∅
∅

Critical Percolation on Any Nonamenable Group Has No Infinite Clusters

BY ITAI BENJAMINI, RUSSELL LYONS, YUVAL PERES, AND ODED SCHRAMM

Abstract. We show that independent percolation on any Cayley graph of a nonamenable group has no infinite components at the critical parameter. This result was obtained in Benjamini, Lyons, Peres, and Schramm (1997) as a corollary of a general study of group-invariant percolation. The goal here is to present a simpler self-contained proof that easily extends to quasi-transitive graphs with a unimodular automorphism group. The key tool is a "mass-transport" method, which is a technique of averaging in nonamenable settings.

s.intro §1. Introduction.

The main long-standing open question in percolation is to show that critical percolation in $\mathbb{Z}^d$ has no infinite components for all $d \geq 2$. The work of Harris (1960) and Kesten (1980) established the two-dimensional case; Hara and Slade (1994) proved it for $d \geq 19$. Recently, a study of percolation on other graphs, such as Cayley graphs, was initiated. The relevant definitions appear below. Here is our main theorem.

t.pc THEOREM 1.1. *Let X be a Cayley graph of a finitely generated nonamenable group, and consider either site or bond percolation on X. Then at the corresponding critical value $p = p_c$, almost surely there is no infinite component.*

In particular, there are no infinite components for critical percolation on any lattice in hyperbolic space, nor on a (k-regular tree) $\times\, \mathbb{Z}$. The latter was previously known when $k \geq 7$ (Wu 1993). Wu's proof goes along the lines of the high-dimensional Euclidean proof and uses the triangle condition.

1991 Mathematics Subject Classification. Primary 60B99. Secondary 60D05, 82B43.
Key words and phrases. Percolation, Cayley graphs, amenability.
Research partially supported by the Institute for Advanced Studies, Jerusalem (Lyons), NSF grant DMS-94-04391 (Peres), and the Sam and Ayala Zacks Professorial Chair (Schramm).

The key tool in our proof is a new technique of "mass transport". This tool is especially valuable in nonamenable settings, where it can sometimes replace the ergodic theorem for approximation of expectations by suitable spatial averages. The mass-transport method was introduced by Häggström (1996) and developed in Benjamini, Lyons, Peres, and Schramm (1997). The latter paper is a study of group-invariant, possibly dependent, percolation on transitive graphs. After completing the first draft of that paper, we realized that combining two of the results there yields Theorem 1.1.

The resulting proof of Theorem 1.1 was not the most direct possible; in this note, we illustrate the technique of mass transport by presenting the simplest proof we know for Theorem 1.1. Another advantage of the proof given here is that it easily extends to the wider setup of quasi-transitive graphs with a unimodular nonamenable automorphism group. Nevertheless, the present proof still uses invariant percolations.

In the study of percolation on nonamenable graphs, new phenomena, such as a double phase transition, appear; the first example of this is due to Grimmett and Newman (1990), who considered percolation on the product of a regular tree and $\mathbb{Z}$. Benjamini and Schramm (1996) present an introduction to the subject and a list of problems, some of which have been recently solved; see the final section of the present paper for an update. In that section, we also survey some further results and conjectures concerning percolation on nonamenable graphs.

Terminology: Let Γ be a finitely generated group. Given a finite symmetric set of generators $S = \{g_1^{\pm 1}, \ldots, g_n^{\pm 1}\}$ for Γ, the (right) **Cayley graph** of Γ is the graph $X(\Gamma, S) := (\mathsf{V}, \mathsf{E})$ with $\mathsf{V} := \Gamma$ and $[g, h] \in \mathsf{E}$ iff $g^{-1}h \in S$. Thus, the graph $X(\Gamma, S)$ depends on the set of generators. Note that any two Cayley graphs of the same group are roughly isometric (quasi-isometric). More generally, a graph X is called **transitive** if for any vertices x, y, there is a graph automorphism mapping x to y. Clearly, any Cayley graph is transitive by the action of left multiplication.

Let the **edge-isoperimetric constant** of X be

$$\iota_{\mathsf{E}}(X) := \inf\left\{\frac{|\partial_{\mathsf{E}} K|}{|K|};\ K \subset \mathsf{V} \text{ is finite}\right\},$$

where $\partial_{\mathsf{E}} K$, the edge boundary of K, consists of all edges $[u, v]$ with $u \in K$ and $v \in \mathsf{V} - K$. A finitely generated group is **nonamenable** if for some finite generating set, the edge-isoperimetric constant of its Cayley graph is positive. The isoperimetric constant of a graph is sometimes called the Cheeger constant.

