T_o is the probability that X_v intersects $\mathsf{fu}(o)$. Now $\mathsf{fu}(o)$ contains vertices at every distance from o; we shall show that for some $c_1 > 0$ and any set $S \subset B(o, n/2)$ that contains precisely one vertex at distance k from o whenever $0 \leqslant k \leqslant n/2$, we have

$$\sum_{v \in B(o,n)} \mathbf{P}[L_v(S) > 0] \geqslant c_1 n^2 , \tag{13.2}$$

where $L_v(S) := \sum_{w \in S} \sum_{k \geqslant 0} \mathbf{1}_{\{X_v(k) \in S\}}$ is the total occupation time of S by X_v.

Since every random walk starting at a vertex $w \in B(o, n/2)$ must visit at least $n/2$ vertices before leaving $B(o,n)$, we clearly have $\sum_{v \in B(o,n)} g(w,v) \geqslant n/2$, and therefore

$$\begin{aligned}\sum_{v \in B(o,n)} \mathbf{E}[L_v(S)] = \sum_{v \in B(o,n)} \sum_{w \in S} g(v,w) &\geqslant D^{-1} \sum_{v \in B(o,n)} \sum_{w \in S} g(w,v) \\ &\geqslant D^{-1} \sum_{w \in S} n/2 \geqslant n^2/4D . \end{aligned} \tag{13.3}$$

By the Markov property,

$$\mathbf{E}[L_v(S) \mid L_v(S) > 0] \leqslant \max_{y \in S} \mathbf{E}[L_y(S)] .$$

For any $x, y \in S$, we have $\operatorname{dist}(x,y) \geqslant |\operatorname{dist}(o,x) - \operatorname{dist}(o,y)|$. Therefore,

$$\mathbf{E}[L_y(S)] \leqslant \sum_{k \geqslant 0} 2 c_0 \rho^k = c_2$$

when $y \in S$, whence $\mathbf{E}[L_v(S)] \leqslant c_2 \mathbf{P}[L_v(S) > 0]$. In conjunction with (13.3), this yields (13.2).

By conditioning on $\mathsf{fu}(o)$, we obtain from (13.2) the bound

$$\forall n \quad \mathbf{E}|B(o,n) \cap T_o| \geqslant c_1 n^2 . \tag{13.4}$$

∎

Remark 13.2. In the proof of the lower bound (13.4) given above, the hypothesis that $\rho(G) < 1$ can be replaced by the weaker hypothesis that there is a summable decreasing sequence $\langle f(k) \rangle$, such that the Green function on G satisfies $g(x,y) \leqslant f(\text{dist}(x,y))$ for all vertices x, y. This weaker hypothesis holds on any Cayley graph satisfying $|B(o,n)| \geqslant cn^4$ for all n; see Hebisch and Saloff-Coste (1993). In conjunction with Corollary 9.6, this implies that (13.4) holds on any transient Cayley graph.

We believe that the components of the WSF are recurrent on any graph (Conjecture 15.1). This was established for trees in Theorem 11.1 and can now be verified for graphs with $\rho < 1$:

Corollary 13.3. *Let G be a graph with $\rho(G) < 1$ and bounded vertex degree. Let $o \in \mathsf{V}$ be some basepoint. Then a.s. the WSF-component T_o is recurrent.*

The corollary follows immediately from Theorem 13.1 and the following general lemma, since the distance from a vertex v to o in T_o is bounded below by the distance in G.

Lemma 13.4. *Let Υ be a random graph with a distinguished vertex o, and denote by Υ_j the set of vertices at distance j from o. If $\mathbf{E} \sum_{j=1}^n |\Upsilon_j| \leqslant cn^2$ for some $c < \infty$ and all $n \geqslant 1$, then simple random walk on Υ is recurrent a.s.*

Proof. Let $\langle a_n \rangle$ be a decreasing positive sequence such that $\sum_n a_n = \infty$ and $\sum_n n a_n^2 < \infty$, e.g., $a_n = (n \log(n+1))^{-1}$. By the Cauchy-Schwarz inequality,

$$\infty = \left(\sum_{n=1}^{\infty} a_n \right)^2 \leqslant \left(\sum_{n=1}^{\infty} \frac{1}{|\Upsilon_n|} \right) \left(\sum_{n=1}^{\infty} a_n^2 |\Upsilon_n| \right) . \tag{13.5}$$

