RANDOM PLANAR METRICS

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Abstract. A discussion regarding aspects of several quite different random planar metrics and related topics is presented.

1. Introduction

In this note we will review some aspects of random planar geometry, starting with random perturbation of the Euclidean metric. In the second section we move on to stationary planar graphs, including unimodular random graphs, distributional local limits and in particular the uniform infinite planar triangulation and its scaling limit. The last section is about a non planar random metric, the critical long range percolation, which arises as a discretization of a Poisson process on the space of lines in the hyperbolic plane. Several open problems are scattered throughout the paper. We only touch a small part of this rather diverse and rich topic.

2. Euclidean perturbed

One natural way to randomly perturb the Euclidean planar metric is that of first passage percolation (FPP), see [25] for background. That is, consider the square grid lattice, denoted $\mathbb{Z}^2$, and to each edge assign an i.i.d. random positive length. There are other ways to randomly perturb the Euclidean metric and many features are not expected to be model dependent. Large balls converge after rescaling to a convex centrally symmetric shape, the boundary fluctuations are conjectured to have a Tracy-Widom distribution. The variance of the distance from origin to $(n, 0)$ is conjectured to be of order $n^{2/3}$. So far only an upper bound of $\frac{n}{\log n}$ was established, see [6]. It is still not known how stable is the shortest path and its length to random perturbation as considered in noise sensitivity theory, see [7, 21]. Also what are the most efficient algorithms to find the shortest path or to estimate its length? When viewed as a random electrical network it is conjectured that the variance of the resistance from the origin to $(n, 0)$ is uniformly bounded, see [9].

Consider random lengths chosen as follows: 1 with probability $p > 1/2$ and ∞ otherwise. Look at the convex hull of all vertices with distance less than n to the origin (assuming the origin is in the infinite cluster). Simulations suggest that as $p \searrow 1/2$ the limiting shape converges to a Euclidean ball. This is still open but heuristically supported by the conformal invariance of critical Bernoulli percolation.

The structure of geodesic rays and two sided infinite geodesics in first passage percolation is still far from understood. Furstenberg asked in the 80's (following a talk by Kesten) to show that almost surely there are no two sided infinite geodesics for natural FPP's, e.g. exponential length on edges.

Häggström and Pemantle introduced [22] competitions based on FPP, see [18] for a survey. Here is a related problem. Start two independent simple random walks on $\mathbb{Z}^2$ walking with the same clock, with the one additional condition, the walkers are not allowed to step on vertices already visited by the other walk, and otherwise chose uniformly among allowed vertices. Show that almost surely, one walker will be trapped in a finite domain. Prove that this is not the case in higher dimensions.

3. Unimodular random graphs, Uniform random triangulations

There is a recent growing interest in graph limits, see e.g. [31] for a diversity of viewpoints. In parallel the theory of random triangulations was developed as a toy model of quantum gravity, initially by physicists. Angel and Schramm [2, 3] constructed the uniform infinite planar triangulation (UIPT), a rooted infinite unimodular random triangulation which is the limit (in the sense of [10]) of finite random triangulations (the uniform measure on all non isomorphic triangulations of the sphere of size n), a model that was studied extensively by many (see e.g. [26]). Exponential of the Gaussian free field (GFF) provides a model of random measure on the plane, see [19].

Therefore in the theory of random uniform planar graphs and triangulations we encounter several view points and many missing links. The general theory of unimodular random graphs [10, 1] is useful in deducing certain properties, giving a notion of "stationary" graph in the spirit of stationary process. This is a measure on graphs rooted at a directed edge which is invariant for rerooting along a random walk path. This rather minimal assumption turned out to be a surprisingly strong generalization of Cayley graphs, or transitive unimodular graphs. Conformal geometry is useful in the bounded degree set up. Enumeration is useful when no restriction on the degree is given. See the recent work [13] and references there, for the success of enumeration techniques. The links to the Gaussian free field is only a conjecture at the moment, and a method of constructing a conformal invariant random path metric on the real plane from the Gaussian free field is still eluding. There are many open problems in any of the models. Here are a few:

