On random graph homomorphisms into $\mathbf{Z}$

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Abstract

Given a bipartite connected finite graph $G = (V, E)$ and a vertex $v_0 \in V$, we consider uniform probability measure on the set of graph homomorphisms $f : V \to \mathbf{Z}$ satisfying $f(v_0) = 0$. This measure can be viewed as a G-indexed random walk on $\mathbf{Z}$, generalizing both the usual time-indexed random walk and tree-indexed random walk. Several general inequalities for G-indexed random walk are derived, including an upper bound on fluctuations implying that the distance $d(f(u), f(v))$ between $f(u)$ and $f(v)$, is stochastically dominated by the distance to 0 of a simple random walk on $\mathbf{Z}$ having run for $d(u, v)$ steps. Various special cases are studied. For instance, when G is an n-level regular tree with all vertices on the last level wired to an additional single vertex, we show that the expected range of the walk is $O(\log n)$. This result can also be rephrased as a statement about conditional branching random walk. To prove it, a power-type Pascal triangle is introduced and exploited.

1 Introduction

The study of Lipschitz functions on graphs and metric spaces is rather advanced. Uniform measure on graph homomorphisms into $\mathbf{Z}$ provides a model for looking at typical Lipschitz functions. It is natural to ask what the properties of such random Lipschitz functions are. For instance, is it true that concentration inequalities for typical Lipschitz function are stronger than those which hold for all Lipschitz functions? The research reported in this paper makes some initial steps in that direction. We start with the definition of the measure.

Let $G = (V_G, E_G)$ be a finite graph. We assume that G is connected and bipartite. Let $v_0 \in V_G$ be a specified vertex of G. Let X_{G,v_0} denote the set of all mappings $f : V_G \to \mathbf{Z}$ with the property that

(i) $f(v_0) = 0$, and

(ii) $|f(u) - f(v)| = 1$ for all $u, v \in V_G$ such that $\{u, v\} \in E_G$

(property (ii) asserts that f is a graph homomorphism from G to $\mathbf{Z}$). Let $\mathbf{P}_{G,v_0}$ be the uniform probability measure on X_{G,v_0}, i.e.

$$\mathbf{P}_{G,v_0}(f) = \frac{1}{|X_{G,v_0}|}$$

for each $f \in X_{G,v_0}$; here $|X_{G,v_0}|$ denotes the cardinality of X_{G,v_0}. We also write $\mathbf{E}_{G,v_0}$ for expectation with respect to $\mathbf{P}_{G,v_0}$. Note that the assumptions of connectedness and bipartiteness of G are necessary and sufficient for $\mathbf{P}_{G,v_0}$ to be well-defined: the bipartiteness ensures that X_{G,v_0} is nonempty, and the connectedness ensures that it is finite.

Note that when we take G to be a path of length n starting at v_0, i.e.

$$V_G = \{v_0, \ldots, v_n\},\ E_G = \{\{v_i, v_{i+1}\} : 0 \le i < n\}\,, \tag{1}$$

then the model reduces to the usual simple random walk (SRW) on $\mathbf{Z}$ up to time N. If we instead take G to be some tree rooted at v_0, then we obtain the usual model for a tree-indexed random walk on $\mathbf{Z}$; see Benjamini and Peres [3]. Hence it is natural to use the term **G-indexed random walk** for our model.

Much of our interest is on the distributions of the range

$$R(f) = |\{f(v) : v \in V_G\}|\,, \tag{2}$$

and of the difference $|f(u) - f(v)|$ for $u, v \in V_G$. Note that these distributions are independent of the choice of v_0, because for any $v_0, v_1 \in V_G$ there is a natural bijection between X_{G,v_0} and X_{G,v_1} which preserves $|f(u) - f(v)|$ for all $u, v \in V_G$.

We will look at some examples of such walks when G is large in the sense that $|V_G|$ is exponentially large in the diameter of the graph. Related models (such as the solid-on-solid model and Shlosman's random staircases) where G is $\mathbf{Z}^2$ have been studied in the physics literature; see e.g. Georgii [6].

One might suspect that the model presented here is just a discrete version of the graph-indexed Gaussian field as defined e.g. in Janson [12], and thus has similar properties. At least for some properties, this does not seems to be the case: Janson [12, page 133] proved that the variance of the field at a vertex v in the Gaussian field is equal to the electrical resistance in the graph (viewed as a network with unit resistors) from the v to the fixed vertex v_0 whose value is fixed to be 0. In particular, the variance of the field value is monotone decreasing in adding edges. The remark following Proposition 2.6 below shows that this monotonicity fails in our model. It is an interesting task to figure out what properties are common to these two models.

In Section 2, we shall obtain some basic correlation and other inequalities for G-indexed random walks. For instance, we will see in Theorem 2.8 that for any $u, v \in V_G$ at distance d from each other, we have

$$\mathbf{E}_{G,v_0}\left(|f(u) - f(v)|^2\right) \le d\,, \tag{3}$$

Thus providing a subdiffusive estimate for the fluctuations. The example in (1) shows that this bound is sharp. More generally, Theorem 2.8 shows that actually for all n and all increasing functions g

$$\sup_{\substack{G \\ u,v \in V_G : d(u,v) = n}} \mathbf{E}_{G,v_0}\left[g\left(|f(u) - f(v)|\right)\right] \tag{4}$$

is attained by G as in example (1).

The subsequent Sections 3–6 are devoted to particular cases. Section 3 deals with the case where G consists of two endpoints connected by m parallel paths of length k.

In Section 4, we treat the more intricate case where G is an n-level regular tree wired at the n'th level, i.e. with all leaves on the last level connected to an additional single vertex. This is tantamount to conditioning a branching random walk (see e.g. Asmussen and Hering [2] or Ney [17]) on the event that all particles occupy the same location at time $n+1$. Somewhat surprisingly, it turns out (Theorem 4.2) that the expected range of this process is as small as $O(\log n)$; in contrast, it is well-known and easy to see that the unconditional branching random walk (i.e. free boundary) has an expected range of order n. As a key tool in the analysis of the conditional branching random walk, we will introduce the **power-type Pascal triangle**, which is a natural generalization of the usual Pascal triangle.

The short Section 5 concerns the case where G is the k-dimensional discrete hypercube. We expect (Conjecture 5.2) the concentration of measure for random Lipschitz functions to be much stronger than the usual concentration of measure phenomenon for the hypercube; in particular, we believe that the expected range of the G-indexed walk is $o(n)$.

In Section 6, we indicate the richness of the G-indexed random walk model by showing how it can be used to emulate the famous Ising model through a particular choice of G.

Finally, in Section 7, we make some concluding remarks about open problems and natural directions of generalization.

2 Correlation and other inequalities

This section contains some general inequalities for G-indexed random walks. These inequalities provide information about unimodality and correlations under $\mathbf{P}_{G,v_0}$, as well as comparisons between G-indexed random walks for different choices of G. For $u, v \in V_G$, let $d(u,v)$ denote graph-theoretic distance between u and v.

We begin with a simple result concerning the marginal distribution of $f(v)$ for a given vertex $v \in V_G$. The distribution of $f(v)$ under $\mathbf{P}_{G,v_0}$ is obviously symmetric around 0. Furthermore, $f(v)$ is either $\mathbf{P}_{G,v_0}$-a.s. even or $\mathbf{P}_{G,v_0}$-a.s. odd depending on whether $d(v_0, v)$ is even or odd. The following result tells us that if we restrict to the even or the odd integers, then the distribution of $f(v)$ is in fact unimodal.

Proposition 2.1 *Fix any bipartite connected finite graph G and any $v_0, v \in V_G$. For any non-negative integers s, t such that $s < t$ and $t - s$ is even, we have*

$$\mathbf{P}_{G,v_0}[f(v) = t] \leq \mathbf{P}_{G,v_0}[f(v) = s]\,. \tag{5}$$

Proof: Set $A_s = \{f \in X_{G,v_0} : f(v) = s\}$ and define A_t similarly. Since $\mathbf{P}_{G,v_0}$ assigns the same probability to each $f \in X_{G,v_0}$, it suffices to show that $|A_s| \geq |A_t|$, and to do this we shall describe an injective mapping from $|A_t|$ to $|A_s|$. For any $f \in A_t$, we define the vertex set $\Pi_f \subset V_G$ as follows. For each $w \in V_G$ we take Π_f to contain w if and only if

(i) $f(w) = \frac{s+t}{2}$, and

(ii) there exists a path from v_0 to w such that all vertices u on the path (except w) satisfy $f(u) < \frac{s+t}{2}$

(note that $\frac{s+t}{2}$ is a strictly positive integer). Pictorially, Π_f is a "cutset" separating v_0 from v, and moreover Π_f is the cutset "closest" to v_0 with the property that all vertices in the cutset take value $\frac{s+t}{2}$. Take $\tilde{\Pi}_f$ to be the set of vertices that can be reached from v_0 through paths that only contain vertices u with $f(u) < \frac{s+t}{2}$. Finally, define $f' \in X_{G,v_0}$ by setting

$$f'(w) = \begin{cases} f(w) & \text{if } w \in \tilde{\Pi}_f \cup \Pi_f \\ t + s - f(w) & \text{otherwise,} \end{cases}$$

for each $w \in V_G$. Clearly, $f' \in A_s$, and moreover it is easy to see that the mapping is invertible, so that any two elements of A_t are mapped on different elements of A_s. $\square$

Remark: The proof is easily extended to show that the inequality in (5) is strict whenever $\mathbf{P}_{G,v_0}[f(v) = s] > 0$.

