A LARGE WIENER SAUSAGE FROM CRUMBS

OMER ANGEL, ITAI BENJAMINI AND YUVAL PERES

ABSTRACT. Let $B(t)$ denote Brownian motion in $\mathbb{R}^d$. It is a classical fact that for any Borel set A in $\mathbb{R}^d$, the volume $V_1(A)$ of the Wiener sausage $B[0,1]+A$ has nonzero expectation iff A is nonpolar. We show that for any nonpolar A, the random variable $V_1(A)$ is unbounded.

1. Introduction

The impetus for this note was the following message, that was sent to one of us (Y. P.) by Harry Kesten:

> "... *First a question, though. It is not of major importance but has bugged me for a while in connection with some large deviation result for the Wiener sausage (with Yuji Hamana). Let* $V_1(A)$ *be the volume of the Wiener sausage at time* 1*, that is,* $V_1(A) = \mathrm{Vol}_d\Big(\cup_{s\le 1} B_s + A\Big)$*, where* B_s *is d-dimensional Brownian motion, and* A *is a d-dimensional set of positive capacity. Is it true that the support of* $V_1(A)$ *is unbounded,* i.e., *is* $\mathbf{P}[V_1(A) > x] > 0$ *for all* x*? This is easy if* A *has a section of positive* $(d-1)$*-dimensional Lebesgue measure, but I cannot prove it in general. Do you have any idea?* "

We were intrigued by this question, because it led us to ponder the source of the volume of the Wiener sausage when A is a "small" set (*e.g.*, a nonpolar set of zero Hausdorff dimension, in the plane). Is it due to the macroscopic movement of B (in which case $V_1(A)$ would not be bounded) or to the microscopic fluctuations (in which case $V_1(A)$ might be bounded, like the quadratic variation)?

Our proof of the following theorem indicates that while the microscopic fluctuations of B are necessary for the positivity of $V_1(A)$, the macroscopic behaviour of B certainly affects the magnitude of $V_1(A)$.

Theorem 1. *If the capacity* $\mathcal{C}(A)$ *of* $A \subset \mathbb{R}^d$ *is positive, then* $V_1(A)$ *is not bounded.*

The relevant capacity can be defined for $A \subset \mathbb{R}^d$ with $d \ge 3$, by

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$$\begin{aligned} \mathcal{C}(A) &= \sup_{\mu} \frac{\mu(A)^2}{\mathcal{E}(\mu)} \\ \text{where} \quad \mathcal{E}(\mu) &= \iint \frac{c_d \, d\mu(x) \, d\mu(y)}{|x-y|^{d-2}} , \end{aligned}$$

and the supremum is over measures supported on A. (the constant c_d is unimportant for our purpose). A similar formula holds for $d = 2$ with a logarithmic kernel; in that case $\mathcal{C}(A)$ is often called Robin's constant, and it will be convenient to restrict attention to sets A of diameter less than 1.

Denote by τ_A the hitting time of A by Brownian motion. By Fubini's theorem

$$\mathbf{E}[V_1(A)] = \int_{\mathbb{R}^d} \mathbf{P}_x[\tau_A \le 1] \, dx \, .$$

It follows from the relation between potential theory and Brownian motion, that $\mathbf{E}[V_1(A)]$ is nonzero if and only if A has positive capacity; see, *e.g.*, [3], [2], or [4].

2. The recipe

For any kernel $K(x,y)$, the corresponding capacity is defined by $\mathcal{C}_K(A) = \sup_\mu \frac{\mu(A)^2}{\mathcal{E}_K(\mu)}$ where $\mathcal{E}_K(\mu) = \iint K(x,y) \, d\mu(x) \, d\mu(y)$ and the supremum is over measures on A. We assume that $K(x,x) = \infty$ for all x, and that for $0 < |x-y| < R_K$, the kernel K is continuous and $K(x,y) > 0$.

The following lemma holds for all such kernels.

