Uniformization and percolation

Itai Benjamini

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Riemann's theorem and probability

In the 1940's Shizuo Kakutani observed that two dimensional Brownian motion is conformal invariant, up to a time reparametrization. Therefore the scaling limit of simple random walks on the Euclidean grid is conformal invariant.

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Riemann surfaces uniformization

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Uniformization and random process

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A discrete uniformization theorem

Every planar graph admits a circle packing Koebe (1936).

He and Schramm (1995): Let G be the 1-skeleton of a triangulation of an open disk. If the random walk on G is *recurrent*, then G is circled packed in the Euclidean plane. Conversely, if the degrees of the vertices in G are bounded and the random walk on G is *transient*, then G is circle packed in the unit disc.

With Oded Schramm (1995) we used an extended version of the Brooks, smith, Stone and Tutte (1940) square tiling theorem, –a related discrete uniformization theorem for graphs using squares.

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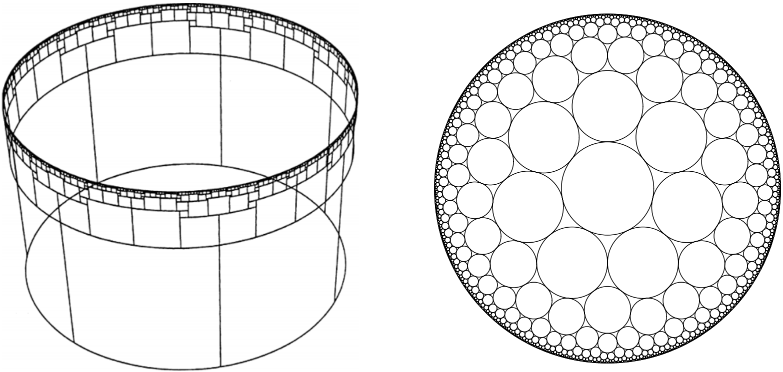
The square tiling and the circle packing of the 7-regular hyperbolic triangulation



Discrete uniformization and random walks

Using discrete uniformization, with Oded (1995) we showed : A bounded degree transient planar graph admits non constant bounded harmonic functions.

Corollary: $\mathbb{Z}^3$ is not planar.

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The space of bounded harmonic functions

Moreover the Poisson boundary of a planar graph coincides with the boundary of its square tiling and with the boundary of its circle packing.
Recent works by Georgakopoulos and by Angel, Barlow, Gurel-Gurevich and Nachmias respectively.

Uniformization and percolation?

Let G be the 1-skeleton of a bounded degree triangulation of an open disk.

Conjecture

Assume G is transient, then $1/2$-Bernoulli site percolation on G admits an infinite cluster a.s.

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One reason to be skeptical about the conjecture is that for critical percolation on the triangular lattice, the probability the cluster of the origin reaches distance r decays polynomially in r, while there are transient triangulations of volume growth $r^2 \log^3 r$.

Motivation for the conjecture, a short detour

Tile the unit square with (possibly infinity number) of squares of varying sizes so that at most three squares meet at corners. Color each square *black* or *white* with equal probability independently.

Conjecture

Show that there is a universal $c > 0$, so that the probability of a black *left right crossing is bigger than c. And as the size of the largest square goes to 0, the crossing probability goes to $1/2$.*

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Comments on the tiling conjecture

If true, the same should hold for a tiling, or a packing of a triangulation, with a set of shapes that are of bounded Hausdorff distance to circles.

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Comments on the tiling conjecture

Behind the tiling conjecture is a rough version of conformal invariance. That is, the crossing probability is balanced if the tiles are of uniformly bounded distance to circles (rotation invariance), and the squares can be of different sizes, (dilation invariance).

From the tiling to the percolation conjecture

Let G the 1-skeleton of bounded degree transient triangulation of an open disk. By *discrete uniformization* it admits a circle packing with similar properties as the tiling in conjecture. And if the conformal invariance heuristic holds, we will a.s. see macroscopic crossings for $1/2$-Bernoulli site percolation.

Non uniqueness at $1/2$

Moreover by same reasoning we will see unboundely many macroscopic clusters for $1/2$-Bernoulli percolation, suggesting that if G is a 1-skeleton of bounded degree transient a triangulation of an open disk, then there are a.s. infinitely many infinite clusters for $1/2$-Bernoulli site percolation?

Since we believe that $p_c \leq 1/2$ for such G's we conjecture that $p_u < 1$ and uniqueness monotonicity.

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Further comments

We *believe* that $p_c \geq 1/2$ for polynomial growth triangulations of the open disk. Note that if all degrees are at least 6, polynomial growth implies that vertices of higher degrees are polynomially sparse, this suggests that their critical probability for percolation is $1/2$, as of the triangular lattice. For nonamenable transitive or sofic triangulations $p_c < 1/2$, *remove the transitivity assumption?*

iff?

What about a converse to conjecture? Does recurrence implies no percolation at $1/2$?

Study similar questions in the context of magnetization.

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