For the remaining results in this section, we need to recall a couple of general inequalities which are widely used in statistical mechanics: variants of Holley's Theorem [11] and the FKG inequality [5].

For a finite set V and a finite set S of reals, we consider two random elements Y and Y' taking values in S^V, and write μ and μ' for their respective distributions. S^V is equipped with the usual coordinatewise partial order $\preceq$. A function $g : S^V \to \mathbf{R}$ is said to be increasing if $g(\xi) \leq g(\eta)$ whenever $\xi \preceq \eta$. The probability measure μ on S^V is said to have **positive correlations** if all increasing functions from S^V to $\mathbf{R}$ are positively correlated under μ. We write $\preceq_d$ for the usual stochastic domination, i.e. $\mu \preceq_d \mu'$ if all increasing $g : S^V \to \mathbf{R}$ have greater expectation under μ' than under μ. We say that μ is **irreducible** if, for any $\xi, \eta \in S^V$ such that both ξ and η have positive μ-probability, we can move from ξ to η through single-site flips without passing through any element of zero μ-probability.

Lemma 2.2 (Holley) *Suppose that the probability measure μ and μ' on S^V are irreducible, and that there exists $\xi \in S^V$ such that $\mu(\xi) > 0$ and $\mu'(\xi) > 0$. If for all $v \in V$, all $s \in S$, μ-a.e. $\xi \in S^{V \setminus \{v\}}$ and μ'-a.e. $\eta \in S^{V \setminus \{v\}}$ such that $\xi \preceq \eta$ we have*

$$\mu(X(v) \geq s \,|\, X(V \setminus \{v\}) = \xi) \leq \mu'(X'(v) \geq s \,|\, X'(V \setminus \{v\}) = \eta)\,, \tag{6}$$

then $\mu \preceq_d \mu'$.

Lemma 2.3 (FKG) *Suppose that μ is irreducible, and for all $v \in V$, all $s \in S$, and μ-a.e. $\xi, \eta \in S^{V \setminus \{v\}}$ such that $\xi \preceq \eta$, we have*

$$\mu(X(v) \geq s \,|\, X(V \setminus \{v\}) = \xi) \leq \mu(X(v) \geq s \,|\, X(V \setminus \{v\}) = \eta)\,. \tag{7}$$

Then μ has positive correlations.

Proofs of these results appear e.g. in Georgii et al. [7]; the same proofs under slightly different conditions can be found in Liggett [15].

As a first application, we have the following result.

Proposition 2.4 *For any bipartite connected finite graph G and any $v_0 \in V_G$, the measure $\mathbf{P}_{G,v_0}$ has positive correlations.*

Proof: This is a trivial matter of checking that $\mathbf{P}_{G,v_0}$ satisfies the conditions in Lemma 2.3. $\Box$

Next, we let $\mathbf{P}^*_{G,v_0}$ be the probability measure on X_{G,v_0} corresponding to picking $f^* \in X_{G,v_0}$ as follows: pick f according to $\mathbf{P}_{G,v_0}$, and let $f^*(v) = |f(v)|$ for each $v \in V$. Define $X^*_{G,v_0} = \{f \in X_{G,v_0} : f(v) \geq 0 \text{ for all } v \in V_G\}$, and note that $\mathbf{P}^*_{G,v_0}$ is concentrated on X^*_{G,v_0}. For $f^* \in X^*_{G,v_0}$, let $k(f^*)$ denote the number of connected components of the vertex set $\{v \in V_G : f^*(v) > 0\}$. By simply counting the number of $f \in X_{G,v_0}$ that give rise to a given $f^* \in X^*_{G,v_0}$, we get that

$$\mathbf{P}^*_{G,v_0}(f^*) = \frac{2^{k(f^*)}}{|X_{G,v_0}|} \tag{8}$$

for each $f^* \in X^*_{G,v_0}$ (note the similarity with the Fortuin–Kateleyn random-cluster model; see e.g. Grimmett [8]). It turns out that not only $\mathbf{P}_{G,v_0}$, but also $\mathbf{P}^*_{G,v_0}$, has positive correlations:

Proposition 2.5 *For any bipartite connected finite graph G and any $v_0 \in V_G$, the measure $\mathbf{P}^*_{G,v_0}$ has positive correlations.*

Proof: Again, it is just a matter of checking that the conditions in Lemma 2.3 hold. To check that (7) holds for $\mathbf{P}^*_{G,v_0}$ is slightly less trivial than for $\mathbf{P}_{G,v_0}$, so we do this explicitly. For $v = v_0$, (7) holds trivially (with equality), so we take $v \in V_G \setminus \{v_0\}$, and some $\xi \in \mathbf{N}^{V_G \setminus \{v\}}$ which arises as a projection on $\mathbf{N}^{V_G \setminus \{v\}}$ of some element of X^*_{G,v_0}. Define

$$N(v,\xi) = \{\xi(w) : w \text{ is a nearest neighbor of } v\}$$

and furthermore let $\kappa(v,\xi)$ be the number of connected components of the vertex set $\{w \in V_G \setminus \{v\} : \xi(w) > 0\}$ that intersect the neighborhood of v. If ξ arises as such a projection, then $N(v,\xi)$ is either $\{i\}$ or $\{i, i+2\}$ for some $i \in \mathbf{N}$. Write $P^*_{v|\xi}$ for the conditional distribution, under $\mathbf{P}^*_{G,v_0}$, of $f^*(v)$ given that $f^*(V_G \setminus \{v\}) = \xi$. $P^*_{v|\xi}$ can be determined directly from (8), and we get the following. If $N(v,\xi) = \{i, i+2\}$ for some $i \in \mathbf{N}$, then

$$P^*_{v|\xi}(i+1) = 1\,.$$

If $N(v,\xi) = \{0\}$, then

$$P^*_{v|\xi}(1) = 1\,,$$

while if $N(v,\xi) = \{1\}$, then

$$\left\{ \begin{array}{l} P^*_{v|\xi}(0) = \frac{2^{\kappa(v,\xi)}}{2^{\kappa(v,\xi)}+2} \\ P^*_{v|\xi}(2) = \frac{2}{2^{\kappa(v,\xi)}+2}\,. \end{array} \right. \tag{9}$$

Finally, if $N(v,\xi) = \{i\}$ for $i > 1$, then

$$\left\{ \begin{array}{l} P^*_{v|\xi}(i-1) = \frac{1}{2} \\ P^*_{v|\xi}(i+1) = \frac{1}{2}\,. \end{array} \right.$$

Since $\kappa(v,\xi)$ is decreasing in ξ, we see that $P^*_{v|\xi}$ is stochastically increasing in ξ, as needed. $\Box$

Next, we give a couple of results that allow us to compare $\mathbf{P}_{G,v_0}$ for different choices of G. Intuitively, one might think that adding edges would make the G-indexed random walk become more concentrated around 0. This is true if we add an edge incident to v_0:

Proposition 2.6 *Let G be a bipartite connected finite graph, and let v_0 and v_1 be two vertices in V_G at odd distance from each other. Let G' be the graph obtained from G by adding an edge between v_0 and v_1. We then have*

$$\mathbf{P}^*_{G',v_0} \preceq_d \mathbf{P}^*_{G,v_0} . \tag{10}$$

Proof: The proof is by applying Lemma 2.2; we need to check that (6) holds with $\mu = \mathbf{P}^*_{G',v_0}$ and $\mu' = \mathbf{P}^*_{G,v_0}$. ¿From the proof of Proposition 2.5, we know that the conditional distribution of $f^*(v)$ given that $f^*(V_G \setminus \{v\}) = \xi$ is stochastically increasing in ξ, both for $\mathbf{P}^*_{G',v_0}$ and for $\mathbf{P}^*_{G,v_0}$. It is therefore enough to show for any (feasible) ξ that the conditional distribution of $f^*(v)$ given that $f^*(V_G \setminus \{v\}) = \xi$ is stochastically greater for $\mathbf{P}^*_{G,v_0}$ than for $\mathbf{P}^*_{G',v_0}$. For $v \neq v_1$ this holds with equality, and it also holds with $v = v_1$ because the effect of adding the edge $\{v_0, v_1\}$ is to force $f^*(v_1)$ to be 1, which is the smallest possible value for a vertex at odd distance from v_0. $\Box$

Remark: Unfortunately, Proposition 2.6 cannot be extended in such a way that (10) can be deduced whenever G' is obtained by adding some (arbitrary) edge that does not destroy the bipartiteness. A simple counterexample is as follows. Define G by taking

$$V_G = \{v_0, \dots, v_4\},\ E_G = \{\{v_0,v_1\},\{v_0,v_3\},\{v_1,v_2\},\{v_1,v_4\},\{v_3,v_4\}\},$$

and take G' to be the same except that the edge $\{v_2, v_3\}$ is added. A calculation shows that the $\mathbf{P}^*_{G,v_0}$-probability of having a nonzero value at v_4 is $1/3$, whereas the $\mathbf{P}^*_{G',v_0}$-probability of having a nonzero value at v_4 is larger: $2/5$.

