Upper Bounds on the Height Difference of the Gaussian Random Field and the Range of Random Graph Homomorphisms into $\mathbb{Z}$

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Abstract

Bounds on the range of random graph homomorphism into $\mathbb{Z}$, and the maximal height difference of the Gaussian random field, are presented.

1 Introduction

In this note we study the range of two related processes on connected graphs, the Gaussian random field and random graph homomorphisms into $\mathbb{Z}$.

The Gaussian random field on a connected graph with n vertices is defined as follows. Given a finite connected graph $G = (V_G, E_G)$, for each directed edge $\overrightarrow{e} \in G$, let $Y_{\overrightarrow{e}}$ be a standard normal random variable, so that for opposite orientation of the same edge $Y_{\overleftarrow{e}} = -Y_{\overrightarrow{e}}$, and otherwise the variables $Y_{\overrightarrow{e}}$ are independent, and let $X_{\overrightarrow{e}}$ be $Y_{\overrightarrow{e}}$ Conditioned on the event that all sums of these variables, around a closed cycle in the graph, vanish. Pick a base vertex $v_0 \in G$, and define the height function on the vertices of the graph: $h(v_0) = 0$ and for any other $v \in G$ set $h(v) = \sum_{\gamma_v} X_e$, where γ_v is a path from v_0 to v in the graph. h is well defined, (See Georgii (1988) or Janson (1998)). We will see below that for any such graph, the distribution of the maximal height difference satisfies

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Proposition 1. *For every non-negative convex increasing function f on* $\mathbb{R}_+$,

$$Ef(\max_{v,u\in G}|h(v)-h(u)|) \le Ef(\max_{1\le j<k\le 2n}|\sum_{i=j}^{k}X_i|),$$

Remarks:
1) We **conjecture** that $2n$ in the theorem can be replaced by n.
2) In particular $E(\max_{v,u\in G}|h(v)-h(u)|) \le Cn^{1/2}$.

Theorem 1 admits an analogue for random graph homomorphisms which we now describe.

We start with definitions following Benjamini, Häggström and Mossel (1998) (BHM (1998) from now on). Let $G=(V_G,E_G)$ be a finite graph. We assume that G is connected and bipartite. Let $v_0\in V_G$ be a specified vertex of G. Let X_{G,v_0} denote the set of all mappings $f:V_G\to\mathbf{Z}$ with the property that

(i) $f(v_0)=0$, and

(ii) $|f(u)-f(v)|=1$ for all $u,v\in V_G$ such that $\{u,v\}\in E_G$

(property (ii) asserts that f is a graph homomorphism from G to $\mathbb{Z}$). Let P_{G,v_0} be the uniform probability measure on X_{G,v_0}, i.e.

$$P_{G,v_0}(f)=\frac{1}{|X_{G,v_0}|}$$

for each $f\in X_{G,v_0}$; here $|X_{G,v_0}|$ denotes the cardinality of X_{G,v_0}. We also write E_{G,v_0} for expectation with respect to P_{G,v_0}. Note that the assumptions of connectedness and bipartiteness of G are necessary and sufficient for P_{G,v_0} to be well-defined: the bipartiteness ensures that X_{G,v_0} is nonempty, and the connectedness ensures that it is finite.

Much of the interest is on the distributions of the range

$$R(f)=|\{f(v):\, v\in V_G\}|,$$

and of the difference $|f(u)-f(v)|$ for $u,v\in V_G$. Note that these distributions are independent of the choice of v_0, because for any $v_0,v_1\in V_G$ there is a natural bijection between X_{G,v_0} and X_{G,v_1} which preserves $|f(u)-f(v)|$ for all $u,v\in V_G$.

In BHM (1998) several general inequalities for random graph homomorphisms were derived, including an upper bound on fluctuations implying that the distance $d(f(u), f(v))$ between $f(u)$ and $f(v)$, is stochastically dominated by the distance to 0 of a simple random walk on $\mathbb{Z}$ having run for $d(u, v)$ steps. It was conjectured there that the supremum of the expected range $E_{G,v_0}(R(f))$ among all bipartite finite connected graphs G on n vertices, is attained when G is a path of length $n-1$.

We have

Theorem 2.

$$P(\max_v |f(v) - f(v_0)| > t) \leq 20e^{-\alpha(\frac{t}{n^{1/2}})^2}$$

for some absolute $\alpha > 0$.

Theorem 2 verifies, up to a constant, Conjecture 2.10 from BHM (1998). The proof uses the subgaussian estimate from their paper, and a technique of Dudley (1967) to bound the maximum of a subgaussian process.