However,

$$\begin{aligned} \mathbf{E} \left(\sum_{n=1}^{\infty} a_n^2 |\Upsilon_n| \right) &= \sum_{n=1}^{\infty} (a_n^2 - a_{n+1}^2) \mathbf{E} \left(\sum_{j=1}^{n} |\Upsilon_j| \right) \\ &\leqslant \sum_{n=1}^{\infty} (a_n^2 - a_{n+1}^2) cn^2 = \sum_{n=1}^{\infty} a_n^2 c(2n-1) < \infty . \end{aligned}$$

Hence, (13.5) gives $\sum_{n=2}^{\infty} 1/|\Upsilon_n| = \infty$ a.s. By the Nash-Williams criterion (see Doyle and Snell (1984)), this implies that Υ is recurrent a.s. ∎

Remark 13.5. The hypothesis of Lemma 13.4 can be replaced by $\mathbf{E} \sum_{j=1}^n |\Upsilon_j| \leqslant n b_n$, where $\langle b_n \rangle$ is an increasing sequence such that $\sum b_n^{-1} = \infty$ and $\sup_n n(b_n - b_{n-1})/b_n < \infty$.

To verify this, define $a_n := (b_n D_n)^{-1}$ where $D_n := \sum_{k=1}^{n} b_k^{-1}$, observe that $\sum a_n = \infty$ but $\sum a_n^2 b_n < \infty$, and mimic the proof above.

In $\mathbb{Z}^d$, $d > 4$, the WSF is not connected. How close is it to being connected? One variant of this question was addressed in Remark 9.7.

Here is another variant. Suppose that $\epsilon \in (0,1)$. Let ω_ϵ be ϵ-Bernoulli percolation with marginal ϵ; that is, for all $e \in \mathsf{E}$, let $e \in \omega$ with probability ϵ, independently for different e's, and independent of the WSF, $\mathfrak{F}$. Is $\mathfrak{F} \cup \omega_\epsilon$ connected?

Burton and Keane (1989) have shown that a random subgraph of $\mathbb{Z}^d$ whose distribution is invariant under translations and satisfies the so-called "finite-energy" condition has a.s. at most one infinite component. Although $\mathfrak{F} \cup \omega_\epsilon$ satisfies only half of that condition, namely, that edges can be added without great penalty, the Burton-Keane argument does show that $\mathfrak{F} \cup \omega_\epsilon$ and also ω_ϵ have a.s. at most one infinite component in any transitive amenable network. One can argue that this demonstrates that uniform spanning forest on $\mathbb{Z}^d$ is a critical model: it exhibits criticality with respect to connectivity.

In contrast, it has been conjectured by Benjamini and Schramm (1996c) that on any nonamenable transitive graph, there is some ϵ such that ω_ϵ has infinitely many infinite components. Although this conjecture is still unresolved, the following theorem shows that for any nonamenable network, $\mathfrak{F} \cup \omega_\epsilon$ has a.s. infinitely many infinite components if $\epsilon > 0$ is sufficiently small.

THEOREM 13.6. ($\mathsf{WSF} + \epsilon$) *Let G be a network with edge conductances bounded below and vertex conductances bounded above. Assume that the spectral radius ρ of G is less than 1. Let $\mathfrak{F}$ be the* WSF *of G. For any $\epsilon \in (0,1)$, let ω_ϵ be ϵ-Bernoulli percolation on G. Then, provided ϵ is sufficiently close to 0, a.s. there are infinitely many components in $\mathfrak{F} \cup \omega_\epsilon$ and each has infinitely many ends.*

We first show:

LEMMA 13.7. *Let G be a network where the probability that two random walk paths starting at v and w will intersect tends to 0 as $\operatorname{dist}(v,w) \to \infty$. Then a.s. each component of the* WSF *has an infinite boundary as a subgraph of G.*

Proof. Consider a finite (possibly empty) vertex set A. We need to show that the probability that A is the boundary of a component of the WSF is zero. Let W be any infinite component of A^c. By applying the hypothesis to two sufficiently far apart vertices v, w in W, we see that the probability that W is in one component of the WSF is arbitrarily small, hence 0. Thus A is a.s. not the boundary of a WSF tree. Since there are only countably many finite sets of vertices, this completes the proof. ∎

Proof of Theorem 13.6. Adding loops if necessary, we may assume for convenience that all the vertex conductances are equal. (This may change ρ, but will not make $\rho(G)$ equal 1.) It then follows that for every $v, u \in \mathsf{V}$ and all $n \geqslant 0$,

$$\mathbf{P}_v[X(n) = u] \leqslant \mathbf{P}_v[X(2n) = v]^{1/2} \leqslant \rho^n\,, \tag{13.6}$$

where X is the random walk on G.