A different way of modifying G into a new graph G' is to glue together all neighbors $v_1, \dots, v_m$ of v_0 into a single vertex. This is equivalent to conditioning $\mathbf{P}_{G,v_0}$ on the event that $f^*(v_1) = \cdots = f^*(v_m)$. Write $\tilde{\mathbf{P}}_{G,v_0}$ for this conditional distribution; the advantage of considering $\tilde{\mathbf{P}}_{G,v_0}$ rather than $\mathbf{P}_{G',v_0}$ is that $\tilde{\mathbf{P}}_{G,v_0}$ is defined on the same space X_{G,v_0} as $\mathbf{P}_{G,v_0}$. Define $\tilde{\mathbf{P}}^*_{G,v_0}$ from $\tilde{\mathbf{P}}_{G,v_0}$ in the same way that $\mathbf{P}^*_{G,v_0}$ was defined from $\mathbf{P}_{G,v_0}$ (i.e. by taking vertexwise absolute values). Also define

$$\tilde{X}_{G,v_0} = \{f \in X_{G,v_0} : f(v_1) = \cdots = f(v_m)\} .$$

Proposition 2.7 *For any bipartite connected finite graph G and any $v_0 \in V_G$, we have*

$$\mathbf{P}^*_{G,v_0} \preceq_d \tilde{\mathbf{P}}^*_{G,v_0} .$$

Proof: This is another application of Lemma 2.2. For the same reason as in the proof of Proposition 2.6, it is enough to show for any (feasible) ξ that the conditional distribution of $f^*(v)$ given that $f^*(V_G \setminus \{v\}) = \xi$ is stochastically greater for $\tilde{\mathbf{P}}^*_{G,v_0}$ than for $\mathbf{P}^*_{G,v_0}$. Analogously to (8), $\tilde{\mathbf{P}}^*_{G,v_0}$ satisfies

$$\tilde{\mathbf{P}}^*_{G,v_0}(f^*) = \frac{2^{\tilde{k}(f^*)}}{|\tilde{X}_{G,v_0}|} .$$

Here $\tilde{k}$ is defined as the number of connected components of the set of nonzeroes in ξ, except that all connected components intersecting $\{v_1, \dots, v_m\}$ count as a single one. Single-site conditional distributions under $\tilde{\mathbf{P}}^*_{G,v_0}$ become identical to those obtained for $\mathbf{P}^*_{G,v_0}$ in the proof of Proposition 2.6, except in (9) where $\kappa(v, \xi)$ is replaced by $\tilde{\kappa}(v, \xi)$. The latter quantity is defined as the number of connected components of nonzeroes in ξ that intersect the neighborhood of v, again counting all connected components intersecting $\{v_1, \ldots, v_m\}$ as just a single one. Clearly, $\tilde{\kappa}(v, \xi) \leq \kappa(v, \xi)$, and it follows that the conditional distribution of $f^*(v)$ given that $f^*(V_G \setminus \{v\}) = \xi$ is stochastically greater for $\tilde{\mathbf{P}}^*_{G,v_0}$ than for $\mathbf{P}^*_{G,v_0}$, as desired. $\square$

Proposition 2.7 is a key ingredient in proving the following upper bound on the fluctuations under $\mathbf{P}_{G,v_0}$. The diffusive bound (3) is an immediate consequence.

Theorem 2.8 *Let G be a bipartite connected finite graph and fix $v_0, v \in V_G$. Let $\{S(k)\}_{k=0,1,\ldots}$ denote a SRW on $\mathbf{Z}$ starting with $S(0) = 0$. Then the distribution of $|f(v)|$ under $\mathbf{P}_{G,v_0}$ is stochastically dominated by the distribution of $|S(d(v_0, v))|$.*

For the proof, it is convenient to isolate the following lemma. A random variable X is said to be symmetric if $-X$ has the same distribution as X.

Lemma 2.9 *Let X and Y be symmetric random variables taking values in $2\mathbf{Z}$. Suppose that $|X|$ is stochastically dominated by $|Y|$. Let Z be a ± 1-valued random variable which is independent of X and Y. Then $|X + Z|$ is stochastically dominated by $|Y + Z|$. The same thing holds if X and Y take values in $2\mathbf{Z} + 1$ rather than in $2\mathbf{Z}$.*

Proof: The fact that $|X|$ is stochastically dominated by $|Y|$ is equivalent to the existence of a coupling P of X and Y such that

$$P\left[|X| \leq |Y|\right] = 1 \tag{11}$$

(this is Strassen's Theorem; see e.g. Lindvall [16]). Since both X and Y are symmetric, (11) implies that there exists a coupling which assigns probability 1 to the event

$$\{0 \leq X \leq Y\} \cup \{Y \leq X \leq 0\}. \tag{12}$$

We now look at $X + Z$ and $Y + Z$ under such a coupling. If $X = Y$ we must have $|X + Z| = |Y + Z|$. If $X \neq Y$ then we have $|X| \leq |Y| + 2$. This implies that again $|X + Z| \leq |Y + Z|$ since Z is ± 1-valued. $\square$

Proof Theorem 2.8: Let $d = d(v_0, v)$. We prove the theorem by induction on d. If $d = 0$ there is nothing to prove. Suppose that $d > 0$. Let G' be the graph obtained from G by gluing together all the neighbours of v_0 into a single vertex v'. By the induction hypothesis we know that the distribution of $|f(v)|$ under $\mathbf{P}_{G',v'}$ is dominated by the distribution of $|S(d-1)|$. Therefore if X is a random variable which takes each of the values $-1, 1$ with probability $1/2$ and is independent of $\mathbf{P}_{G',v'}$, then by Lemma 2.9 the distribution of $|X + f(v)|$ under $\mathbf{P}_{G',v'}$ is dominated by the distribution of $|X + S(d-1)|$. However, the distribution of $|X + S(d-1)|$ is nothing but the distribution of $|S(d)|$. Moreover, by Proposition 2.7 the distribution of $|f(v)|$ under $\mathbf{P}_{G,v}$ is stochastically dominated by the distribution of $|X + f(v)|$ under $\mathbf{P}_{G',v'}$. Putting these observations together, we have that the distribution of $|f(v)|$ under $\mathbf{P}_{G,v}$ is dominated by the distribution of $|S(d)|$, as desired. $\square$

Another way to state Theorem 2.8 is the following. Fix a positive integer d and any increasing function g (taking $g(x) = x^2$ corresponds to (3)). The supremum of $\mathbf{E}_{G,v_0}(g(|f(v)|))$ among all choices of bipartite connected finite G and $v_0, v \in V_G$ with $d(v_0, v) \leq d$, is attained when G is simply a path of length d, and v and v_0 are the two endpoints of the path. This maximum is clearly not unique; it is e.g. attained whenever G is a tree.

Somewhat related is the following conjecture.

Conjecture 2.10 *The supremum of the expected range* $\mathbf{E}_{G,v_0}(R(f))$ *among all bipartite finite connected graphs* G *on* n *vertices, is attained when* G *is a path of length* $n-1$*.*

Perhaps even the same is true for $\mathbf{E}_{G,v_0}(g(R(f)))$ for any increasing g.

3 Parallel paths

In this section, we investigate the series-parallel behavior of the G-indexed random walk model, by considering the case where $G = G_{k,m} = \{0\} \cup \{1,\ldots,k\} \times \{1,\ldots,m\} \cup \{k+1\}$, and there are edges between (i,s) and $(i+1,s)$ for all $1 \le i < k$ and $1 \le s \le m$. There are also edges between 0 and $(1,s)$ for all s, and between (k,s) and $k+1$ for all s. See Figure 1. Note that when $m = 2$ we get a SRW bridge.

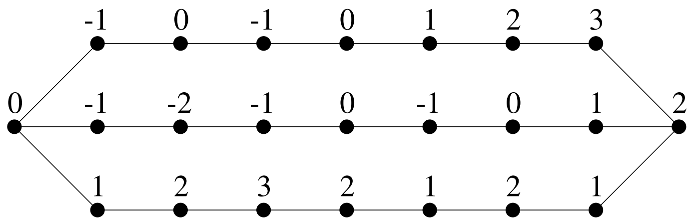

Figure 1: A typical $G_{7,3}$-indexed walk.