(1) Angel and Schramm [3] conjectured that the UIPT is a.s. recurrent. At what rate does the resistance grow? Note that the local limit of bounded degree finite planar graphs is recurrent [10]. The degree distribution of UIPT has an exponential tail. It is of interest to understand the limit of large random triangulations conditioned on having degree smaller than some fixed constant.
(2) View a large finite triangulation as an electrical network. Understanding the effective resistance will make it possible to study the Gaussian free field on the triangulation. The Laplacian spectrum and eigenfunctions nodal domain and level sets are of interest, see [20] for background.
(3) Show that the simple random walk on the UIPT is subdiffusive. What is the (sub)-diffusivity exponent?
(4) Show that if G is a distributional limit (in the sense of [10]) of finite planar graphs then the critical probability for percolation on G satisfies $p_c(G, site) \geq 1/2$ a.s. and no percolation at the critical probability. This last fact should hold for any unimodular random graph.
(5) Consider the $n \times n$ grid equipped with the Gaussian free field with no boundary conditions. The exponential of the field gives a positive "length" to each vertex. We get a random metric on the square grid. Let $\gamma_1(n)$ be the shortest path between the top corners and $\gamma_2(n)$ the shortest path between the bottom corners. Show there is $c > 0$, so that for any n, $P(\{\gamma_1(n) \bigcap \gamma_2(n) \neq \emptyset\}) > c$. Identify the scaling limit of $\gamma_1(n)$? Establish and study the scaling limit of these metric spaces. How do geodesic concentrate around a fixed height of the field? What is the dimension of the geodesics? Since scaling limits of geodesics likely have Euclidean dimension strictly bigger than one, it suggests that geodesics wind a every scale and therefore "forget" the starting point. Thus likely the limit is rotationally invariant and maybe close to Schramm's SLE_κ curve, for what κ?
(6) The most natural way to generate a quadrangulation from a sequence of $2n$ bits is using Schaeffer's bijection. How stable are natural properties of the quadrangulation to independent noise of the bits? E.g. what is the probability the diameter drops from above to below median after applying ϵ noise? See [7] and [21] for a recent survey on noise.
(7) Try to formulate and/or prove something in higher dimensions, see [5].

The coming three subsections discuss the scaling limit of finite random planar maps and harmonic measure for random walks on random triangulations.

3.1. **Scaling limit of Planar maps.** A planar map m is a proper embedding of a planar graph into the two dimensional sphere $\mathbb{S}_2$ seen up to deformations. A *quadrangulation* is a rooted planar map such that all faces have degree 4. For sake of simplicity we will only deal with these maps (see universality results). Let m_n be a uniform variable on the set $\mathcal{Q}_n$ of all roted quadrangulations with n faces. The radius of m_n is

$$r_n = \max_{v \in \mathrm{Vertices}(m_n)} \mathrm{d_{gr}}(\rho, v),$$

where ρ denotes the root vertex of m_n. In their pioneering work, Chassaing and Schaeffer [17] showed that the rescaled radii converge in law towards the diameter r of the one-dimensional Integrated Super Brownian Excursion (ISE),

$$n^{-1/4} r_n \xrightarrow{\text{(law)}} \left(\frac{8}{9}\right)^{1/4} r.$$

The key ingredient is a bijective encoding of rooted quadrangulations by labelled trees due to Cori-Vauquelin and Schaeffer [33]. This was the first proof of the physicist's conjecture that the distance in a typical map of size n should behave like $n^{1/4}$. Nevertheless this convergence does not allow us to understand the whole metric structure of a large map. To do this, we should consider a map endowed with its graph distance $\mathrm{d_{gr}}$ as a metric space and ask for convergence in the sense of Gromov-Hausdorff metric (see [15]). In other words, if m_n is uniform on $\mathcal{Q}_n$, we wonder whether the following weak convergence for the Gromov-Hausdorff metric occurs

$$\left(m_n, n^{-1/4}\mathrm{d_{gr}}\right) \xrightarrow{?} (m_\infty, \mathrm{d}_\infty), \tag{3.1}$$

where $(m_\infty, \mathrm{d}_\infty)$ is a random compact metric space. Unfortunately, the convergence (3.1) is still unproved and constitutes the main open problem in this area. Nevertheless, Le Gall has shown in [28] that (3.1) is true along subsequences. Thus we are left with a family of random metric spaces called Brownian maps which are precisely the limiting points of the the sequence $\left(m_n, n^{-1/4}\mathrm{d_{gr}}\right)$ for the weak convergence of probability measures with respect to Gromov-Hausdorff distance. Moreover any Brownian map carries a natural volume measure, which is the limit of the uniform probability measure on the vertex set of m_n. One conjectures that there is no need to take a subsequence, that is all Brownian maps have the same law. Still one can establish properties shared by all Brownian maps e.g.