Given $o, v \in \mathsf{V}$, let $N(o, v)$ be the number of distinct simple paths from o to v in $\mathfrak{F} \cup \omega_\epsilon$. We shall show that for all $\epsilon > 0$ sufficiently small,

$$\forall o, v \in \mathsf{V} \quad \mathbf{E}[N(o, v)] < \infty \tag{13.7}$$

and

$$\lim_{\mathrm{dist}(o,v)\to\infty} \mathbf{E}[N(o, v)] = 0\,. \tag{13.8}$$

Let us first see that this will suffice to prove the theorem. Note that $\mathbf{E}[N(o, v)]$ bounds the probability that o and v are in the same $\mathfrak{F} \cup \omega_\epsilon$ component. Consequently, (13.8) implies that a.s. $\mathfrak{F} \cup \omega_\epsilon$ has infinitely many components. There is an obvious map ψ from the set of ends of components of $\mathfrak{F}$ to the set of ends of components of $\mathfrak{F} \cup \omega_\epsilon$. Suppose that v and o belong to distinct components of $\mathfrak{F}$, but to the same component of $\mathfrak{F} \cup \omega_\epsilon$. Let ξ_v be an end of the component of $\mathfrak{F}$ containing v and let ξ_o be an end of the component of $\mathfrak{F}$ containing o. If $\psi(\xi_o) = \psi(\xi_v)$, then it easily follows that $N(o, v) = \infty$. Consequently, ψ is a.s. injective when (13.7) holds. By the preceding lemma, each component of $\mathfrak{F}$ has infinite boundary a.s. Therefore, a.s. each component of $\mathfrak{F}$ has infinitely many edges in ω_ϵ connecting it to other components of $\mathfrak{F}$. For every pair T_1, T_2 of components of $\mathfrak{F}$, there are only finitely many edges in ω_ϵ joining them, by injectivity of ψ. Consequently, every component T_* of $\mathfrak{F} \cup \omega_\epsilon$ contains infinitely many components of $\mathfrak{F}$; therefore, T_* must have infinitely many ends, by another application of injectivity of ψ. Consequently, it is enough to prove (13.7) and (13.8).

We now think of $\mathfrak{F}$ as oriented towards infinity, that is, we consider the OWSF. Recall that at every vertex w of $\mathfrak{F}$, there is precisely one outgoing edge of $\mathfrak{F}$.

Consider a basepoint $o \in \mathsf{V}$ and some $z \in \mathsf{V} - \{o\}$. Suppose that $\mathcal{P}$ is a simple path from o to z in $\mathfrak{F} \cup \omega_\epsilon$. Then there is a unique sequence $\gamma(\mathfrak{F}, \omega_\epsilon, \mathcal{P})$ of the form

$$(v_1, w_1, u_1, v_2, w_2, u_2, \dots, u_n) \tag{13.9}$$

with the following properties:

(a) $v_1 = o$, $u_n = z$;

(b) for each $j = 1, \ldots, n$, there is a path $\mathcal{P}_j^+$ in $\mathfrak{F} \cap \mathcal{P}$ from v_j to w_j, and the orientation of this path agrees with the orientation of $\mathfrak{F}$ and of $\mathcal{P}$;

(c) for each $j = 1, \ldots, n$, there is a path $\mathcal{P}_j^-$ in $\mathfrak{F} \cap \mathcal{P}$ from w_j to u_j, which agrees with the orientation of $\mathcal{P}$ and goes opposite to the orientation on $\mathfrak{F}$;

(d) all the paths $\mathcal{P}_j^\pm$ are pairwise vertex disjoint, except that $\mathcal{P}_j^+$ and $\mathcal{P}_j^-$ share the vertex w_j; and

(e) for each $j = 1, \ldots, n-1$, there is an edge in ω_ϵ connecting u_j and v_{j+1}, but there is no such edge in $\mathfrak{F}$.

Note that we may have $v_j = w_j$ or $w_j = u_j$; that is, some of the paths $\mathcal{P}_j^\pm$ may have only one vertex. To obtain this sequence $\gamma(\mathfrak{F}, \omega_\epsilon, \mathcal{P})$, just follow $\mathcal{P}$ and record every vertex where the orientation changes or where an edge of $\omega_\epsilon - \mathfrak{F}$ is used. Note that given $\mathfrak{F}$, ω_ϵ, and β of the form (13.9), there is at most one simple path $\mathcal{P}$ from v_1 to u_n in $\mathfrak{F} \cup \omega_\epsilon$ such that $\beta = \gamma(\mathfrak{F}, \omega_\epsilon, \mathcal{P})$.