We are interested in the range of the walk and in the $\mathbf{P}_{G_{k,m},0}$-distribution of $f(k+1)$, which we call the **top** (despite the orientation of Figure 1!). We consider the asymptotic behavior as $k \to \infty$ and $m = m(k)$ may depend on k in various ways.. When $m(k)$ is small we have the following result:

Proposition 3.1 *If* $m(k)$ *satisfies*

$$\lim_{k\to\infty} \frac{m(k)}{k+1} = 0\,, \tag{13}$$

then the distribution of $f(k+1)\sqrt{\frac{m(k)}{k+1}}$ *under* $\mathbf{P}_{G_{k,m},0}$ *converges to a standard normal distribution.*

Proof: Let $p_{k+1,x}$ be the probability that a SRW is at site x at time $k+1$. Assume first that $k+1$ is even. We then have

$$\mathbf{P}_{G_{k,m},0}\left[f(k+1) \in [a,b]\right] = \frac{\sum_{x\in[a,b]} p_{k+1,x}^{m(k)}}{\sum_{y\in\mathbf{Z}} p_{k+1,y}^{m(k)}}. \tag{14}$$

Fix $\epsilon > 0$. By the CLT, we have a finite $A > 0$ such that

$$\sum_{x\in[-A\sqrt{k+1},A\sqrt{k+1}]} p_{k+1,x} > 1-\epsilon. \tag{15}$$

By (14) and the monotonicity properties of $\{p_{k+1,x}\}$ in x, (15) implies that

$$\mathbf{P}_{G_{k,m},0}\left[f(k+1) \in [-A\sqrt{k+1}, A\sqrt{k+1}]\right] > 1-\epsilon. \tag{16}$$

By the local CLT (see e.g. Lawler [14]) we have for all even $x \in [-A\sqrt{k+1}, A\sqrt{k+1}]$ that

$$p_{k+1,x} = \sqrt{\frac{2\pi}{k+1}} e^{-\frac{x^2}{2k+2}} \left(1 + O\left(\frac{1}{k+1}\right)\right)$$

so that if (13) holds, then

$$p_{k+1,x}^{m(k)} = \left(\frac{2\pi}{k+1}\right)^{\frac{m(k)}{2}} e^{-\frac{m(k)x^2}{2k+2}} (1 + o(1)). \tag{17}$$

By (16),(14) and (17) we have

$$\begin{aligned}
&\mathbf{P}_{G_{k,m},0}\left[f(k+1)\sqrt{\frac{m(k)}{k+1}} \in [a,b]\right] \\
&= \mathbf{P}_{G_{k,m},0}\left[f(k+1)\sqrt{\frac{m(k)}{k+1}} \in [a,b] \Big| f(k+1) \in [-A\sqrt{k+1}, A\sqrt{k+1}]\right] + O(\epsilon) \\
&= \frac{\sum_{x\in[a\sqrt{\frac{k+1}{m(k)}}, b\sqrt{\frac{k+1}{m(k)}}]\cap[-A\sqrt{k+1}, A\sqrt{k+1}]\cap 2\mathbf{Z}} p_{k+1,x}^{m(k)}}{\sum_{y\in[-A\sqrt{k+1}, A\sqrt{k+1}]\cap 2\mathbf{Z}} p_{k+1,y}^{m(k)}} + O(\epsilon) \\
&= \frac{(1+o(1))\sum_{x\in[a\sqrt{\frac{k+1}{m(k)}}, b\sqrt{\frac{k+1}{m(k)}}]\cap[-A\sqrt{k+1}, A\sqrt{k+1}]\cap 2\mathbf{Z}} e^{-\frac{m(k)x^2}{2k+2}}}{(1+o(1))\sum_{y\in[-A\sqrt{k+1}, A\sqrt{k+1}]\cap 2\mathbf{Z}} e^{-\frac{m(k)y^2}{2k+2}}} + O(\epsilon) \\
&= \frac{\sum_{x\in[a\sqrt{\frac{k+1}{m(k)}}, b\sqrt{\frac{k+1}{m(k)}}]\cap 2\mathbf{Z}} e^{-\frac{m(k)x^2}{2k+2}}}{\sum_{y\in[-A\sqrt{k+1}, A\sqrt{k+1}]\cap 2\mathbf{Z}} e^{-\frac{m(k)y^2}{2k+2}}} + O(\epsilon) \\
&= \frac{\int_a^b e^{-\frac{x^2}{2}}\,dx}{\int_{-A}^A e^{-\frac{y^2}{2}}\,dy} + O(\epsilon) = \frac{\int_a^b e^{-\frac{x^2}{2}}\,dx}{\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\,dy} + O(\epsilon),
\end{aligned} \tag{18}$$

where we have used the assumption (13) in the first equality in (18). The case with $k+1$ being odd is treated similarly. □

When $m(k)$ is larger we get a tight limit:

Proposition 3.2 *The distribution of $f(k+1)$ under $\mathbf{P}_{G_{k,m},0}$ is tight as $k \to \infty$ if and only if*

$$\liminf_{k\to\infty} \frac{m(k)}{k} > 0. \tag{19}$$

Proof: Assume first that $k+1 = 2r$ is even. As before, we denote by $p_{k+1,x}$ the probability that SRW is at x at time $k+1$. Note that we have:

$$\mathbf{P}_{G_{k,m},0}[f(k+1) = x] = \frac{p_{k+1,x}^{m(k)}}{\sum_{y\in\mathbf{Z}} p_{k+1,y}^{m(k)}}. \tag{20}$$

In particular,

$$1 \geq \frac{\mathbf{P}_{G_{k,m},0}[f(k+1)=\pm 2]}{\mathbf{P}_{G_{k,m},0}[f(k+1)=\pm 0]} \geq \frac{\mathbf{P}_{G_{k,m},0}[f(k+1)=\pm 4]}{\mathbf{P}_{G_{k,m},0}[f(k+1)=\pm 2]} \geq \dots \tag{21}$$

Therefore, the distribution is tight if and only if there exists an integer t such that

$$\limsup_{k\to\infty} \frac{\mathbf{P}_{G_{k,m},0}[f(k+1)=2t+2]}{\mathbf{P}_{G_{k,m},0}[f(k+1)=2t]} < 1. \tag{22}$$

Using (20) we see that (22) is equivalent to

$$\limsup_{k\to\infty} \left(\frac{r+t}{r+t+1}\right)^{m(k)} < 1\,.$$

This, in turn, is equivalent to (19). The case where $k+1$ is odd is similar. □.

Proposition 3.3 *The distribution of $f(k+1)$ under $\mathbf{P}_{G_{k,m},0}$ converges to δ_0 as $k = 2r \to \infty$ if and only if*

$$\liminf_{k=2r\to\infty} \frac{m(k)}{k} = \infty. \tag{23}$$

Proof: We use the same notation as in the proof of Proposition 3.2. Note that by (21) the distribution converges to δ_0 if and only if

$$\limsup_{k=2r\to\infty} \frac{\mathbf{P}_{G_{k,m},0}[f(k+1)=2]}{\mathbf{P}_{G_{k,m},0}[f(k+1)=0]} = 0,$$

which is equivalent to

$$\limsup_{k=2r\to\infty} \left(\frac{r}{r+1}\right)^{m(k)} = 0,$$

which is equivalent to (23). □.

Remark: Similarly, for odd k condition (23) is equivalent to convergence of $f(k+1)$ to $\frac{1}{2}(\delta_1+\delta_{-1})$.

We next consider the range $R(f)$ of the $G_{k,m}$-indexed random walk; recall the definition in (2).

Proposition 3.4 *If $m(k) \geq C\lambda^k$ for some $C>0$ and $\lambda>1$ then there exists a constant $D>0$ such that*

$$\lim_{k\to\infty} \mathbf{P}_{G_{k,m},0}[R(f) > Dk] = 1 \tag{24}$$

where f is a $G_{k,m}$-indexed walk. If $\lim_{k\to\infty} \frac{\log(m(k))}{k} = 0$, then for all $D>0$

$$\lim_{k\to\infty} \mathbf{P}_{G_{k,m},0}[R(f) > Dk] = 0. \tag{25}$$

Proof: We let $\{S^k(n))\}_{n=0,\dots,k}$ denote SRW, and $\{S^k_x(n)\}_{n=0,\dots,k}$ denote S condition on $S(k)=x$. Assume first that $m(k) \geq C\lambda^k$ for $C>0, \lambda>1$. By Proposition 3.3 and the remark following that proposition, in this case for k odd and for all D:

$$\lim_{k\to\infty} \mathbf{P}_{G_{k,m},0}[R(f) > Dk] = \lim_{k\to\infty} \mathbf{P}_{G_{k,m},0}[R(f) > Dk | f(k+1)=0], \tag{26}$$

and for k even and all D:

$$\lim_{k\to\infty} \mathbf{P}_{G_{k,m},0}[R(f) > Dk] = \lim_{k\to\infty} \frac{1}{2}\mathbf{P}_{G_{k,m},0}[R(f) > Dk | f(k+1) = 1] + \lim_{k\to\infty} \frac{1}{2}\mathbf{P}_{G_{k,m},0}[R(f) > Dk | f(k+1) = -1]. \quad (27)$$

On the other hand, from well-known results on SRW bridges, there exists $C' > 0, D > 0, \delta > \lambda^{-1}$, such that for $x \in \{-1, 0, 1\}$,

$$\mathbf{P}_{G_{k,m},0}\left[\max_{n\in\{0,\dots k\}} |S_x^k(n)| > Dk\right] > C'\delta^k. \quad (28)$$

Moreover, $\mathbf{P}_{G_{k,m},0}[R(f) > Dk | f(k+1) = x]$ is the probability that if we take $m(k)$ independent copies of S_x^{k+1}, there exists at least one of them for which $\max_{n\in\{0,\dots k\}} |S_x^k(n)| > Dk$. Since

$$m(k) * C\delta^{k+1} \geq CC'\delta(\lambda\delta)^k \to \infty$$

(28), (26) and (27) imply (24).