In Bernoulli bond percolation on a graph $X = \big(\mathsf{V}(X), \mathsf{E}(X)\big)$, the edges are **open** with probability p independently. Those edges that are not open are **closed**. The corresponding product measure on the edge configurations is denoted $\mathbf{P}_p$. The **percolation subgraph** is the random graph whose vertices are $\mathsf{V}(X)$ and whose edges are the open edges. Let $K(v)$ be the **cluster** of v, that is, the connected component of v in the percolation subgraph. We write

$$\theta_v(p) := \mathbf{P}_p\big[K(v) \text{ is infinite}\big]\,.$$

On a Cayley graph (or, more generally, on a transitive graph), the value of $\theta_v(p)$ is independent of the choice of v, whence the subscript v is dropped. Let

$$p_c := \inf\big\{p\,;\ \theta(p) > 0\big\}$$

be the critical probability for percolation. Bernoulli site percolation is defined similarly with vertices replacing edges. See Grimmett (1989) for more information about Bernoulli percolation.

s.mt §2. Mass Transport.

Bernoulli percolation is one example of invariant percolation. If a group Γ acts by automorphisms on a graph X, we call a probability measure on the subgraphs of X a **Γ-invariant percolation** if it is invariant under the action of Γ. An invariant percolation $\mathbf{P}$ is a **bond percolation** if $\mathbf{P}[\mathsf{V}(\omega) = \mathsf{V}(X)] = 1$, where ω denotes a subgraph of X.

We now present the mass-transport principle. Let $m(x, y; \omega)$ be a nonnegative function of three variables: two vertices x, y and a subgraph ω of X. Intuitively, $m(x, y; \omega)$ represents the mass transported from x to y when the percolation subgraph is ω. We suppose that $m(\bullet, \bullet; \bullet)$ is invariant under the diagonal action of Γ, i.e., $m(x, y; \omega) = m(gx, gy; g\omega)$ for all x, y, g and ω. The **Mass-Transport Principle** asserts that for any invariant percolation, the expected total mass transported out of any vertex x equals the expected total mass transported into x, i.e.,

$$\forall x \in \mathsf{V} \quad \sum_{y \in \mathsf{V}} M(x, y) = \sum_{y \in \mathsf{V}} M(y, x)\,, \tag{2.1}$$ e.mass

where $M(x, y) := \mathbf{E}m(x, y; \omega)$ is also invariant under the diagonal action of Γ.

On a Cayley graph, it is straightforward to verify (2.1):

$$\sum_{y \in \mathsf{V}} M(x, y) = \sum_{g \in \Gamma} M(x, gx) = \sum_{g \in \Gamma} M(g^{-1}x, x) = \sum_{y \in \mathsf{V}} M(y, x)\,.$$

The creative element in applying the mass-transport method is to make a judicous choice of the transport function $m(x, y; \omega)$. This depends on the particular application; three such applications are given in this note.

The identity (2.1) is valid on many transitive graphs that are not Cayley graphs, provided their automorphism group is unimodular; see BLPS (1997). The proof of Theorem 1.1 below uses only transitivity of X and (2.1), so it applies to these graphs as well. With minor modifications, the proof also applies to unimodular **quasi-transitive** graphs, i.e., graphs $X = (\mathsf{V}, \mathsf{E})$ whose the automorphism group Γ is unimodular and has finitely many orbits in V.

Let S be a finite set of generators for a nonamenable group Γ and let $X := X(\Gamma, S)$ be the corresponding Cayley graph. For a finite subgraph $K \subset X$, set

$$\alpha_K := \frac{1}{|K|} \sum_{x \in \mathsf{V}(K)} \deg_K(x)\,,$$

where $\deg_K(x)$ refers to the degree of x as a vertex in the graph K. Let

$$\alpha(X) := \sup\{\alpha_K\,;\; K \subset X \text{ is finite}\}\,.$$

Since every vertex of X has the same degree, it follows that

$$\alpha(X) + \iota_{\mathsf{E}}(X) = \deg_X(o)\,, \tag{2.2}$$ e.isoalpha

where $o \in \mathsf{V}(X)$ is the identity of Γ.