We are going to compare the probability of finding a sequence $\gamma(\mathfrak{F}, \omega_\epsilon, \mathcal{P})$ in $\mathfrak{F} \cup \omega_\epsilon$ to the probability of finding it in the image of the network random walk. Let $q(\beta)$ be the probability that there is some simple path $\mathcal{P}$ in $\mathfrak{F} \cup \omega_\epsilon$ from o to z such that $\beta = \gamma(\mathfrak{F}, \omega_\epsilon, \mathcal{P})$.

Say that a sequence β of the form (13.9) is **adapted** to a finite or infinite path $y(0), y(1), \ldots$ in G if $y(0) = v_1$ and there are integers $0 \leqslant t_1 \leqslant t_1' < t_2 \leqslant t_2' < \ldots \leqslant t_n'$ such that $y(t_j) = w_j$ and $y(t_j') = u_j$ for $j = 1, \ldots, n$ and $y(t_j' + 1) = v_{j+1}$ for $j = 1, \ldots, n-1$. Let $q_X(\beta)$ be the probability that β is adapted to X, where $X(0), X(1), \ldots$ denotes the network random walk that starts at $X(0) = o$.

LEMMA 13.8. *There is a constant $c > 0$, depending only on the network G, such that for all β of the form (13.9) such that each u_j neighbors in G with v_{j+1} ($j = 1, \ldots, n-1$),*

$$q(\beta) \leqslant (c\epsilon)^{n-1} q_X(\beta).$$

Proof. Construct $\mathfrak{F}$ by Wilson's algorithm rooted at infinity, starting with the vertices

$$w_1, w_2, \ldots, w_n, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$$

in this order. For β to occur as $\gamma(\mathfrak{F}, \omega_\epsilon, \mathcal{P})$ for a simple path $\mathcal{P} \subset \mathfrak{F} \cup \omega_\epsilon$ from v_1 to u_n, the random walk starting at each v_j and each u_j must hit the corresponding w_j, and $[u_j, v_{j+1}]$ must be in ω_ϵ for each approprite j. Let $\varphi(v, w)$ denote the probability that a network random walk that starts at v will hit w. Then we get

$$q(\beta) \leqslant \epsilon^{n-1} \prod_{j=1}^{n} \left(\varphi(v_j, w_j) \varphi(u_j, w_j) \right).$$

Reversibility and the equality of conductances at vertices imply that $\varphi(v,w) = \varphi(w,v)$. Since u_j neighbors with v_{j+1} when $j = 1, \ldots, n-1$, the transition probabilities satisfy $p(u_j, v_{j+1}) \geqslant 1/c$ for some constant $c > 0$ depending only on G. Consequently,

$$\begin{aligned} q(\beta) &\leqslant \epsilon^{n-1} \prod_{j=1}^{n} \Big(\varphi(v_j, w_j)\varphi(w_j, u_j)\Big) \\ &\leqslant (c\epsilon)^{n-1} \prod_{j=1}^{n} \big(\varphi(v_j, w_j)\varphi(w_j, u_j)\big) \prod_{j=1}^{n-1} p(u_j, v_{j+1}) \\ &\leqslant (c\epsilon)^{n-1} q_X(\beta), \end{aligned}$$

which proves the lemma. ∎

We now continue with the proof of Theorem 13.6. Let $\mathcal{Y}_m$ be the set of all walks $y = \langle y(0), \ldots, y(m)\rangle$ with $y(0) = o$, $y(j) \sim y(j-1)$ for $j = 1, \ldots, m$, and $y(m) = z$. Let $\mathcal{B}_n$ be the set of all sequences β of the form (13.9) with $v_1 = o$, $u_n = z$, $u_j \sim v_{j+1}$, all u_j distinct, all v_j are distinct, and all w_j distinct. Given $y \in \mathcal{Y}_m$, let $\mathcal{B}_n(y)$ be the set of $\beta \in \mathcal{B}_n$ adapted to y. Note that

$$\forall y \in \mathcal{Y}_m \qquad \big|\mathcal{B}_n(y)\big| \leqslant \binom{m}{n}^3 \leqslant \binom{3m}{3n}.$$

