PERCOLATION AND COARSE CONFORMAL UNIFORMIZATION

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Abstract. We formulate conjectures regarding percolation on planar triangulations suggested by assuming (quasi) invariance under coarse conformal uniformization.

1. Introduction

We start with a quick panoramic overview. A *conformal map*, between planar domains, is a function that infinitesimally preserves angles. The derivative of a conformal map is everywhere a scalar times a rotation.

Riemann's mapping theorem states that any open simply connected domain of the Euclidean plane admits a bijective conformal map to the open unit disk. In the 1940's Shizuo Kakutani observed that two dimensional Brownian motion is conformal invariant, up to a time reparametrization. Therefore the scaling limit of simple random walks on the Euclidean grid is conformal invariant.

In 2000 Stas Smirnov [5] proved that the scaling limit of critical Bernoulli site percolation on the triangular lattice is conformal invariant.

Poincaré (1907) proved that every simply connected Riemann surface is conformally equivalent to one of the following three surfaces: the open unit disk, the Euclidean plane, or the Riemann sphere. In particular it admits a Riemannian metric of constant curvature. This classifies Riemannian surfaces as elliptic (the shpere), parabolic (Euclidean), and hyperbolic (negatively curved).

The uniformization theorem is a generalization of the Riemann mapping theorem from proper simply connected open subsets of the plane to arbitrary simply connected Riemannian surfaces.

Conformal invariance of Brownian motion extends to the context of the uniformization. A simply connected Riemann surface is conformally equivalent to the hyperbolic plane iff the Brownian motion is transient.

How does surface uniformiztion manifest itself in the context of percolation?

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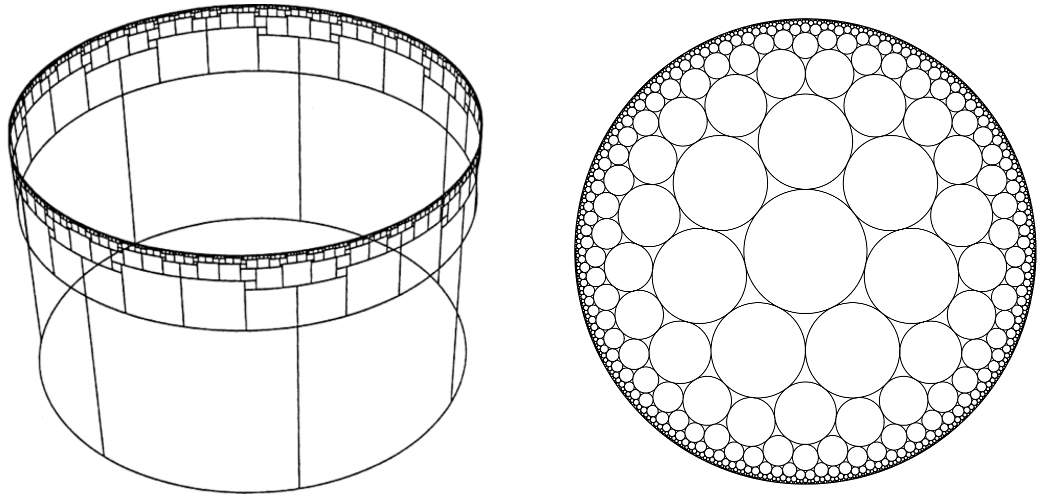

FIGURE 1. The square tiling and the circle packing of the 7-regular triangulation.

Below we suggest that the discrete setup of planar triangulations is natural for this problem.

Recall, every planar graph admits a circle packing, Koebe (1936).

In 1995 He and Schramm [4] proved a discrete uniformization theorem for triangulations: Let G be the 1-skeleton of a triangulation of an open disk. If the random walk on G is *recurrent*, then G is circled packed in the Euclidean plane. Conversely, if the degrees of the vertices in G are bounded and the random walk on G is *transient*, then G is circle packed in the unit disc.

For an extended version of the Brooks, smith, Stone and Tutte (1940) square tiling theorem, a related discrete uniformization theorem for graphs using squares, see [1].