t.threshold THEOREM 2.1. *Let X by a Cayley graph of a nonamenable group Γ with respect to some finite set of generators. Let $o \in \mathsf{V}(X)$ denote the identity of Γ. Suppose that ξ is an invariant bond percolation on X. If $\mathbf{E}\deg_\xi(o) > \alpha(X)$, then ξ has an infinite component with positive probability.*

r.threshold REMARK 2.2. It suffices that $\mathbf{E}\deg_\xi(o) \ge \alpha(X)$. See BLPS (1997) for details.

c.threshold COROLLARY 2.3. *If $\mathbf{P}[e \notin \xi] < \iota_{\mathsf{E}}(X)/\deg_X(o)$ for all $e \in \mathsf{E}(X)$, then ξ has an infinite component with positive probability.*

Proof. This follows immediately from Theorem 2.1 and (2.2). ∎

Proof of Theorem 2.1. For each $v \in \mathsf{V}(X)$, let $K(v)$ denote the component of ξ containing v. Set

$$m(v, u; \xi) := \begin{cases} \deg_\xi(v)/|K(v)| & \text{if } K(v) \text{ is finite and } u \in K(v), \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$\sum_{u \in \mathsf{V}(X)} m(u, o; \xi) = \alpha_{K(o)} \le \alpha(X)$$

whenever $K(o)$ is finite, and $\sum_{u\in\mathsf{V}(X)} m(u,o;\xi) = 0$ otherwise. On the other hand,

$$\sum_{u\in\mathsf{V}(X)} m(o,u;\xi) = \begin{cases} \deg_\xi(o) & \text{when } K(o) \text{ is finite,} \\ 0 & \text{otherwise.} \end{cases}$$

Since $m(\bullet,\bullet;\bullet)$ is diagonally invariant, the Mass-Transport Principle gives

$$\begin{aligned} &\mathbf{E}\big[\deg_\xi(o) \bigm| |K(o)| < \infty\big]\mathbf{P}\big[|K(o) < \infty|\big] \\ &\qquad = \mathbf{E} \sum_{u\in\mathsf{V}(X)} m(o,u;\xi) = \mathbf{E} \sum_{u\in\mathsf{V}(X)} m(u,o;\xi) \le \alpha(X)\mathbf{P}\big[|K(o) < \infty|\big]\,. \end{aligned}$$

When $\mathbf{P}\big[|K(o)| < \infty\big] = 1$, we get $\mathbf{E}\deg_\xi(o) \le \alpha(X)$, as required. ∎

s.unique §3. Unique Percolation.

Proof of Theorem 1.1. For simplicity, we restrict our attention to critical bond percolation; the proof for critical site percolation requires only minor modifications.

The number of infinite clusters is invariant under the action of Γ. By ergodicity, the number of infinite clusters is constant a.s. It follows that this constant must be 0, 1 or ∞; see Newman and Schulman (1981). In this section, we rule out a unique infinite cluster at $p = p_c$, and in the next section, we rule out infinitely many infinite clusters.

Let ω be critical Bernoulli bond percolation on X. Suppose that almost surely there is a unique infinite cluster U of ω. Let γ_ϵ be an ϵ-Bernoulli bond percolation on X that is independent of ω. We use γ_ϵ to define a bond percolation ξ_ϵ on X as follows. For $v \in \mathsf{V}$, let $U(v) \subset U$ be the set of vertices $u \in U$ such that $d_X(v,u) = d_X(v,U)$; that is, $U(v)$ is the set of vertices of U that are closest to v. Given two neighbors $v, w \in \mathsf{V}$, put the edge $[v,w]$ in ξ_ϵ iff $d_X(v,U) < 1/\epsilon$, $d_X(w,U) < 1/\epsilon$, and $U(v) \cup U(w)$ is contained in a connected component of $\omega - \gamma_\epsilon$. If we fix ω, then

$$\lim_{\epsilon\to 0} \mathbf{P}\big[[v,w] \notin \xi_\epsilon \bigm| \omega\big] = 0\,,$$

because every pair of vertices in U is joined by some finite path in U that is unlikely to intersect γ_ϵ when ϵ is small. Consequently, the Bounded Convergence Theorem gives

$$\lim_{\epsilon\to 0} \mathbf{P}\big[[v,w] \notin \xi_\epsilon\big] = 0\,.$$

By Corollary 2.3, it follows that for $\epsilon > 0$ sufficiently small, with positive probability, there will be an infinite component of ξ_ϵ. However, when there is an infinite ξ_ϵ-component, there must be an infinite component in $\omega - \gamma_\epsilon$. As $\omega - \gamma_\epsilon$ is a $(1-\epsilon)p_c$-Bernoulli bond percolation, this contradicts the definition of p_c and proves that ω cannot have a unique infinite cluster.

s.nonunique §4. Nonunique Percolation.