In order to prove (25), note that if $S^{k+1,1}, \dots, S^{k+1,m(k)}$ are $m(k)$ independent copies of SRW, and if $\lim \frac{\log(m(k))}{k} = 0$, then for all $D > 0$

$$\mathbf{P}_{G_{k,m},0}\left[\max_{n\in\{0,\dots,k+1\}, i\in\{1,\dots,m(k)\}} |S^{k+1,i}(n)| > Dk\right] \to 0.$$

However, if we set $B^{k+1,i} = (f(0), f((1,i)), f((2,i)), \dots, f((k,i)), f(k+1))$, i.e. $B^{k+1,i}$ is the i:th of the parallel paths, then

$$\begin{aligned}
&\mathbf{P}_{G_{k,m},0}[R(f) > Dk] \\
&= \mathbf{P}_{G_{k,m},0}\left[\max_{0\leq n\leq k+1, 1\leq i\leq m(k)} |B^{k+1,i}(n)| > Dk\right] \\
&= \sum_{j=1}^{m(k)} \mathbf{P}_{G_{k,m},0}\left[\max_{0\leq n\leq k+1} |B^{k+1,j}(n)| > Dk \Big| \max_{0\leq n\leq k+1, 0\leq i\leq j-1} |B^{k+1,i}(n)| \leq Dk\right] \\
&\leq \sum_{j=1}^{m(k)} \mathbf{P}_{G_{k,m},0}\left[\max_{0\leq n\leq k+1} |S^{k+1,j}(n)| > Dk \Big| \max_{0\leq n\leq k+1, 0\leq i\leq j-1} |S^{k+1,i}(n)| \leq Dk\right] \\
&= \mathbf{P}_{G_{k,m},0}\left[\max_{0\leq n\leq k+1, 1\leq i\leq m(k)} |S^{k+1,i}(n)| > Dk\right] \to 0
\end{aligned}$$

as needed. □

4 Wired regular trees

4.1 Main results

In this section we discuss the case where G_k^d is a k-level d-ary tree ($d \geq 2$) rooted at v_0, with all the leaves at the last (k'th) level connected to a single node v^* (which is distinct from all the nodes of the tree). This process may be described as a conditional branching random walk (with deterministic branching mechanism, so that all the randomness is in the displacement of the particles) where the condition is that all particles occupy the same location at time $k+1$.

When T_k^d is the k-level d-ary tree rooted at h_0 (with no additional vertices), the behavior of T_k^d-indexed walks is well known. If f is a T_k^d-indexed random walk and $h \in T_k^d$ is at level l, then it is trivial that $f(h)$ has the same distribution as the distribution of SRW started at 0 at time l. Moreover, using e.g. the second moment arguments of Benjamini and Peres [3], one may see that there exists a constant $D > 0$ such that for T_k^d-indexed walks

$$\lim_{k\to\infty} \mathbf{P}_{T_k^d, h_0}[R(f) > Dk] = 1. \tag{29}$$

Note that (29) also holds for d^k parallel paths (by Proposition 3.3). However, we will see that (29) does not hold for G_k^d-indexed walks. The first result we have is:

Proposition 4.1 *For all k, we have for G_k^d-indexed walks f that*

$$\mathbf{P}_{G_k^d, v_0}[|f(v^*)| > n] \le 2t^{d^n}$$

for some $t = t(d) \le 2^{-d+1}$. In particular, the distribution of $f(v^*)$ is tight as $k \to \infty$.

The proof of this proposition is based on properties of power-type Pascal triangles which are developed in the next subsection. Our main result is:

Theorem 4.2 *For all $c > 0$, we have that for G_k^d-indexed walks*

$$\lim_{k\to\infty} \mathbf{P}_{G_k^d, v_0}\left[\frac{(1-c)\log k}{2\log d} < R(f) < \frac{(1+c)\log k}{\log d}\right] = 1. \tag{30}$$

Remarks:

1. Proposition 4.1 holds if we replace the tree of G_k^d by any k-level tree in which the degrees of the internal vertices are at least d. In particular consider the following two step process. At the first step a super-critical branching process for which the children distribution is supported on $\{2, \dots, \}$ is used to produce a k-level tree. All the leaves of that tree are connected to some vertex v^* to obtain some (random) graph G. At the second step we consider a G indexed walk on the graph obtained. Then, Proposition 4.1 hold (with t^{d^n} replaced with t^{2^n}). The proof for these generalizations follows the lines of the proof given below.

2. Theorem 4.2 holds if we replace the tree of G_k^d by any k-level tree in which the degrees of the internal vertices are bounded below by d and above by M. More formally, the exist constants C_1, C_2 which depend on d and M such that for any sequence of such trees:

$$\lim_{k\to\infty} \mathbf{P}_{G_k^d, v_0}[C_1 \log k < R(f) < C_2 \log k] = 1.$$

   Again, this implies the result for super-critical branching processes in which the child distribution is supported on $\{2, \dots, M\}$. The proof is similar to the proof of the theorem given below.

3. If we consider supercritical branching processes in which the children distribution is supported on $\{1, \dots, M\}$ with positive probability on 1, then Theorem 4.2 is no longer true. Instead, we have for a positive constant D,

$$\lim_{k\to\infty} \mathbf{P}[R(f) > Dk] = 1. \tag{31}$$

This follows from the fact that in such a tree with high probability there are exponential number of pipes of linear length. If the child distribution of the super-critical process is supported on $\{0, \dots, M\}$ with positive probability of 0, and we consider the back-bone of the tree, then (31) is still true where $\mathbf{P}$ denotes the probability conditioned on survival. We omit the details.

Recall that $X_{G_k^d, v_0}$ is the set of all G_k^d-indexed walks. What can be said about the cardinality of $X_{G_k^d, v_0}$? The corresponding question for the discrete cube is well known, see Kahn [13]. Since nearest-neighbours in G_k^d are mapped to nearest-neighbours in $\mathbf{Z}$, we must have: $|X_{G_k^d, v_0}| \le 2^{|G_k^d|-1}$. On the other hand by mapping all the vertices in odd (even) levels to the same element in $\mathbf{Z}$, and all vertices in even (odd) levels to one of the two neighbours of this element, we have $|X_{G_k^d, v_0}| \ge \max\{2^{even(G_k^d)}, 2^{odd(G_k^d)}\}$, where $even(G)$ ($odd(G)$) denotes the number of vertices in even (odd) levels of G, excluding the root. It is easy to see that this bound is not optimal: If we fix every 4:th level to be mapped to 0 we get a somewhat better result, if we fix every 8:th level to be mapped to 0 we do even better and so on. However, using entropy methods (as in Kahn [13]) and Proposition 4.1, we improve the trivial upper bound. For a discrete random variable X taking k different values with probabilities $p_1, \dots, p_k$ we define the entropy $H(X)$ as

$$H(X) = H(p_1, \dots, p_k) = -\sum_{i=1}^{k} p_i \log_2 p_i \tag{32}$$

(see e.g [1] for basic properties of entropy).

Proposition 4.3 *We have*

$$|X_{G_k^d, v_0}| \le 4 \max\left\{2^{odd(G_k^d)+even(G_k^d)h(d)}, 2^{even(G_k^d)+odd(G_k^d)h(d)}\right\}$$

where $h(d) = H\left(\frac{1}{1+t(d)}, \frac{t(d)}{1+t(d)}\right)$, *and* $t(d) \le 2^{-d+1}$ *(note that* $\lim_{d\to\infty} h(d) = 0$*).*

4.2 Power-type Pascal triangles

Definition 4.4 *Fix an integer* $d \ge 1$. *The* **power-d Pascal triangle** *is the array*

$$\tilde{P}_d = \{\tilde{P}_d(k, n)\}_{k=0,1,\dots,\, n\in\mathbf{Z}},$$

defined by the recursion

$$\tilde{P}_d(n, k) = \left(\tilde{P}_d(k-1, n-1) + \tilde{P}_d(k-1, n+1)\right)^d, \tag{33}$$

with initial values

$$\tilde{P}_d(0, n) = \begin{cases} 1 & \textit{for } n \in \{-1, 1\} \\ 0 & \textit{for } n \notin \{-1, 1\}. \end{cases} \tag{34}$$

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 4 & & 1 & & \\
 & 1 & & 25 & & 25 & & 1 & \\
1 & & 676 & & 2500 & & 676 & & 1
\end{array}
\qquad\qquad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 1 & & 8 & & 1 & \\
1 & & 729 & & 729 & & 1
\end{array}
$$

Figure 2: The first few elements in the power-2 and power-3 Pascal triangles. (The numbers quickly become too large to fit typographically in such arrays!)