Therefore, by Lemma 13.8, we have

$$\begin{aligned} \mathbf{E}[N(o,z)] &= \sum_{n=1}^{\infty} \sum_{\beta \in \mathcal{B}_n} q(\beta) \\ &\leqslant \sum_{n=1}^{\infty} \sum_{\beta \in \mathcal{B}_n} (c\epsilon)^{n-1} q_X(\beta) \\ &\leqslant \sum_{m=1}^{\infty} \sum_{y \in \mathcal{Y}_m} \mathbf{P}_o[\forall j = 1, \ldots, m \ X(j) = y(j)] \sum_{n=1}^{m/3} (c\epsilon)^{n-1} \big|\mathcal{B}_n(y)\big| \\ &\leqslant \sum_{m=1}^{\infty} \mathbf{P}_o[X(m) = z] \sum_{n=1}^{m/3} (c\epsilon)^{n-1} \binom{3m}{3n} \\ &\leqslant (c\epsilon)^{-1} \sum_{m=1}^{\infty} \mathbf{P}_o[X(m) = z] \Big(1 + (c\epsilon)^{1/3}\Big)^{3m}. \end{aligned}$$

With (13.6), this gives

$$\mathbf{E}[N(o,z)] \leqslant (c\epsilon)^{-1} \sum_{m \geqslant \mathrm{dist}(o,z)} \rho^m \Big(1 + (c\epsilon)^{1/3}\Big)^{3m}.$$

This implies (13.7) and (13.8), which complete the proof. ∎

We call a random subgraph of G an **automorphism-invariant percolation** if its distribution is invariant under the automorphism group of G. The Burton-Keane (1989) argument has two parts; one shows that all automorphism-invariant percolations on an amenable transitive graph have a.s. at most 2 ends in each component. Theorem 13.6 provides a converse, which can be sharpened as follows:

COROLLARY 13.9. (BURTON-KEANE CONVERSE) *Let G be a nonamenable connected graph with a unimodular transitive automorphism group. Then there is an automorphism-invariant percolation process ψ on G in which each component is a tree with infinitely many ends.*

Proof. By Lemma 7.2 of BLPS (1997), any invariant percolation (in our case, $\mathfrak{F} \cup \omega_\epsilon$) with more than two ends in some component can be thinned out to a forest ψ' with more than two ends in some component. By Theorem 7.1 of that paper, some component of ψ' will have $p_c < 1$ with positive probability, hence infinitely many ends. Condition on that event, and let ψ'' be ψ' with all trees that have finitely many ends removed. For each vertex $v \in \mathsf{V}$ that is not in an infinite component of ψ'', choose randomly, uniformly and independently an edge $e = e(v)$ that connects it to a vertex closer to the infinite components of ψ''. Let ψ be the union of ψ'' and all such edges $e(v)$. Then ψ satisfies the requirements. ∎

In Benjamini, Lyons and Schramm (1998), Theorem 13.6 is used to show that under the same assumptions as in Corollary 13.9, there is an automorphism-invariant random forest $\mathfrak{F} \subset G$ with $\rho(\mathfrak{F}) < 1$ a.s.

We can extend Proposition 10.11 to the planar non-transitive setting:

COROLLARY 13.10. *Let G be a proper planar graph with bounded degrees and a bounded number of sides to its faces. If $\rho(G) < 1$, then a.s. some component T of the* FSF *on G has $p_c(T) < 1$.*

Proof. Let $\mathfrak{F}$ be the FSF of G, and let ω_ϵ be independent ϵ-Bernoulli percolation. Recall from Theorem 12.2 that $\mathfrak{F}^* := \{e^\dagger : e \notin \mathfrak{F}\}$ has the same distribution as the WSF on $G^\dagger$. Consequently, $\mathfrak{F} - \omega_\epsilon$ has the same distribution as $\mathfrak{F}^* \cup \{e^\dagger : e \in \omega_\epsilon\}$. Since $\{e^\dagger : e \in \omega_\epsilon\}$ is ϵ-Bernoulli percolation on $G^\dagger$, the result follows from Theorem 13.6, because $\mathfrak{F} - \omega_\epsilon$ has an infinite component whenever $\mathfrak{F}^* \cup \{e^\dagger : e \in \omega_\epsilon\}$ has more than one infinite component. ∎

§14. Applications to Loop-Erased Walks and Harmonic Measure.