Theorem 3.1 ([28],[30])**.** *Let $(m_\infty, \mathrm{d}_\infty)$ be a Brownian map. Then*

- **(a):** *Almost surely, the Hausdorff dimension of $(m_\infty, \mathrm{d}_\infty)$ is 4.*
- **(b):** *Almost surely, $(m_\infty, \mathrm{d}_\infty)$ is homeomorphic to $\mathbb{S}_2$.*

In a recent work [29], Le Gall completely described the geodesics towards a distinguished point and the description is independent of the Brownian map considered. Here are some extensions and open problems:

(1) Although we know that Brownian maps share numerous properties, they do not seem sufficient to identify the law and thus prove (3.1). In a forthcoming paper by Curien, Le Gall and Miermont, they show the convergence (without taking any subsequence) of the so-called "Cactus" associated to m_n.

(2) The law of the matrix of mutual distances between p points chosen uniformly at random is sufficient to characterize the law of a random measured metric space. For $p = 2$, the law of the distance in any Brownian map between two independent random points can be expressed in terms of ISE. Recently the physicists Bouttier and Guitter [16] obtained a similar expression in the case $p = 3$. Unfortunately their techniques do not seem to extend to higher values of p.

3.2. **QG and GFF.** Let $\mathcal{T}_n$ be the set of all triangulations of the sphere $\mathbb{S}_2$ with n faces with no loops or multiple edges. We recall the well known circle packing theorem (see Wikipedia, [23]):

Theorem 3.2. *If T is a finite triangulation without loops or multiple edges then there exists a circle packing $P = (P_c)_{c \in C}$ in the sphere $\mathbb{S}_2$ such that the contact graph of P is T. This packing is unique up to Möbius transformations.*

Recall that the group of Möbius transformations $z \mapsto \frac{az+b}{cz+d}$ for $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ can be identified with $\mathrm{PSL}_2(\mathbb{C})$ and act transitively on triplets (x, y, z) of $\mathbb{S}_2$. The circle packing enables us to take a "nice" representation of a triangulation $T \in T_n$, nevertheless the non-uniqueness is somehow disturbing because to fix a representation we can, for example, fix the images of three vertices of a distinguished face of T. This specification breaks all the symmetry, because sizes of some circles are chosen arbitrarily. Here is how to proceed:

Barycenter of a measure on $\mathbb{S}_2$. The action on $\mathbb{S}_2$ of an element $\gamma \in \mathrm{PSL}_2(\mathbb{C})$ can be continuously extended to $\mathbb{B}_3 := \{(x, y, z) \in \mathbb{R}^3, x^2 + y^2 + z^2 \leq 1\}$: this is the Poincaré-Beardon extension. We will keep the notation γ for transformations $\mathbb{B}_3 \to \mathbb{B}_3$.

The action of $\mathrm{PSL}_2(\mathbb{C})$ on $\mathbb{B}_3$ is now transitive on points. The group of transformations that leave 0 fixed is precisely the group $\mathrm{SO}_2(\mathbb{R})$ of rotations of $\mathbb{R}^3$.

Theorem 3.3 (Douady-Earle)**.** *Let μ be a measure on $\mathbb{S}_2$ such that $\#\mathrm{supp}(\mu) \geq 2$. Then we can associate to μ a "barycenter" denoted by $\mathrm{Bar}(\mu) \in \mathbb{B}_3$ such that for all $\gamma \in \mathrm{PSL}_2(\mathbb{C})$ we have*

$$\mathrm{Bar}(\gamma^{-1}\mu) = \gamma(\mathrm{Bar}(\mu)).$$

We can now describe the renormalization of a circle packing. If P is a circle packing associated to a triangulation $T \in \mathcal{T}_n$, we can consider the atomic measure μ_P formed by the Dirac's at centers of the spheres in P

$$\mu_P := \frac{1}{\#P} \sum_{x \text{ centers of } P} \delta_x.$$

By transitivity there exists a conformal map $\gamma \in \mathrm{PSL}_2(\mathbb{C})$ such that $\mathrm{Bar}(\gamma^{-1}\mu_P) = 0$. The renormalized circle packing is by Definition $\gamma(P)$, this circle packing is unique up to rotation of $\mathrm{SO}_2(\mathbb{R})$, we will denote it by $\mathbf{P}_T$. This constitutes a canonical discrete conformal structure for the triangulation.