In the usual ($d = 1$) Pascal triangle each term is the sum of the two terms above it. In the power-d Pascal triangle, each term is the d:th power of the two terms above it. See Figure 2.

The connection between the G_k^d-indexed walks and power-type Pascal triangles is given in the following proposition.

Proposition 4.5 *For all d, n, k, we have*

$$\mathbf{P}_{G_k^d, v_0}[f(v^*) = n] = \frac{\tilde{P}(n,k)}{\sum_{j=-\infty}^{\infty} \tilde{P}(n,j)} .$$

Proof: Immediate by induction. □

Next, we give the main tool for the proof of Proposition 4.1. Define $P_d(k,n) = \mathbf{P}_{G_k^d, v_0}[f(v^*) = n]$.

Lemma 4.6 *There exists a constant $t(d) \le 2^{-d+1} < 1$ such that for all $k \ge 1$ and $n \ge 0$*

$$P_d(k, n+2) \le t(d) P_d(k,n). \tag{35}$$

Similarly, for $n \le 0$,

$$P_d(k, n-2) \le t(d) P_d(k,n). \tag{36}$$

Proof: We will prove the lemma for $k = 2m+1, m \ge 1$. The proof for even k is similar. By Proposition 4.5 we may prove (35) and (36) for $\tilde{P}_d$ instead of P_d. We prove these inequalities by induction on m for $t(d) = 2^{-d+1}$. For $m = 0$ we have $\tilde{P}_d(1,-2) = \tilde{P}_d(1,2) = 1$, and $\tilde{P}_d(1,0) = 2^d$, so (35) and (36) hold. We now deduce (35) and (36) for $k = 2m+3$ from (35) and (36) for $k = 2m+1$. Iterating (33) we have

$$\tilde{P}_d(k,n) = \left(\left(\tilde{P}_d(k-2,n-2) + \tilde{P}_d(k-2,n) \right)^d + \left(\tilde{P}_d(k-2,n) + \tilde{P}_d(k-2,n+2) \right)^d \right)^d . \tag{37}$$

Assume first that $n > 0$. In this case, by the induction hypothesis we have

$$\tilde{P}_d(k-2,n) \le t(d) \tilde{P}_d(k-2,n-2),$$

$$\tilde{P}_d(k-2,n+2) \le t(d) \tilde{P}_d(k-2,n),$$

$$\tilde{P}_d(k-2,n+4) \le t(d) \tilde{P}_d(k-2,n+2),$$

so (37) generates:

$$\tilde{P}_d(k,n+2) \leq \left(t(d)^d\right)^d \tilde{P}_d(k,n) \leq t(d)\tilde{P}_d(k,n). \tag{38}$$

The critical case is when is when $n = 0$. There we get

$$\begin{aligned}
&\tilde{P}_d(k,2) = \\
&= \left(\left(\tilde{P}_d(k-2,0) + \tilde{P}_d(k-2,2)\right)^d + \left(\tilde{P}_d(k-2,2) + \tilde{P}_d(k-2,4)\right)^d\right)^d \\
&\leq \left(\frac{1+t(d)^d}{2}\right)^d \left(\left(\tilde{P}_d(k-2,0) + \tilde{P}_d(k-2,2)\right)^d + \left(\tilde{P}_d(k-2,-2) + \tilde{P}_d(k-2,0)\right)^d\right)^d \\
&\leq t(d)P_d(k,0).
\end{aligned}$$

We have proved (35); (36) follows since $P_d(k,-n) = P_d(k,n)$. □

Now we use Lemma 4.6 to obtain tail estimates:

Lemma 4.7 *There exists a constant $t(d) \leq 2^{-d+1} < 1$ such that for all $k \geq 1$ and $n \geq 0$*

$$P_d(k,n+2) \leq t(d)^{d^n} P_d(k,n). \tag{39}$$

Similarly for $n \leq 0$,

$$P_d(k,n-2) \leq t(d)^{d^{-n}} P_d(k,n). \tag{40}$$

Proof: Once more we will prove for $k = 2m+1$ by induction on m. Here also we may prove (39) and (40) for $\tilde{P}_d$ instead of P_d. When $m = 0, k = 1$, the inequalities hold. We now deduce the claim for $k+2$ from the claim for k. The case of $n = 0$ is covered by Lemma 4.6. Hence, we may assume $n > 0$. By the induction hypothesis

$$\tilde{P}_d(k-2,n) \leq t(d)^{n-2}\tilde{P}_d(k-2,n-2),$$

$$\tilde{P}_d(k-2,n+2) \leq t(d)^n\tilde{P}_d(k-2,n) \leq t(d)^{n-2}\tilde{P}_d(k-2,n),$$

$$\tilde{P}_d(k-2,n+4) \leq t(d)^{n+2}\tilde{P}_d(k-2,n+2) \leq t(d)^{n-2}\tilde{P}_d(k-2,n+2),$$

so (37) generates

$$\tilde{P}_d(k+2,n) \leq \left(\left(t(d)^{d^{n-2}}\right)^d\right)^d P_d(k,n) = t(d)^{d^n} P_d(k,n)$$

as needed. □.

4.3 The range and the top

We now prove Proposition 4.1 and Theorem 4.2.

Proof of Proposition 4.1: Immediate from Proposition 4.5 and Lemma 4.7. □

In order to prove Theorem 4.2 we need some more lemmas.

Lemma 4.8 *Letting $v_1^l, \ldots, v_{d^l}^l$ denote the vertices of the l:th level of G_k^d (or T_k^d), we have*

$$\begin{aligned} & \mathbf{P}_{G_k^d, v_0}\left[\Big(f(v_1^l), \ldots, f(v_{d^l}^l)\Big) = (x_1, \ldots, x_{d^l}) | f(v^*) = x\right] = \\ = \quad & \frac{1}{Z}\mathbf{P}_{T_l^d, v_0}\left[\Big(f(v_1^l), \ldots, f(v_{d^l}^l)\Big) = (x_1, \ldots, x_{d^l})\right] \prod_{i=1}^{d^l} \mathbf{P}_{G_{k-l}^d, v_0}\left[f(v^*) = x - x_i\right] . \end{aligned}$$

for some positive constant Z.

Proof: Immediate. □

To lighten the notation in what follows, we write Q_k for $\mathbf{P}_{T_k^d, v_0}$. For an integer $t \geq 0$, we also write Q_k^t for Q_k conditioned on the event that $|v_1^k| = t$.

Lemma 4.9 *For $t \geq s$ there exists a coupling $Q_k^{t,s}$ of the measures Q_k^t and Q_k^s satisfying*

$$Q_k^{t,s}\left[\Big\{(f,g) : |f(v)| \leq |g(v)| \text{ for all } v \in V_{T_k^d}\Big\}\right] = 1\,. \tag{41}$$

Proof: The result follows by considering $\mathbf{P}^*_{T_k^d, v_0}$ conditioned on $f(v_1^k) = t$ (or on $f(v_1^k) = s$), calculating conditional probabilities as in the proof of Proposition 2.5 and applying Lemma 2.2. □

Proof of Theorem 4.2: We first claim that it suffices to prove (30) for even k and condition on $f(v^*) = 0$. Indeed, suppose we have proven (30) under these conditions, and we have for some $c > 0$,

$$\lim_{k \to \infty} \mathbf{P}_{G_k^d, v_0}\left[R(f) > \frac{(1-c)\log k}{2 \log d}\right] \neq 1.$$

Thus, from Proposition 4.1 there exists an integer r, an $\epsilon > 0$, and an infinite number of k_i's such that,

$$\mathbf{P}_{G_{k_i}^d, v_0}\left[R(f) \leq \frac{(1-c)\log k_i}{2\log d}\Big| f(v^*) = r\right] > \epsilon\,. \tag{42}$$

For such k_i, let $l_i \in \{k_i + |r|, k_i + |r| + 1\}$ be even. We claim that

$$\mathbf{P}_{G_{l_i}^d, v_0}\left[R(f) \leq \frac{(1-c)\log k_i}{2\log d} + |r| + 1 \Big| f(v^*) = 0\right] > 2^{-d^{|r|+2}+1} y^{d^{|r|+1}} \epsilon^{d^{|r|+1}} \tag{43}$$

for some $0 < y < 1$. This implies that for $c' = c/2$ we have

$$\lim_{k \to \infty} \mathbf{P}_{G_k^d, v_0}\left[R(f) > \frac{(1-c')\log k}{2\log d}\Big| f(v^*) = 0\right] \neq 1,$$

where the limit is taken over even k, in contradiction to our assumption. In order to show that (42) implies (43), let A_{l_i} be the event that the G_{l_i}-indexed walk maps all v in level $j \leq l_i - k_i$ to j. ¿From Lemma 4.8 we have that

$$\mathbf{P}_{G_{l_i}^d, v_0}[A_{l_i} | f(v^*) = 0] > 2^{-d^{|r|+2}+1} y^{d^{|r|+1}}, \tag{44}$$

for some $y > 0$ (which depends on r but not on k_i or l_i), and it is clear that

$$\mathbf{P}_{G^d_{l_i},v_0}\left[R(f) \leq \frac{(1-c)\log k_i}{2\log d} + |r| + 1 \Big| A_{l_i}, f(v^*) = 0\right] > \epsilon^{d^{|r|+1}}. \tag{45}$$

Combining (44) and (45) we see that (42) implies (43). The proof that for the other bound it is enough to assume that $f(v^*) = 0$ and that k is even is similar (but easier).