Infinite loop-erased random walk is defined in any transient network by chronologically erasing cycles from the random walk path. On a recurrent network, the natural substitute is to run random walk until it first reaches distance n from its starting point, erase cycles, and take a weak limit as $n \to \infty$. On a general recurrent network, such a weak limit need not exist; in $\mathbb{Z}^2$, weak convergence was established by Lawler (1988) using Harnack inequalities (see Lawler (1991), Prop. 7.4.2). Lawler's approach yields explicit estimates of the rate of convergence, but is difficult to extend to other networks. Using spanning trees, we obtain the following general result.

Proposition 14.1. *Let $\langle G_n \rangle$ be an exhaustion of a recurrent network G. Consider the network random walk $\langle X_o(k) \rangle$ started from $o \in G$. Denote by $\tau_{G_n^c}$ the first exit time of G_n, and let L_n be the loop erasure of the path $\langle X_o(k) : 0 \leqslant k \leqslant \tau_{G_n^c} \rangle$. If the random spanning tree T_G in G has one end a.s., then the random paths L_n converge weakly to the law of the unique ray from o in T_G. In particular, this applies if G is a proper planar network with a locally finite recurrent dual.*

Proof. This is immediate from the definition of the WSF, Wilson's algorithm applied to the wired graph G_n^W, and Proposition 5.6. The final assertion uses Theorem 12.4. ∎

Let A be a finite set of vertices in a recurrent network G. Denote by τ_A the hitting time of A, and by h_v^A the harmonic measure from v on A:

$$\forall B \subseteq A \quad h_v^A(B) := \mathbf{P}_v[X_v(\tau_A) \in B] .$$

If the measures h_v^A converge when $\mathrm{dist}(v, A) \to \infty$, then it is natural to refer to the limit as **harmonic measure from ∞ on A**. This convergence fails in some recurrent networks (e.g., in $\mathbb{Z}$), but it does hold in $\mathbb{Z}^2$; see Lawler (1991, Thm. 2.1.3). As above, random spanning trees yield a very general result.

Theorem 14.2. *Let G be a recurrent network and A be a finite set of vertices. Suppose that random spanning tree T_G in G has one end a.s. Then the harmonic measures h_v^A converge as $\mathrm{dist}(v, A) \to \infty$.*

Proof. Add a finite set B of edges to G to form a graph G' in which the subgraph (A, B) is connected, and suppose that B is minimal with respect to this property. Assign unit conductance to the edges of B. Note that having at most one end in T is a tail event. Since $T_{G'}$ conditioned on $T_{G'} \cap B = \emptyset$ has the same distribution as T_G, by tail triviality, a.s. $T_{G'}$ has one end. Similarly, because $T_{G'}$ conditioned on $T_{G'} \cap B = B$ has the same distribution as $T_{G'/B}$, also $T_{G'/B}$ has one end a.s.

The path from v to A in $T_{G'/B}$ is constructed by running a random walk from v until it hits A and then loop erasing. Thus, when $\mathrm{dist}(v, A) \to \infty$, the measures h_v^A must tend to the conditional distribution, given $T_{G'} \cap B = B$, of the point in A that is closest (in $T_{G'/B}$) to the unique end of $T_{G'/B}$. ∎

If the FSF in a network G is a tree T, then for any two adjacent vertices $v, w \in G$, there is a unique simple path $\mathcal{P}(v, w)$ in T that connects them; when G is recurrent, Proposition 5.6 shows that this path can be obtained by loop erasing the network random walk started at v and stopped at w. An easy lower bound for the tail probabilities of the random variable diam $\mathcal{P}(v, w)$ is given in the following theorem; it would be interesting to obtain precise estimates for these probabilities; see Lawler (1998) for a recent improvement.

THEOREM 14.3. (CONNECTION DIAMETER TAIL) *Consider the uniform spanning forest in $\mathbb{Z}^d$, $2 \leqslant d \leqslant 4$, and let v, w be adjacent vertices in $\mathbb{Z}^d$. Then*

$$\forall n \geqslant 1 \quad \mathbf{P}[\mathrm{diam}\ \mathcal{P}(v, w) \geqslant n] \geqslant \frac{1}{8n}\,. \tag{14.1}$$

Proof. We use the coordinates $(x_1, \ldots, x_d)$ in $\mathbb{Z}^d$. Let ∂S be the perimeter of the square $S := [-n, n]^2 \times \{0\}^{d-2}$; we think of ∂S both as a closed path and as a set of $8n$ edges. For each edge $e = [v, w]$ in ∂S, we may consider $\mathcal{P}(e) = \mathcal{P}(v, w)$ as a detour for e. If for all e in ∂S, the path $\mathcal{P}(e)$ had diameter less than n, then the concatenation of these paths would be a closed path, homotopic to ∂S in $\mathbb{R}^d - \{x : x_1 = x_2 = 0\}$. Consequently, T would contain a cycle, which is a contradiction. Thus

$$\sum_{e \in \partial S} \mathbf{1}_{\{\mathrm{diam}\ \mathcal{P}(e) \geqslant n\}} \geqslant 1\,.$$

Taking expectations and using the edge-transitivity of $\mathbb{Z}^d$ proves (14.1). ∎

§15. Open Questions.