In this section, we continue with the proof of Theorem 1.1, dealing with the case that the critical percolation has more than one infinite cluster.

d.enc DEFINITION 4.1. Let G be an infinite graph. A vertex $v \in \mathsf{V}(G)$ is an **encounter point** if there are three or more neighbors of v that belong to distinct infinite connected components of $G - \{v\}$.

Suppose that with positive probability, ω has more than one infinite cluster. It is then well known (Burton and Keane 1989) that the set Y of encounter points of ω is nonempty almost surely.

We say that C has an **ω-neighbor** of v if there is some $x \in C$ with $[x, v] \in \omega$.

l.infenc LEMMA 4.2. *Consider Bernoulli bond percolation ω with $p \geq p_c$ on a Cayley graph. With $\mathbf{P}_p$-probability 1, for every encounter point v of ω and every infinite component C of $\omega - \{v\}$ having an ω-neighbor of v, there is some encounter point of ω in C.*

Proof. Let $u, v \in \mathsf{V}$. Set $m(u, v; \omega) := 1$ if u and v are in the same cluster of ω and v is the unique encounter point of ω with minimal distance to u (where distances are measured in ω). Otherwise, set $m(u, v; \omega) := 0$. Then $\sum_{u \in \mathsf{V}} m(v, u; \omega) \leq 1$. By the Mass-Transport Principle, it follows that almost surely $\sum_{u \in \mathsf{V}} m(u, v; \omega) < \infty$ for all $v \in \mathsf{V}$. This implies the lemma. ∎

We now construct an invariant forest ξ whose vertices are Y, the encounter points of ω. The edges of ξ will not necessarily be in E. For each $v \in \mathsf{V}$, let $x(v)$ be a uniformly distributed random variable in $[0, 1]$ such that ω and all $x(v)$ are mutually independent. Suppose that $v \in Y$. For every infinite connected component Z of $\omega - \{v\}$ having an ω-neighbor of v, let $Y(Z, v) \subset Y \cap Z$ be the set of encounter points of ω in Z that are closest (in ω) to v, and let $u(Z, v)$ be the vertex $u \in Y(Z, v)$ that minimizes $x(u)$. Let ξ be the set of all edges $[v, u(Z, v)]$, where $v \in Y$ and Z is an infinite connected component of $\omega - \{v\}$ having an ω-neighbor of v. It is clear that ξ is an invariant graph on V. For every $v \in Y$, the degree of v in ξ is at least 3 by Lemma 4.2.

We now verify that ξ is almost surely a forest, that is, it contains no cycles. Assume otherwise, and suppose that C is a cycle in ξ of length ≥ 3. When $u = u(Z, v)$, consider edges in ξ of the form $[v, u]$ to be directed from v to u. Let v be a vertex in C and note that $C - \{v\}$ is contained in some connected component Z of $\omega - \{v\}$, because v appears only once in C. This shows that the number of edges in C that are directed away from v is at most one. Since the number of edges in C equals the number of vertices in C, every vertex must have one edge directed towards it and one edge directed away from it.

If $[a, b]$ and $[b, c]$ are directed edges in C, then the distance in ω from b to c cannot be larger than the distance from a to b. As C is a cycle, it follows that these distances must be the same. Let z be the vertex in C that maximizes $x(z)$. Then there must be vertices a, b in C such that $[a, b]$ and $[b, z]$ are directed edges of C and $a \neq z$. But a and z are at the same distance from b and $x(a) < x(z)$. This contradicts the existence of the directed edge $[b, z]$ in C, and this contradiction proves that ξ is a forest.

Let $\epsilon > 0$ and let γ_ϵ be an ϵ-Bernoulli bond percolation process. Since ω is critical, almost surely $\omega - \gamma_\epsilon$ has only finite components. Now define an invariant bond percolation $\xi_\epsilon \subset \xi$ as follows. For any edge $[v, u] \in \xi$, let $[v, u] \in \xi_\epsilon$ iff v and u are in the same component of $\omega - \gamma_\epsilon$. Then almost surely ξ_ϵ has only finite components.