It remains to prove that for even k and for all $c > 0$, we have

$$\lim_{k\to\infty} \mathbf{P}_{G^d_k,v_0}\left[R(f) < \frac{(1+c)\log k}{\log d} \Big| f(v^*) = 0\right] = 1, \tag{46}$$

and

$$\lim_{k\to\infty} \mathbf{P}_{G^d_k,v_0}\left[R(f) > \frac{(1-c)\log k}{2\log d} \Big| f(v^*) = 0\right] = 1. \tag{47}$$

We start with a proof of (46). Let v be any vertex. We will show that there exist $r \in (0,1)$ such that if $t, s \in \mathbf{Z}$, and $t > s$, then

$$\mathbf{P}_{G^d_k,v_0}[|f(v)| = t | f(v^*) = 0] \leq r^{d^t} \mathbf{P}_{G^d_k,v_0}[|f(v)| = s | f(v^*) = 0]\,. \tag{48}$$

¿From this it follows that

$$\mathbf{P}_{G^d_k,v_0}[|f(v)| > s] \leq r^{d^s}, \tag{49}$$

(for some other $r \in (0,1)$) and therefore if

$$\lim_{k\to\infty} r^{d^{s(k)}} d^k = 0,$$

then

$$\lim_{k\to\infty} \mathbf{P}_{G^d_k,v_0}[R(f) < s(k) | f(v^*) = 0] = 1.$$

In particular, (46) holds for all $c > 0$.

In order to prove (48), assume that $v = v^l_i$ is at level l, at index i. We denote $w = (w_1, \ldots, w_{d^l})$, and $v = (v^l_1, \ldots, v^l_{d^l})$. Lemmas 4.7, 4.8 and 4.9 imply that

$$\begin{aligned}
&\frac{\mathbf{P}_{G^d_k,v_0}[|f(v^l_i)| = t | f(v^*) = 0]}{\mathbf{P}_{G^d_k,v_0}[|f(v^l_i)| = s | f(v^*) = 0]} = \\
&= \frac{Z^{-1}\sum_{w:|w_i|=t} Q_l[f(v) = w] \prod_{j=1}^{d^l} \mathbf{P}_{G^d_{k-l},v_0}[f(v^*) = -w_j]}{Z^{-1}\sum_{w:|w_i|=s} Q_l[f(v) = w] \prod_{j=1}^{d^l} \mathbf{P}_{G^d_{k-l},v_0}[f(v^*) = -w_j]} \\
&= \frac{Q_l[|f(v^l_i)| = t] \sum_w Q^t_l[w] \prod_{j=1}^{d^l} \mathbf{P}_{G^d_{k-l},v_0}[f(v^*) = -w_j]}{Q_l[|f(v^l_i)| = s] \sum_w Q^s_l[w] \prod_{j=1}^{d^l} \mathbf{P}_{G^d_{k-l},v_0}[f(v^*) = -w_j]} \leq r^{d^t}.
\end{aligned}$$

The last equality follows from the fact that for all w with $|w_i| = t$, we have

$$Q_l[f(v) = w] = Q_l[|f(v^l_i)| = t] Q^t_l[f(v) = w]\,,$$

whereas the last inequality follows from the fact that since $t > s > 0$,

$$Q_l[|f(v^l_i)| = t] \leq Q_l[|f(v^l_i)| = s].$$

Moreover, using the coupling of Lemma 4.9, we get

$$\frac{\sum_w Q_l^t[w]\prod_{j=1}^{d^l}\mathbf{P}_{G^d_{k-l},v_0}[f(v^*)=-w_j]}{\sum_w Q_l^s[w]\prod_{j=1}^{d^l}\mathbf{P}_{G^d_{k-l},v_0}[f(v^*)=-w_j]}\le\frac{\mathbf{P}_{G^d_{k-l},v_0}[f(v^*)=t]}{\mathbf{P}_{G^d_{k-l},v_0}[f(v^*)=s]}\le r^{d^t}.$$

The proof of the upper bound (46) is now complete.

We turn to the proof of the lower bound. For a moment fix h. Let A_k be the event that $R(f)\ge h$. We denote the set of nodes at level i by L_i, and let B_k be the following event:

$$B_k=\{\max\{f(v):v\in\cup_i L_{2ih}\}-\min\{f(v):v\in\cup_i L_{2ih}\}<h\}.$$

Clearly,

$$\begin{aligned}\mathbf{P}_{G^d_k,v_0}[A_k] &= \mathbf{P}_{G^d_k,v_0}[B_k]\mathbf{P}_{G^d_k,v_0}[A_k|B_k]+\mathbf{P}_{G^d_k,v_0}[\bar{B}_k]\mathbf{P}_{G^d_k,v_0}[A_k|\bar{B}_k]\\ &= \mathbf{P}_{G^d_k,v_0}[\bar{B}_k]+\mathbf{P}_{G^d_k,v_0}[B_k]\mathbf{P}_{G^d_k,v_0}[A_k|B_k]\\ &\ge \mathbf{P}_{G^d_k,v_0}[A_k|B_k].\end{aligned}$$

We now estimate $\mathbf{P}_{G^d_k,v_0}[A_k|B_k]$. We note that

$$\mathbf{P}_{G^d_k,v_0}[A_k|B_k]\ge\min\mathbf{P}_{G^d_k,v_0}[A_k|B_k,\{f(v)\}_{v\in\cup_i L_{2ih}}],\tag{50}$$

where the minimum is taken over all $\{f(v)\}_{v\in\cup_i L_{2ih}}$ for which B_k hold. For each $v\in\cup_i L_{2ih}$, let $T_{2h}(v)$, be the subtree rooted at v of $2h$ levels, and let

$$A_k^v=\{|\{f(v):v\in T_{2h}(v)\}|\ge h\}.$$

The events A_k^v are independent given $\{f(v)\}_{v\in\cup_i L_{2ih}}$. Moreover, it is easy to see that for all $v\in\cup_i L_{2ih}$,

$$\mathbf{P}_{G^d_k,v_0}[A_k^v|B_k,\{f(v)\}_{v\in\cup_i L_{2ih}}]\ge 2^{-d^{2h+1}+1}.$$

Therefore, if we have for $h=h(k)$ that

$$\lim_{k\to\infty}d^{k-h}2^{-d^{2h+1}+1}\to\infty,$$

then also

$$\lim_{k\to\infty}\mathbf{P}_{G^d_k,v_0}[A_k|B_k]\to 1.$$

Taking

$$h(k)=\frac{(1-c)\log k}{2\log d}$$

we obtain the desired result. $\square$

4.4 Number of G^d_k Walks

Proof of Proposition 4.3: We will prove the proposition for odd k. The proof for even k is similar. Since k is odd, the task is to prove that

$$|X_{G^d_k,v_0}|\le 4\times 2^{odd(G^d_k)+even(G^d_k)h(d)}.\tag{51}$$

Let $X^0_{G^d_k,v_0}$ be the set of G^d_k-indexed walks which satisfy $f_k(v^*)=0$. From Lemma 4.7 it follows that

$$|X_{G^d_k,v_0}| \le 4|X^0_{G^d_k,v_0}|.$$

Therefore, in order to prove (51) it suffices to show that

$$|X^0_{G^d_k,v_0}| \le 2^{odd(G^d_k)+even(G^d_k)h(d)}. \tag{52}$$

Let X be a uniform variable on $X^0_{G^d_k,v_0}$. Let H be the entropy function. It is clear that (52) is equivalent to

$$H(X) \le odd(G^d_k) + even(G^d_k)h(d). \tag{53}$$

However,

$$H(X) \le \sum_{l=1}^{k}\sum_{i=1}^{l^d} H\left(X(v^l_i)\middle|X(v^{l'}_i)\right) \tag{54}$$

where v' denotes the parent of v.