There are many tantalizing open questions related to uniform spanning forests. We present a sample here.

CONJECTURE 15.1. *Each component tree of the wired uniform spanning forest on any graph G is recurrent a.s. for simple random walk. More generally, if the edges in the components of the* WSF *on a network are given the conductances they have in the network, then all the components are recurrent a.s.*

This holds when G is a transitive network with $\mathrm{Aut}(G)$ unimodular, when G is a tree, and when G is a graph satisfying $\rho(G) < 1$, by Theorem 10.1, Theorem 11.1 and Corollary 13.3, respectively.

QUESTION 15.2. Let G be a proper transient planar graph with bounded degree and a bounded number of sides to its faces. Is the free spanning forest a single tree a.s.? If true, this would strengthen Corollary 12.9.

QUESTION 15.3. If a graph has spectral radius < 1, must each tree in its wired uniform spanning forest have one end a.s.? We know that each tree is recurrent a.s., by Corollary 13.3.

QUESTION 15.4. Does each component in the wired uniform spanning forest on an infinite supercritical Galton-Watson tree have one end a.s.?

QUESTION 15.5. Let G be a transitive network whose automorphism group is not unimodular. Does every tree of the WSF on G have one end a.s.?

QUESTION 15.6. Let G be an infinite network. Is the number of trees of the FSF a.s. a constant?

QUESTION 15.7. Let G be a transitive network with $\mathsf{WSF} \neq \mathsf{FSF}$. Must *all* components of the FSF have infinitely many ends a.s.?

In view of Proposition 10.11, this would follow in the unimodular case from a proof of the following conjecture:

CONJECTURE 15.8. *The components of the* FSF *on a transitive graph are indistinguishable in the sense that for every automorphism-invariant property* $\mathcal{A}$ *of subgraphs, either a.s. all components satisfy* $\mathcal{A}$ *or a.s. they all do not. The same holds for the* WSF.

QUESTION 15.9. Is there a "natural" coupling of FSF and WSF? For example, if G is a Cayley graph, is there a coupling that is invariant under multiplication by elements of G?

QUESTION 15.10. Let G be a graph with bounded degree and $\rho(G) < 1$. Let T_o denote the component of the vertex o in the WSF on G, as in Theorem 13.1. Does the inequality

$$\liminf_{n\to\infty} n^{-2}|T_o \cap B(o,n)| > 0$$

hold almost surely?

CONJECTURE 15.11. *Let* T_o *be the component of the identity* o *in the* WSF *on a Cayley graph, and let* $\xi = \langle v_n : n \geqslant 0\rangle$ *be the unique ray from* o *in* T_o*. The sequence of "bushes"* $\langle b_n\rangle$ *observed along* ξ *converges in distribution. (Formally,* b_n *is the connected component of* v_n *in* $T - v_{n-1} - v_{n+1}$*, multiplied on the left by* v_n^{-1}*.)*

QUESTION 15.12. (This question was also asked by J. Propp; see Propp (1998).) One may consider the uniform spanning tree on $\mathbb{Z}^2$ embedded in $\mathbb{R}^2$. In fact, consider it on $\epsilon\mathbb{Z}^2$ in

$\mathbb{R}^2$ and let $\epsilon \to 0$. In what sense should a limit be taken, and how can one show the limit exists? Does the limit have some conformal invariance property?

The invariance asked for in Question 15.12 could be expected on the basis of conformal invariance of simple random walk. It is also supported by some computer simulations of O. Schramm. In addition, the uniform spanning tree is intimately tied to random domino tiling of $\mathbb{Z}^2$: see, e.g., Burton and Pemantle (1993). New work of Rick Kenyon (1997) shows that domino tilings have a strong form of conformal invariance. Nonrigorous conformal field theory was used by Duplantier (1992) and Majumdar (1992) to estimate the rate of escape of loop-erased walks in $\mathbb{Z}^2$. Finally, critical Bernoulli percolation is believed to have conformal invariance in the limit (see, e.g., Langlands, Pouliot and Saint-Aubin (1994) and Benjamini and Schramm (1996d)) and the uniform spanning tree is a critical model, as explained in Section 13.