We now perform mass transport on V. For any $v \in \mathsf{V}$, let $K(v)$ denote its ξ_ϵ-connected component (which is almost surely finite) and let $K\partial(v)$ denote the set of vertices $u \in K(v)$ that have some ξ-neighbor outside of $K(v)$. Set

$$m(v, u; \xi) := \begin{cases} 1/|K\partial(v)| & \text{if } v \in Y \text{ and } u \in K\partial(v), \\ 0 & \text{otherwise.} \end{cases}$$

By the Mass-Transport Principle, we know that

$$\sum_{u \in \mathsf{V}} \mathbf{E}[m(v, u; \xi)] = \sum_{u \in \mathsf{V}} \mathbf{E}[m(u, v; \xi)] \,. \tag{4.1}$$ e.mtt

Note that

$$\sum_{u \in \mathsf{V}} m(u, v; \xi) = \begin{cases} |K(v)|/|K\partial(v)| & \text{when } v \in Y \cap K\partial(v), \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$ e.mtget

Because ξ is a forest and each vertex in Y has ξ-degree at least 3, it easily follows that $2|K\partial(v)| \geq |K(v)|$. Consequently, (4.2) shows that $\sum_{u \in \mathsf{V}} m(u, v; \xi) \leq 2$ almost surely. On the other hand, $\sum_{u \in \mathsf{V}} m(v, u; \xi)$ is 1 if $v \in Y$ and 0 otherwise. Therefore, (4.1) and (4.2) give

$$\mathbf{P}[v \in Y] \leq 2\mathbf{P}[v \in Y,\ v \in K\partial(v)] \,. \tag{4.3}$$ e.pp

If we fix ω, ξ, and $e \in \xi$, then the probability that $e \notin \xi_\epsilon$ tends to 0 as $\epsilon \to 0$: there is some finite path in ω joining the endpoints of e, and when ϵ is small, this path is unlikely to meet γ_ϵ. From the Bounded Convergence Theorem, it therefore follows that for any $v \in \mathsf{V}$, the probability that v has some edge in $\xi - \xi_\epsilon$ tends to 0 as $\epsilon \to 0$. In other words,

$$\lim_{\epsilon \to 0} \mathbf{P}[v \in Y,\ v \in K\partial(v)] = 0.$$

This contradicts (4.3) since $\mathbf{P}[v \in Y]$ is positive and does not depend on ϵ. Hence the proof of Theorem 1.1 is complete. ∎

s.fur §5. Further Results and Problems on Percolation in Nonamenable Groups.

Recall that for percolation on $\mathbb{Z}^d$, if $\theta(p) > 0$, then there exists with probability 1 a unique infinite open cluster. (See Aizenman, Kesten, and Newman (1987) and the elegant proof in Burton and Keane (1989).) As was shown by Grimmett and Newman (1990), for percolation on a (k-regular tree) $\times \mathbb{Z}$, this is not the case if k is sufficiently large: For some values of p, uniqueness holds, while for others, there are infinitely many infinite clusters. Benjamini and Schramm (1996) defined

$$p_u := \inf \Big\{ p\,;\ \mathbf{P}_p[\text{there is exactly one infinite cluster}] = 1 \Big\}$$

and conjectured that $p_c < p_u$ for percolation on nonamenable Cayley graphs. This was established by Lalley (1996) for certain planar Cayley graphs, and then by Benjamini and Schramm (1997) for all nonamenable planar Cayley graphs. Häggström and Peres (1997), in response to another question from Benjamini and Schramm (1996), showed that for any $p > p_u$, uniqueness holds almost surely. Benjamini and Schramm (1996) also asked whether there is uniqueness at p_u in transitive nonamenable graphs. Since this is true for planar graphs, we now

Conjecture: For any nonamenable Cayley graph, there is a unique infinite cluster almost surely for Bernoulli percolation at p_u.

We note that for Cayley graphs of finitely presented groups with one end, Babson and Benjamini (1996) showed that $p_u < 1$.

r.noisol REMARK 5.1. Using Lemma 4.2, it is not hard to show that almost surely, the space of ends of any infinite component of ω is topologically a Cantor set; in particular, it is uncountable. (Details will appear elsewhere.) This was conjectured by Benjamini and Schramm (1996) and first established by Häggström and Peres (1997) using a more complicated argument.

Theorem 1.1 says that $\theta(p_c) = 0$ for nonamenable Cayley graphs. One may wish for quantitative versions of this result. For example:

Problem: Bound the modulus of continuity of $\theta(p)$ at p_c.