A word on motivation (again from BHM (1998)). Uniform measure on graph homomorphisms into $\mathbb{Z}$ provides a model for looking at typical Lipschitz functions. It is natural to ask what the properties of such random Lipschitz functions are. For instance, is it true that concentration inequalities for typical Lipschitz function are stronger than those which hold for all Lipschitz functions? See BHM (1998) for several examples and further problems. In particular it is conjectured there that the expected range of a random graph homomorphism of the n-dimensional discrete hypercube into $\mathbb{Z}$, is $o(n)$.

Remark: It is of interest to clear the relations between random graph homomorphisms into $\mathbb{Z}$ and the Gaussian random field. The proof of Proposition 1 shows that the maximal height difference of the Gaussian random field is monotone non-increasing in adding edges. While the example after Proposition 2.6 in BHM (1998), shows that this is not the case for random graph homomorphisms into $\mathbb{Z}$.

2 Gaussian Random Fields

Proof of Proposition 1: Fernique (1975) proved that if X and Y are Gaussian random vectors in $\mathbb{R}^n$ such that for every i, j

$$E|Y_i - Y_j|^2 \leq E|X_i - X_j|^2,$$

then for every non-negative convex increasing function f on $\mathbb{R}_+$,

$$Ef(\max_{i,j}|Y_i - Y_j|) \leq Ef(\max_{i,j}|X_i - X_j|),$$

$\{h(v), v \in G\}$ is a Gaussian random vector in $\mathbb{R}^n$. Janson (1998) proved that $E|h(v) - h(u)|^2$ is equal to the resistance between v and u, when you replace each edge of G by a resistor of unit resistant. By Rayleigh's monotonicity principle (see Doyle and Snell (1984)) removing edges from G will increase the resistance between any two vertices. That is, if G' is a connected subgraph of G, obtained from G by removing edges and if we denote by h' the height function for G', then

$$E|h(v) - h(u)|^2 \leq E|h'(v) - h'(u)|^2.$$

Another way to prove this inequality is as follows. The Gaussian random field is obtained by projecting the standard normal vector onto the subspace on which sums along cycles in the graph vanish. Removing edges enlarges this subspace.

If G admits a Hamiltonian path, remove all the edges not on the path and one of the edges on the path to get a stronger result then the proposition, with n rather than $2n$. Otherwise, pick a spanning tree for G, and denote by h' the height function for an independent field on the tree. We get then, that for every $v, u \in G$,

$$E|h(v) - h(u)|^2 \leq E|h'(v) - h'(u)|^2. \tag{1}$$

By Fernique's theorem mentioned above, it is now enough to prove the proposition for trees.

Consider the depth first search path on the tree (starting at the root) of length $2n$, denoted by $\{\alpha(i)\}_{i=1}^{2n}$. This path travels along all the edges and visits every edge twice, first crossing it away from the root and later crossing it back towards the root. Let S_i^α be the values of the sums of the Gaussian field on the tree along the depth first search path, naturally oriented, from the root to $\alpha(i)$. The values of the heights of the Gaussian random field on the tree coincide with the values of $S_i^\alpha, 1 \leq i \leq 2n$. Let $S_i = \sum_{k=1}^{i} X_k, 1 \leq i \leq 2n$, for $X_k, 1 \leq i \leq 2n$, independent standard normals. Since for any $1 \leq i < j \leq 2n$

$$E|S_j^\alpha - S_i^\alpha|^2 \leq j - i,$$

one can apply again Fernique's (1975) comparison, between $\{S_i^\alpha, 1 \leq i \leq 2n\}$ and $\{S_i, 1 \leq i \leq 2n\}$ to finish. $\square$

2.1 Examples

It seems that on graphs admitting strong connectivity properties, the range of the random Gaussian field will be rather concentrated, much narrower than that of n vertices in a row. Indeed, on the complete graph on n vertices, the resistance between any two vertices is bounded by c/n. Thus the height difference will be dominated by $C|\mathcal{N}(0, \log n/n)|$. The same bound will hold for typical random graphs of size n (i.e. graphs chosen according to $G(n, 1/2)$, for instance). For the discrete cube $\{0,1\}^n$, it is easy to see that the resistance between any two vertices is uniformly bounded by c/n. Hence, the maximal height difference of the field on the cube will be dominated by $\max_{1\le i\le 2^n}|X_i|$, where the X_i's are independent normals with variance c/n. In particular the expected maximal height difference is $O(1)$.

3 Random Graph Homomorphisms

We use a technique of Dudley (1967) for estimating the maximum of a Gaussian process in terms of a metric entropy integral. His technique works also for subgaussian processes (see for example Jain and Marcus (1978)). We prefer to repeat the argument adapted to our situation.