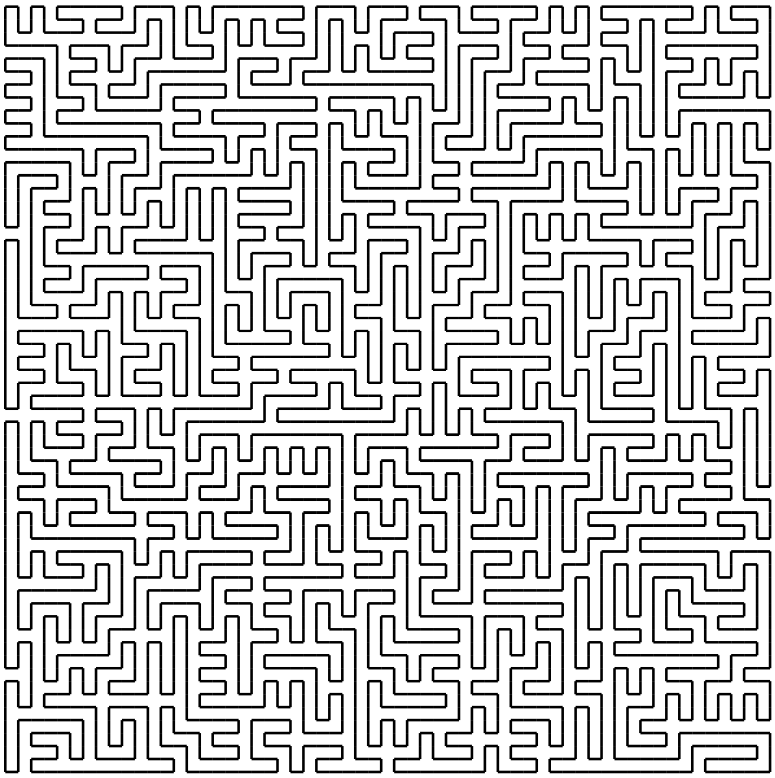

Figure 4. The inside of this lattice-filling curve is a uniform spanning tree on a square grid, while the outside is another on the dual grid (wired).

Tóth and Werner (1997), §11, explain the connection between certain self-repelling random walks on $\mathbb{Z}$ and simple oriented random walk on $\mathbb{Z}^2$ that is oriented so that only steps to the right and up are possible. Of course, simple oriented random walk never visits any state more than once. Tóth and Werner consider coalescing paths of this Markov chain; these paths form the wired random spanning tree of $\mathbb{Z}^2$ that is associated to this chain by Wilson's method rooted at infinity as in Remark 5.4. The dual of this tree is considered, as well as the lattice-filling curve that "threads" between the two trees; see Fig. 4 for the lattice-filling curve of a *uniform* spanning tree. Their paper is devoted to analyzing the continuous analogue of these objects in the *oriented* case. In particular, a stochastic differential equation describes their space-filling curve. Note that the lattice-filling curve completely determines the tree and its dual. There is also a stochastic differential equation that is conjectured to determine the space-filling curve that presumably is the limit as $\epsilon \to 0$ of the lattice-filling curves threading between the uniform spanning tree and its dual on $\epsilon\mathbb{Z}^2$. More details on this conjecture will appear in forthcoming work of O. Schramm.

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MATHEMATICS DEPARTMENT, THE WEIZMANN INSTITUTE OF SCIENCE, REHOVOT 76100, ISRAEL
itai@wisdom.weizmann.ac.il
http://www.wisdom.weizmann.ac.il/~itai/

DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, IN 47405-5701, USA
rdlyons@indiana.edu
http://php.indiana.edu/~rdlyons/

INSTITUTE OF MATHEMATICS, THE HEBREW UNIVERSITY, GIVAT RAM, JERUSALEM 91904, ISRAEL AND DEPARTMENT OF STATISTICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720-3860, USA
peres@stat.berkeley.edu
http://www.stat.berkeley.edu/~peres/

MATHEMATICS DEPARTMENT, THE WEIZMANN INSTITUTE OF SCIENCE, REHOVOT 76100, ISRAEL
schramm@wisdom.weizmann.ac.il
http://www.wisdom.weizmann.ac.il/~schramm/