From $\theta(p_c) = 0$, it follows that

$$\lim_{d\to\infty} \mathbf{P}_{p_c}[\mathrm{diam} K(o) > d] = 0\,. \tag{5.1}$$ e.conc

Problem: For a nonamenable Cayley graph X, find a bound for $\mathbf{P}_{p_c}[\mathrm{diam} K(o) > d]$ that tends to zero as $d \to \infty$.

It seems reasonable that such a bound would involve only the isoperimetric constant of X and perhaps the degree of the vertices.

Let $\tau_p(u,v)$ denote the connectivity function, i.e., the $\mathbf{P}_p$-probability that u and v are in the same cluster.

Problem: For a nonamenable Cayley graph X, find a bound for $\tau_{p_c}(u,v)$ that goes to zero as $d(u,v) \to \infty$.

We expect that $\tau_{p_c}(u,v) \le \alpha^{d(u,v)}$ for some $\alpha < 1$. In fact, this may hold also for all $p < p_u$, which is indeed the case when X is planar.

REFERENCES

AIZENMAN, M., KESTEN, H., and NEWMAN, C. M. (1987). Uniqueness of the infinite cluster and continuity of connectivity functions for short and long range percolation. *Commun. Math. Phys.* **111**, 505–531.

BABSON, E. and BENJAMINI, I. (1997). Cut sets and normed cohomology with application to percolation, *Proc A.M.S.*, to appear.

BENJAMINI, I., LYONS, R., PERES, Y., and SCHRAMM, O. (1997). Group-invariant percolation on graphs, *preprint.*

BENJAMINI, I. and SCHRAMM, O. (1996). Percolation beyond $\mathbb{Z}^d$, many questions and a few answers, *Electronic Commun. Probab.* **1**, 71–82.

BENJAMINI, I. and SCHRAMM, O. (1997). Percolation in the hyperbolic plane, *in preparation.*

BURTON, R. M. and KEANE, M. (1989). Density and uniqueness in percolation. *Commun. Math. Phys.* **121**, 501–505.

GRIMMETT, G. R. (1989). *Percolation.* Springer, New York.

GRIMMETT, G. R. and NEWMAN, C. M. (1990). Percolation in $\infty + 1$ dimensions, in *Disorder in physical systems*, (G. R. Grimmett and D. J. A. Welsh, eds.). Clarendon Press, Oxford, pp. 219–240.

HÄGGSTRÖM, O. (1996). Infinite clusters in dependent automorphism invariant percolation on trees, *Ann. Probab.*, to appear.

HÄGGSTRÖM, O. and PERES, Y. (1997). Monotonicity of uniqueness for percolation on unimodular transitive graphs: all infinite clusters are born simultaneously, *preprint.*

HARA, T. and SLADE, G. (1994). Mean-field behaviour and the lace expansion, in *Probability and Phase Transition (Cambridge, 1993)*, (G. R. Grimmett ed.). Kluwer Acad. Publ., Dordrecht, pp. 87–122.

HARRIS, T. E. (1960). A lower bound for the critical probability in a certain percolation process, *Proc. Cambridge Philos. Soc.* **56**, 13–20.

KESTEN, H. (1980). The critical probability of bond percolation on the square lattice equals $\frac{1}{2}$, *Commun. Math. Phys.* **74**, 41–59.

LALLEY, S. P. (1996). Percolation on Fuchsian groups, *Ann. Institut H. Poincaré, Probab. et Stat.*, to appear.

Newman, C. M. and Schulman, L. S. (1981). Infinite clusters in percolation models, *J. Statist. Phys.* **26**, 613–628.

Wu, C. C. (1993). Critical behavior of percolation and Markov fields on branching planes, *J. Appl. Probab.* **30**, 538–547.

Mathematics Department, The Weizmann Institute of Science, Rehovot 76100, Israel
itai@wisdom.weizmann.ac.il
http://www.wisdom.weizmann.ac.il/~itai/

Department of Mathematics, Indiana University, Bloomington, IN 47405-5701, USA
rdlyons@indiana.edu
http://php.indiana.edu/~rdlyons/

Department of Statistics, 367 Evans Hall, University of California, Berkeley, CA 94720-3860, USA
peres@stat.berkeley.edu
http://www.stat.berkeley.edu/pub/users/peres/

Mathematics Department, The Weizmann Institute of Science, Rehovot 76100, Israel
schramm@wisdom.weizmann.ac.il
http://www.wisdom.weizmann.ac.il/~schramm/