Proof of Theorem 2: Let $D \le n$ be the diameter of G and for $i = [\log_2 n/D], \dots, 2 + [\log_2 n]$ let $\mathcal{N}_i$ be an $\frac{n}{2^i}$ -net of minimal cardinality (i.e., a minimal set of vertices such that any other vertex is of distance at most $\frac{n}{2^i}$ to that set of vertices). Note that necessarily $\mathcal{N}_{2+[\log_2 n]} = V_G$ and that $\mathcal{N}_{[\log_2 n/D]}$ can be chosen to be any singleton so we chose it to be $\{v_0\}$. Let $\{\delta_i\}_{i=[1+\log_2 n/D]}^{[2+\log_2 n]}$ be positive with $\sum_i \delta_i \le 1$.

For any $v \in \mathcal{N}_i$, we choose $v' \in \mathcal{N}_{i-1}$ so that $d(v, v') \le \frac{n}{2^{i-1}}$. Given a $v \in V_G = \mathcal{N}_{2+[\log_2 n]}$ let $v = u_{2+[\log_2 n]}, \dots, u_{[\log_2 n/D]} = v_0$ be a sequence of vertices with $u_{i-1} = u_i'$. Then

$$\{f;\ |f(v) - f(v_0)| > t\} \subseteq \bigcup_{i=[1+\log_2 n/D]}^{[2+\log_2 n]} \{f;\ |f(u_i) - f(u_{i-1})| > \delta_i t\}$$

and consequently,

$$P(\max_v |f(v) - f(v_0)| > t) \le \sum_{i=[1+\log_2 n/D]}^{[2+\log_2 n]} P(\max_{v\in\mathcal{N}_i} |f(v) - f(v')| > \delta_i t). \quad (2)$$

Note that $|\mathcal{N}_i| \le 2^{i+1}$. Indeed, let $\mathcal{M}_i$ be a maximal set of vertices s.t. $d(v,u) > \frac{n}{2^i}$, then $\mathcal{M}_i$ is (not necessarily minimal) $\frac{n}{2^i}$-net and for all $v \in \mathcal{M}_i$, $|B(v, \frac{n}{2^{i+1}})| \ge \frac{n}{2^{i+1}}$, since G is connected. The balls $\{B(v, \frac{n}{2^{i+1}})\}_{v \in \mathcal{M}_i}$ are disjoint so $n \ge |\mathcal{M}_i| \frac{n}{2^{i+1}}$. Thus $|\mathcal{N}_i| \le 2^{i+1}$.

Now the right hand side of (2) is smaller than

$$\begin{aligned}\sum_{i=[1+\log_2 n/D]}^{[2+\log_2 n]} |\mathcal{N}_i| \max_{v \in \mathcal{N}_i} P(|f(v) - f(v')| > \delta_i t) &\le 2 \sum_{i=[1+\log_2 n/D]}^{[2+\log_2 n]} 2^{i+1} e^{-\frac{\delta_i^2 t^2}{2\frac{n}{2^{i-1}}}} \\ &= 4 \sum_{i=[1+\log_2 n/D]}^{[2+\log_2 n]} e^{i \log 2 - \delta_i^2 t^2 \frac{2^{i-2}}{n}},\end{aligned}$$

where the first inequality follows from the subgaussian bound of Theorem 2.8 in BHM (1998). For $t \ge cn^{1/2}, c \ge 1$, the above is at most

$$4 \sum_{i=[1+\log_2 n/D]}^{[2+\log_2 n]} e^{i \log 2 - c^2 \delta_i^2 2^{i-2}}.$$

Choose $\delta_i = i^{1/2} 2^{-(i-2)/2} a^{-1}$ then, if a is chosen properly to be some absolute constant, $\sum_i \delta_i = 1$ and if $c \ge c_0$

$$i \log 2 - c^2 \delta_i^2 2^{i-2} \le -c^2 \delta_i^2 2^{i-3} = \frac{-c^2 i}{2a^2}.$$

So the right hand side of (2) is smaller than

$$4 \sum_{i=[1+\log_2 n/D]}^{[2+\log_2 n]} e^{\frac{-c^2 i}{2a^2}} \le 20 e^{\frac{-c^2}{2a^2}}.$$

Thus

$$P(\max_v |f(v) - f(v_0)| > t) \le 20 e^{-\alpha(\frac{t}{n^{1/2}})^2}$$

for some absolute $\alpha > 0$. □

Remark: A similar theorem holds for the Height of the Gaussian random field, namely

$$P(\max_v |h(v) - h(v_0)| > t) \le 20 e^{-\alpha(\frac{t}{n^{1/2}})^2}.$$

The proof is very similar except that the use of the theorem from BHM (1998) is replaced by the much simpler bound (1).

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