Since given $X(v^{l'}_i)$, $X(v^l_i)$ has two possible values, we have for all v^l_i,

$$H\left(X(v^l_i)|X(v^{l'}_i)\right) \le 1. \tag{55}$$

Moreover, if l is even, then from Lemma 4.6, we have that if $X(v^{i'}_l) > 0$ then

$$\mathbf{P}_{G^d_k,v_0}\left[X(v^l_i) = X(v^{l'}_i)+1\right] \le t(d)\mathbf{P}_{G^d_k,v_0}\left[X(v^l_i) = X(v^{l'}_i)-1\right]. \tag{56}$$

Similarly if $X(v^{i'}_l) < 0$, then,

$$\mathbf{P}_{G^d_k,v_0}\left[X(v^l_i) = X(v^{l'}_i)-1\right] \le t(d)\mathbf{P}_{G^d_k,v_0}\left[X(v^l_i) = X(v^{l'}_i)+1\right]. \tag{57}$$

Equations (56) and (57) imply that for l even,

$$H\left(X(v^l_i)|X(v^{l'}_i)\right) \le H\left(\frac{1}{1+t(d)}, \frac{t(d)}{1+t(d)}\right). \tag{58}$$

In (54) we now take the bound (55) for odd l and (58) for even l, to obtain (53). $\square$

5 The discrete cube

In this short section we discuss the case of the k-dimensional discrete cube: $G_k = (V_{G_k}, E_{G_k})$ where $V_{G_k} = \{0,1\}^k, E_{G_k} = \{(x,y) : h(x,y) = 1\}$ and h denotes Hamming distance. In this case we let $v_0 = (0,\dots,0)$. By a direct application of Theorem 2.8 and well-known large deviations behavior of SRW (see e.g. Durrett [4], p. 76), we get the following.

Corollary 5.1 *For any integer k and any $t > 0$, we have for G_k-indexed walks that*

$$\mathbf{P}_{G_k,v_0}[|f(v)| \ge tk] \le 2e^{-\frac{kt^2}{4}}$$

for all $v \in V_{G_k}$.

Remark: Instead of using Theorem 2.8, one may utilize measure concentration results for the discrete cube (see e.g. Talagrand [18]) to obtain a similar result (with somewhat worse constants). We outline the argument below: Fix v and define $S(f)(u) = S_v(f)(u) = f(u) - f(u \oplus v)$, where $\oplus$ is the addition in the group $(\mathbf{Z}/2\mathbf{Z})^k$. It is easy to see that for all $f \in G_k$, $S(f)$ is a Lipschitz function with constant 2. A moment's reflection reveals that for all $w_1, w_2 \in V_{G_k}$ and all $t \in \mathbf{Z}$, we have

$$\mathbf{P}_{G_k,v_0}[S(f)(w_1) = t] = \mathbf{P}_{G_k,v_0}[S(f)(w_2) = t]. \tag{59}$$

On the other hand, from measure concentration results for the discrete cube (see e.g. Talagrand [18]) we have for all fixed $f \in X_{G_k,v_0}$ that

$$\frac{|\{x : |S(f)(x)| > tk\}|}{2^k} \leq \frac{1}{2}e^{-\frac{kt^2}{8}}. \tag{60}$$

Combining (59) with (60) we have

$$\begin{aligned} \mathbf{P}_{G_k,v_0}[|f(v)| > tk] &= \mathbf{P}_{G_k,v_0}[|S(f)(v)| > tk] = \frac{1}{2^k}\sum_{u \in V_G} \mathbf{P}_{G_k,v_0}[|S(f)(u)| > tk] \\ &= \mathbf{E}_{G_k,v_0}\left[\frac{1}{2^k}\left|\{u \in V_G : |S(f)(u)| > tk\}\right|\right] \leq \frac{1}{2}e^{-\frac{kt^2}{8}} \end{aligned}$$

as desired.

We conjecture that the concentration of measure for a typical G_k-indexed random walk should be much stronger than the deterministic bound $R(f) \leq k+1$. In particular, a modest achievement in that direction would be to prove the following.

Conjecture 5.2 *For all $t > 0$, we have*

$$\lim_{k \to \infty} \mathbf{P}_{G_k,v_0}[R(f) > tk] = 0.$$

Remark: We note that the analogue of Conjecture 5.2 for the Gaussian field model holds. Since the resistance between any two vertices is bounded by some global constant (independent of k), the variance of $f(v)$ is also bounded by some global constant. However, $f(v)$ is Gaussian and therefore it follows that for all k, and all $v \in G_k$,

$$\mathbf{P}_k[|f(v)| \geq t] \leq C_1 e^{-C_2 t^2}$$

for some positive C_1, C_2, where $\mathbf{P}_k$ denotes the Gaussian field measure on the k-dimensional discrete cube. Therefore, for the G_k Gaussian fields we have

$$\lim_{k \to \infty} \mathbf{P}_k[|R(f)| \geq C\sqrt{k}] = 0$$

for some positive $C > 0$.

An obvious attempt to bound $R(f)$ would be to use Corollary 5.1 to bound the expected number of vertices taking value above tk, but unfortunately this does not give any useful bound.

Kahn [13] give bounds on the number of G_k-indexed walks. We do not see how to use these bounds for our purpose.

6 Emulating the Ising model

Propositions 2.1, 2.4 and 2.5 are all indications that $\mathbf{P}_{G,v_0}$ is, in various respects, well-behaved. A pessimistic interpretation would be to conclude that G-indexed random walks are "dull". As an argument that this is not the case, we will now demonstrate how the ferromagnetic Ising model on any finite graph H can be emulated by a graph-indexed walk on a different graph G.

The Ising model is one of the most fundamental models in statistical mechanics. It has been the subject of countless studies, and many intricate phenomena have been revealed; the reader may turn e.g. to Liggett [15], Georgii [6] or Georgii et al. [7] for a start.

Let $H = (V_H, E_H)$ be any finite graph. The Gibbs measure μ_β^H for the Ising model on H at reciprocal temperature $\beta \geq 0$ is the probability measure on $\{-1, 1\}^{V_H}$ which to each $\omega \in \{-1, 1\}^{V_H}$ assigns probability

$$\mu_\beta^H(\omega) = \frac{1}{Z_\beta^H} \exp\left(\beta \sum_{\{u,v\}} \omega(u)\omega(v)\right) . \tag{61}$$

Here $\{u, v\}$ means that we sum over all (unordered) pairs of vertices sharing an edge, and Z_β^H is a normalizing constant.

Given H, we define another graph $G = (V_G, E_G)$ from H by

(i) replacing each edge in H by two edges in series,

(ii) adding an additional vertex v_0, and

(iii) including an edge between v_0 and v for each $v \in V_H$.

In other words, $V_G = V_H \cup E_H \cup \{v_0\}$ and

$$E_H = \{\{v, e\} : v \in V_H, e \in E_H, e \text{ is incident to } v\} \cup \{\{v_0, v\} : v \in V_H\} .$$

A direct counting argument shows the following.

Proposition 6.1 *With G and H as above, the $\mathbf{P}_{G,v_0}$-distribution of $f(V_H)$ equals the Ising model Gibbs measure μ_β^H with $\beta = \frac{1}{2}\log 2$.*

If we modify G further by placing k paths of length 2 in parallel between v_0 and each $e \in E_H$, then the $\mathbf{P}_{G,v_0}$-distribution of $f(V_H)$ instead equals μ_β^H with $\beta = \frac{1}{2}\log(1+2^{-k})$. By placing n such "decorations" in parallel between each pair of vertices $u, v \in V_H$ with $\{u, v\} \in E_H$, we get distribution μ_β^H with $\beta = \frac{n}{2}\log(1 + 2^{-k})$. The set of reciprocal temperatures for which we can emulate the Ising model on H is therefore dense in $(0, \infty)$.

This construction has some resemblance with the subshift of finite type imitations of Gibbs models obtained by Häggström [9, 10]. Since there are only countably many ways to construct H, the restriction to a countable dense set of β-values cannot be removed. One may also ask whether it is possible to do the same thing for $\beta < 0$ (this is the so called antiferromagnetic Ising model), but it follows from Proposition 2.4 that this cannot be done.

7 Final remarks

We expect that a lot remains to be revealed about G-indexed random walks. Among open problems, we have already mentioned Conjectures 2.10 and 5.2. Another problem which may be of interest is the following.

Open problem: Let the graphs G and H satisfy the usual assumptions (finite, connected, bipartite) and suppose that G and H are roughly isometric with constant $k < \infty$ (that is, there is a function g from V_G to V_H such that $k^{-1}d(x,y) - k \leq d(g(x), g(y)) \leq kd(x,y) + k$ for any $x, y \in V_G$, and for every $z \in V_H$ there is some $x \in V_G$ so that $d(g(x), z) \leq k$). What is the relationship between G- and H-indexed random walks? In particular, suppose the we have two familis of graphs $\{G_n\}$ and $\{H_n\}$, and that each G_n is roughly isometric to H_n with the same constant k. Can it happen that

$$\lim_{n\to\infty} \frac{\mathbf{E}_{G_n,v_0} R(f)}{\mathbf{E}_{H_n,v_0} R(f)} = \infty$$

or is there some constant $C = C(k)$ bounding $\frac{\mathbf{E}_{G_n,v_0} R(f)}{\mathbf{E}_{H_n,v_0} R(f)}$?

There are of course also various ways in which our model may be extended. The image $\mathbf{Z}$ of our graph homomorphisms may be replaced by any other graph. For instance, if we replace it by a complete graph on k vertices, then we obtain the usual random k-coloring model.

Generalizing further, the underlying simple random walk can be replaced by any reversible Markov chain. Uniform measure is then replaced by some weighted measure where each f gets a weight proportional to $\prod_{\{u,v\}\in E_G} C(f(u), f(v))$ for some interaction function C, thus putting us in the familiar generality of Gibbs measures with nearest-neighbor pair interactions.

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