Lemma 1. *If a set $A \subset \mathbb{R}^d$ has $\mathcal{C}_K(A) > 0$, then for any $L < \infty$ there exists $\epsilon > 0$ and subsets $A_1, A_2, \dots, A_m$ of A such that $\sum_{i=1}^m \mathcal{C}_K(A_i) \ge L$, and the distance between A_i and A_j is at least ϵ for all $i \ne j$. (m and ϵ depend on A and L).*

Proof: We can assume that $\operatorname{diam}(A) < R_K$, for otherwise we can replace A by a subset of positive capacity and diameter less than R_K.

Let μ be a measure supported on A such that $\mu(A) = 1$ and $\mathcal{E}_K(\mu) < \infty$. By dominated convergence,

$$\lim_{\delta \to 0} \iint_{|x-y| \le \delta} K(x,y) \, d\mu(x) \, d\mu(y) = 0.$$

Choose δ so that this integral is less then $2^{-2d} L^{-1}$. Let $\epsilon = \delta d^{-1/2}$ and let $\mathcal{F}$ be a grid of cubes of side ϵ, *i.e.*,

$$\mathcal{F} = \left\{ \prod_{i=1}^d [\epsilon \ell_i, \epsilon \ell_i + \epsilon) \, : \, (\ell_1, \dots, \ell_d) \in \mathbb{Z}^d \right\}.$$

We can partition $\mathcal{F}$ into 2^d subcollections $\left\{ \mathcal{F}_v \, : v \in \{0,1\}^d \right\}$ according to the vector of parities of $(\ell_1, \dots, \ell_d)$. Then the distance between any two cubes in the same $\mathcal{F}_v$ is at least ϵ. Since μ is a probability measure, there exists $v \in \{0,1\}^d$ such that

$$\sum_{Q \in \mathcal{F}_v} \mu(Q) \geq 2^{-d} \,. \tag{1}$$

Let $A_1, A_2, \ldots, A_m$ be all the nonempty sets among $\{A \cap Q : Q \in \mathcal{F}_v\}$. Since μ is supported on A, we can rewrite (1) as $\sum_{i=1}^m \mu(A_i) \geq 2^{-d}$.

Denote by $e_i = \iint_{A_i \times A_i} K(x,y)\, d\mu\, d\mu$ the energy in A_i. Then

$$\sum_{i=1}^m e_i \leq \iint_{|x-y| \leq \delta} K(x,y)\, d\mu\, d\mu < 2^{-2d} L^{-1} \,. \tag{2}$$

By Cauchy-Schwarz,

$$\Big(\sum_{i=1}^m e_i\Big)\Big(\sum_{i=1}^m \frac{\mu(A_i)^2}{e_i}\Big) \geq \Big(\sum_{i=1}^m \mu(A_i)\Big)^2 \geq 2^{-2d} \,. \tag{3}$$

We have $\mathcal{C}_K(A_i) \geq \mu(A_i)^2/e_i$, whence

$$\sum_{i=1}^m \mathcal{C}_K(A_i) \geq \sum_{i=1}^m \frac{\mu(A_i)^2}{e_i} \geq L \,,$$

by (2) and (3).

$\square$

Proof of Theorem 1: Suppose that

$$\operatorname{esssup} V_1(A) = M < \infty \,. \tag{4}$$

Let $V_t(A)$ denote the volume of the Wiener sausage $B[0,t]+A$. From Spitzer [3] (see also [2] or [1]) it follows that $\mathbf{E}[V_1(A)] > 2\alpha_d \mathcal{C}(A)$ for some absolute constant α_d. (If $d = 2$ we assume that $\operatorname{diam} A < 1$). We infer that $\mathbf{E}[V_t(A)] > \alpha_d t \mathcal{C}(A)$ for $0 < t < 1$, by subadditivity of Lebesgue measure and monotonicity of $V_t(A)$,

Fix $L > 6M/\alpha_d$, and let $A_1, \ldots, A_m$ be the subsets of A given by the lemma. A Wiener sausage on A contains the union of Wiener sausages on the A_i, and the sum of their volumes is expected to be large. If we can arrange for the intersections to be small, then $V_1(A)$ will be large as well.