Open problems. If T_n is a random variable uniform over the set $\mathcal{T}_n$, then the variable $\mu_{\mathbf{P}_{T_n}}$ is a random probability measure over $\mathbb{S}_2$ seen up to rotations of $\mathrm{SO}_2(\mathbb{R})$. By classical arguments there exist weak limits μ_∞ of $\mu_{\mathbf{P}_{T_n}}$.

(1) (Schramm [Talk about QG]) Determine coarse properties (invariant under $\mathrm{SO}_2(\mathbb{R})$) of μ_∞, e.g. what is the dimension of the support? Start with showing singularity.
(2) Uniqueness (in law) of μ_∞? In particular can we describe μ_∞ in terms of GFF? Is it $\exp((8/3)^{1/2}GFF)$, does KPZ hold? see [19].
(3) The random measure μ_∞ can come together with d_∞ a random distance on $\mathbb{S}_2$ (in the spirit of [28]). Can you describe links between μ_∞ and d_∞? Does one characterize the other?

3.3. **Harmonic measure and recurrence.** Our goal in this subsection is to remark that if a graph is recurrent then harmonic measure on boundaries of domains can not be very spread and supported uniformly on (too) large sets. We have in mind random triangulations. We first discuss general graphs.

Let G denote a bounded degree infinite graph. Fix a base vertex v and denote by $B(r)$ the ball of radius r centered at v, by $\partial B(r)$ the boundary of the ball, that is vertices with distance r from v. Denote by μ_r the harmonic measure for simple random walk starting at v on $\partial B(r)$.

Assume simple random walk (SRW) on G is recurrent. Further assume that there are arbitrarily large excursions attaining the maximum distance once, this happens in many natural examples but not always (e.g. consider the graph obtained by starting with a ray and adding to the ray a full n levels binary tree rooted at the vertex on the ray with distance n to the root, for all n). The maximum of SRW excursion on $\mathbb{Z}$ is attained a tight number of times. It is reasonable to believe that if each of the vertices in $\partial B(r)$ admit a neighbor in $\partial B(r+1)$, then the same conclusion will hold.

Proposition 3.4. *Under the stronger further assumption above, for infinitely many r's,*

$$\sum_{u \in \partial B(r)} \mu_r(u)^2 > \frac{1}{r \log^2 r}.$$

Note that for the uniform measure, U_r, on $\partial B(r)$, $\sum_{u \in \partial B(r)} U_r(u)^2 = |\partial B(r)|^{-1}$.

Gady Kozma constructed a recurrent bounded degree planar graph (not a triangulation) for which harmonic measure on any minimal cutsets outside $B(r)$ for any r is supported on a set of size at least $r^{4/3}$, or even larger exponents. The example is very "irregular", it will be useful to come up with a natural general condition that will guarantee a linear support.

Proof. What is the probability SRW will reach maximal distance r once, before returning back to v? By summing all paths from v to $\partial B(r)$ and back to v visiting $\partial B(r)$ and v once, we get that up to a constant depending on the degree the answer is

$$\sum_{u \in \partial B(r)} \mu_r(u)^2.$$

But by our assumptions then,

$$\sum_r \sum_{u \in \partial B(r)} \mu_r(u)^2 = \infty.$$

Observe that the events "excursion to maximal distance n from the origin" are independent for different n's. ☐

We next consider planar triangulations. Rather than working in the context of abstract graph it is natural to circle pack them and use conformal geometry. Assume G is a bounded degree recurrent infinite planar triangulation. By He and Schramm [23], G admits a circle packing in the whole Euclidean plane. Fix a root for G.

