Expanders are not Hyperbolic

Itai Benjamini

May 4, 1997

Abstract

We bound from above the number of vertices of a graph in terms of the Cheeger constant and the δ-hyperbolicity of the graph. As a corollary we get that expanders are not uniformly hyperbolic.

1 Introduction

Recently the notion of hyperbolic graphs and thin triangles was introduced (Gromov (1987)), generalizing geometric properties of hyperbolic spaces and using them in the study of metric spaces, and Cayley graphs in particular. In this note, we will bound from above the number of vertices of a graph by it's Cheeger Constant and hyperbolicity constant. Essentially we show that

$$\delta(G) > \mathrm{const}(d, h) \log |G| / \log \log |G|,$$

where $\delta(G)$ is the hyperbolicity constant of G, h is the Cheeger constant of G, d is a bound on the degrees of G (see definitions below) and $\mathrm{const}(d, h)$ is a constant depending only on d and h. This inequality will imply that a family of expanders is not uniformly hyperbolic. At first look this might be somewhat surprising, as examples of expanders can be obtained by taking quotients of some hyperbolic groups (see Lubotzky (1994)). We start with definitions.

Definition (Cheeger constant). Given a finite graph $G = (V, E)$, let

$$h(G) = \inf_{S} \frac{|\partial S|}{|S|}$$

where S runs over all non empty subsets of G with $|S| \leq |V|/2$, and ∂S, the boundary of S, consists of all vertices in $V \setminus S$ that have a neighbor in S.

Recall that a family $\{G_n\}_{n>0}$ of finite graphs with a uniform bound on the degrees and $|G_n| \nearrow \infty$ is a family of **expanders** iff $\inf_n h(G_n) > 0$.

Definition (δ-hyperbolic). Let $G = (V, E)$ be a graph. Given three vertices $u, v, w \in V$, pick geodesics between any two to get a geodesic triangle. Denote the geodesics by $[u, v], [v, w], [w, u]$. Say the triangle is **δ-thin** if for any $v' \in [u, v]$

$$\min(d(v', [w, u]), d(v', [v, w])) \leq \delta,$$

and the same for $v' \in [w, u]$ or $[v, w]$.
G is said to be **δ-hyperbolic** if all geodesic triangles in G are δ-thin. Let

$$\delta(G) = \inf\{\delta | G \text{ is } \delta\text{-thin}\}.$$

There are many recent publications on hyperbolic metric spaces and graphs, see for instance Ancona (1988) or Ghys et al (1991). The definition of hyperbolicity via δ-thin triangles is due to Rips.

2 The Inequality

Given G let $h = h(G)$, $\delta = \delta(G)$ and d a bound on the degrees of G. Set $R = \log_d |G|$ and pick C such that $(h/2)(1 + h)^{(1/2-C)R} > d^{CR}$. We have,

Theorem 1

$$|G| \geq (1 + h/2)^{\delta(|G|^{C/(\delta \log_2 d)}-2)/2}.$$

The idea of the proof is the following. Pick two vertices that realize the diameter of G. Remove from G a ball B, centered at the middle of a geodesic between these two vertices. By hyperbolicity the diameter of $G \setminus B$, is exponential in the radius B, with exponent depending only on δ. Yet if B is not to large, $G \setminus B$ has approximately the same Cheeger constant as G. Optimizing on the size of B, gives the inequality.

Proof: Let vu be the diameter of G, and γ a geodesic between u and v. Let m be the midpoint of γ. Let B the ball of radius CR around m. The distance from v to u is at least R. Hence the distance from v to B is at least

$(1/2 - C)R$. The same is true for the distance from u to B. Let $G \setminus B$ be the graph obtained from G by removing all the vertices of B. By Gromov (1987) 7.1.A

$$d_{G\setminus B}(v,u) \geq \delta(2^{CR/\delta} - 2).$$

Now assume

$$|B_{G\setminus B}(v, d_{G\setminus B}(v,u)/2)| \leq |B_{G\setminus B}(u, d_{G\setminus B}(v,u)/2)|.$$

That is the Volume of the ball in $G \setminus B$ centred at v of half the distance in $G \setminus B$, from v to u, is not bigger than the volume of the similar ball centered at u. Thus for any $r < d_{G\setminus B}(v,u)/2$,

$$|\partial_{G\setminus B} B_{G\setminus B}(v,r)| \geq h|B_{G\setminus B}(v,r)| - |\partial B_{G\setminus B}(v,r) \cap B|.$$

Yet C was chosen so that for $(1/2 - C)R \leq r \leq d_{G\setminus B}(v,u)/2$,

$$|\partial B_{G\setminus B}(v,r)| - |B| \geq h/2|B_{G\setminus B}(v,r)|.$$

(For $r < (1/2 - C)R$), $\partial B_{G\setminus B}(v,r)$ is disjoint from B.)
So

$$\begin{aligned} |G| &\geq (1 + h/2)^{d_{G\setminus B}(v,u)/2} \\ &\geq (1 + h/2)^{\delta(2^{CR/\delta} - 2)/2} \\ &= (1 + h/2)^{\delta(|G|^{C/(\delta \log_2 d)} - 2)/2}. \end{aligned}$$

□

Abbreviating the constants that depend on the degree and the Cheeger constant, the inequality has the following form:

$$|G| \geq c_1^{\delta|G|^{c_2/\delta} + c_3}.$$

By taking log twice and rearranging we get that,

$$\delta(G) > \text{const}(d,h) \log|G| / \log\log|G|.$$

Probably this lower bound is not sharp. Does $\delta(G) > c(h(G)) \operatorname{diam}(G)$? Note that if G contains a closed geodesic of length m, then $\delta(G) \geq m/2 - 1$.

Corollary 2 *Assume* $|G_n| \nearrow \infty$ $\forall n$ G_n *is* δ*-hyperbolic for some* $\delta < \infty$*, and* $\forall n$ $deg(G_n) < d$*, then*

$$\lim_{n \to \infty} h(G_n) = 0.$$

That is, a family of expanders is not uniformly hyperbolic. In particular a sequence of balls, or other convex subgraphs, in an infinite hyperbolic graph, is not a family of expanders.

Question: Is there a Cayley graph, so that the nested sequence of balls in the graph, is a family of expanders?

Remark: An analogous result might be formulated for Riemannian manifolds with bounded geometry.

Acknowlegements: I thank Oded Schramm for useful discussions and advice.

References

[1] Ancona, A. (1988), Positive harmonic functions and hyperbolicity, *LNM 1344* Springer, 1–23.

[2] Ghys, E. Hafliger, A. and Verjovsky, A. (1991), Group theory from a geometrical viewpoint, *World Scientific*

[3] Gromov, M. (1987), Hyperbolic groups, *Math. Sci. Res. Inst. Publ.* **8**, Springer, 75–263.

[4] Lubotzky, A. (1994), Discrete groups, expanding graphs and invariant measures, Brikhauser.

Itai Benjamini
Weizmann Institute
Rehovot, Israel 76100
itai@wisdom.weizmann.ac.il