Consider the event

$$H_n = \Big\{ \max_{0 \leq s \leq \frac{1}{2n}} |B_s| < \frac{\epsilon}{2} \Big\} \,.$$

By Brownian scaling and standard estimates for the maximum of Brownian motion,

$$\mathbf{P}[H_n^c] \leq 4d \exp\!\Big(-\frac{n\epsilon^2}{4d}\Big) \,.$$

Choose n large enough so that the right-hand side is less than $\frac{1}{nm}$. For each i, we have $\mathbf{E}[V_{\frac{1}{2n}}(A_i)|H_n^c] \leq M$ by (4), so

$$\mathbf{E}[V_{\frac{1}{2n}}(A_i)|H_n] \geq \mathbf{E}[V_{\frac{1}{2n}}(A_i)] - M\mathbf{P}[H_n^c] \geq \frac{\alpha_d}{2n}\mathcal{C}(A_i) - \frac{M}{mn}. \tag{5}$$

For $0 \leq j < n$, denote by G_j the event that

$$\max_{\frac{2j}{2n} \leq s \leq \frac{2j+1}{2n}} |B_s - B_{\frac{2j}{2n}}| < \epsilon/2$$

and the first coordinate of the increment $B_{\frac{2j+2}{2n}} - B_{\frac{2j+1}{2n}}$ is greater than the $\mathrm{diam}(A) + 2\epsilon$. Define $G = \cap_{j=0}^{n-1} G_j$. We will see that the expectation of $V_1(A)$ given G is large.

On the event G, for each fixed j, the m sausages $\{B[\frac{2j}{2n}, \frac{2j+1}{2n}] + A_i\}_{i=1}^m$ are pairwise disjoint due to the separation of the A_i and the localization of B in the time interval $[\frac{2j}{2n}, \frac{2j+1}{2n}]$. Therefore,

$$\mathbf{E}\Big[\mathrm{Vol}_d\Big(B[\frac{2j}{2n}, \frac{2j+1}{2n}] + A\Big)\Big|G\Big] \geq \sum_{i=1}^m \Big(\frac{\alpha_d}{2n}\mathcal{C}(A_i) - \frac{M}{mn}\Big) \geq \frac{\alpha_d L}{2n} - \frac{M}{n} > \frac{2M}{n}.$$

Also, on G, the sausages on the odd intervals, $B[\frac{2j}{2n}, \frac{2j+1}{2n}] + A$ for $0 \leq j < n$, are pairwise disjoint due to the large increments of B (in the first coordinate) on the even intervals. We conclude that

$$\mathbf{E}[V_1(A)|G] \geq \sum_{j=0}^{n-1} \mathbf{E}\Big[\mathrm{Vol}_d\Big(B[\frac{2j}{2n}, \frac{2j+1}{2n}] + A\Big)\Big|G\Big] > 2M. \tag{6}$$

This contradicts the assumption (4) and completes the proof. □

Questions:

- Can the event G that we conditioned on at the end of the preceding proof, be replaced by a simpler event involving just the endpoint of the Brownian path?
  In particular, does every nonpolar $A \subset \mathbb{R}^d$ satisfy
  $$\lim_{R \to \infty} \mathbf{E}\Big[V_1(A)\Big| |B(1)| > R\Big] = \infty\,?$$
- Can one estimate precisely the tail probabilities $\mathbf{P}[V_1(A) > v]$ for specific nonpolar fractal sets A and large v, *e.g.*, when $d = 2$ and A is the middle-third Cantor set on the x-axis ?

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Omer Angel and Itai Benjamini
Department of Mathematics, The Weizmann Institute of Science,
Rehovot, Israel 76100
omer@wisdom.weizmann.ac.il itai@wisdom.weizmann.ac.il
http://www.wisdom.weizmann.ac.il/~itai/

Yuval Peres
Institute of Mathematics, The Hebrew University, Jerusalem
and Department of Statistics, University of California, Berkeley.
peres@math.huji.ac.il http://www.math.huji.ac.il/~peres/