Question: Is it the case that for arbitrarily large radii r, there are domains containing a ball of radius r around the root, so that harmonic measure on the domain boundary is supported on $r^{1+o(1)}$ circles?

By *supported* we mean $1 - o(1)$ of the measure is supported on the set. Here is a possible approach: Consider a huge ball in the infinite recurrent triangulation. Circle pack the infinite recurrent triangulation in the whole plane [23]. Look at the Euclidean domain which is the image of this ball. Random walk on the triangulation will be close to SRW on hexagonal packing inside this domain. By the discrete adaptation of Makarov's theorem [27, 32], harmonic measure on the boundary circles will be supported on a linear number of hexagonal circles. How can we see that no more original circles are needed for some of the domains, using recurrence? Note that for hyperbolic triangulations this is not the case.

It might be the case that this is not true for general triangulation but further assuming unimodularity will do the job. In particular is it true for the UIPT?

4. Random hyperbolic lines

Following the Euclidean random graph and the conjecturally recurrent UIPT we move on to the hyperbolic plane.

In [8] it was shown that a.s. the components of the complement of a Poisson process on the space of hyperbolic geodesics in the hyperbolic plane are bounded iff the intensity of the process is bigger or equal one, when the hyperbolic plane is scaled to have -1 curvature. This sharp transition and rapid mining of the geodesic flow suggests that when removing from a compact hyperbolic surface the initial segment of a random geodesic, then the size of the largest component of the complement drops in a sharp transition from order the size of the surface to a logarithmic in the size of the surface. In the coming subsections we will discuss two different direction inspired by this poisson process of hyperbolic lines.

4.1. Vacant sets.

Random geodesics on an hyperbolic surface mix rapidly, this further suggests that the vacant set of non backtracking or even simple walk path on a "well connected" graph will also admit a sharp percolation-like transition. That is, the amount of randomness and independence in processes such as random walk on uniformly transient graphs are sufficient to create phase transitions usually seen in the context of independent percolation or other spin systems such as the random cluster, or Potts models. In [11] there are initial results towards understanding this phenomena.

Let G_n be a sequence of finite transitive graphs $|G_n| \to \infty$ which are uniformly transient (that is, when viewing the edges as one Ohm conductors the electric resistance between any pair of vertices in any of the G_n's is uniformly bounded).

Conjecture 4.1. *Show that the size of the largest vacant component of simple random walk on G_n's drops from order $|G_n|$ to $o(|G_n|)$ after less than $C|G_n|$ steps, for some $C < \infty$ fixed and in an interval of width $o(|G_n|)$.*

Note that the n^d-Euclidean grid tori satisfies the assumption when $d > 2$. With Ariel Yadin [12] we established the case of large girth expanders. In the proof we needed a special case of the following *conjecture* which is still open. The probability to cover a graph by SRW in order size steps is exponentially small. Formally, for any $C < \infty$ there is $c < 1$, so that for any graph G of size n and no double edges, the probability Simple Random Walk covers G in Cn steps is smaller than c^n.

4.2. **Long range percolation.** Consider this Poisson line process (from [8]) with intensity λ on the upper half plane model for the hyperbolic plane. For each pair $x, y \in \mathbb{Z}$, let there be an edge between x and y (independently for different pairs) iff there is a line in the line process with one endpoint in $[x, x+1]$ and the other in $[y, y+1]$. Then a calculation shows that the probability that there is an edge between x and y is asymptotic to $\lambda/|x-y|^2$ as $|x-y| \to \infty$. We just recovered the standard *long range percolation* model on $\mathbb{Z}$ with critical exponent 2 (see [4]). The critical case of long range percolation is not well understood. The fact that it is a discretization of the Möbius invariant process hopefully will be useful and already indicates that the process is somewhat natural.

Here is a direct formulation. Start with the one dimensional grid $\mathbb{Z}$ with the nearest neighbor edges, add to it additional edges as follows. Between, i and j add an edge with probability $\beta|i-j|^{-2}$, independently for each pair. The main open problem is how does the distance between 0 and n grow typically in this random graph? The answer is believed to be of the form $\theta(n^{f(\beta)})$, where f is strictly between 0 and 1 and is strictly monotone in β. When -2 is replaced by another exponent the answers are known, see [4, 14].

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