In p-Bernoulli site percolation, each vertex is declared open independently with probability p, and clusters are connected components of open vertices. A graph is transient if simple random on the graph returns to the origin finitely many times almost surely and otherwise is called recurrent.

2. Two conjectures

Let G be the 1-skeleton of a bounded degree triangulation of an open disk.

Conjecture 2.1. *Assume G is transient, then $1/2$-Bernoulli site percolation on G admits an infinite cluster a.s.*

We don't know it even for any fixed $p > 1/2$.

The motivation for the conjecture is outlined below and is based on conformal invariance of percolation. After more than two decades of thorough research, conformal invariance of critical Bernoulli percolation was established essentially only for the triangular lattice [5].

One reason to be slightly skeptical about the conjecture is that for critical percolation on the triangular lattice, the probability the cluster of the origin reaches distance r decays polynomially in r [6], while there are transient triangulations of volume growth $r^2 \log^3 r$.

A heuristic

Tile the unit square with (possibly infinity number) of squares of varying sizes so that at most three squares meet at corners. Color each square black or white with equal probability independently.

Conjecture 2.2. *Show that there is a universal $c > 0$, so that the probability of a black left right crossing is bigger than c.*

If true, the same should hold for a tiling, or a packing of a triangulation, with a set of shapes that are of bounded Hausdorff distance to circles. At the moment we don't have a proof of the conjecture even when the squares are colored black with probability 2/3.

Behind the second conjecture is a coarse version of conformal invariance. That is, the crossing probability is balanced if the tiles are of uniformly bounded distance to circles (rotation invariance), and the squares can be of different sizes, (dilation invariance).

Let G the 1-skeleton of bounded degree transient a triangulation of an open disk. By [4] it admits a circle packing with similar properties as the tiling in in conjecture 2.2. And if the conformal invariance heuristic holds, we will a.s. see macroscopic crossings for 1/2-Bernoulli percolation.

We *believe* that $p_c \geq 1/2$ for polynomial growth triangulations of the open disk. Note that if all degrees are at least 6, polynomial growth implies that vertices of higher degrees are polynomially sparse, this suggests that their critical probability for percolation is 1/2, as of the triangular lattice. For nonamenable transitive or sofic triangulations $p_c < 1/2$ [3], remove the transitivity assumption.

What about a converse to conjecture 1.1?

Does recurrence implies no percolation at 1/2?

By same reasoning we will see unboundely many macroscopic clusters for 1/2-Bernoulli percolation, suggesting that if G is a 1-skeleton of bounded degree transient a triangulation of an open disk, then there are a.s. infinitely many infinite clusters for 1/2-Bernoulli site percolation?

Since we believe that $p_c > 0$ for such G's, by planar duality we conjecture that $p_u < 1$ and *uniqueness monotonicity* holds as well. Where p_u is the threshold for uniqueness of the infinite cluster.

3. Conformal invariance and hyperbolicity

Consider the Poincare disc model of the hyperbolic plane. Pick four points a, b, c, d on the circle at infinity, dividing the circle to four intervals, A, B, C, D.

What is the probability that when placing λ-intensity Poisson process in the disc, with respect to the hyperbolic metric, and coloring each Voronoi cell black or white independently with equal probability, there is a black crossing between intervals A and C on the boundary?

Since this process is invariant with respect to hyperbolic isometries, we get that this probability is a function of the crossratio of a, b, c, d and λ. There is no scale invariance for the Poisson process on the hyperbolic plane and increasing λ corresponds to the curvature approaching 0.

Fix the boundary intervals. It is reasonable to conjecture that the (annealed) crossing probabilities converge as λ increases to infinity. In particular they converge along a subsequence. We get that the subsequential limit is conformal invariant. The limit is Euclidean and should be given by Cardy's formula [5]. The argument above gives the conformal invariance of a subsequential limit.

The point is that conformal invariance of the subsequential limit follows from the hyperbolicity. Note that in [3], it is shown that $p_c(\lambda) < 1/2$ for any λ and suggested that this can be used to show that $p_c \leq 1/2$ in Euclidean Voronoi percolation. In [2], together with Oded Schramm we conjectured that changing the uniform measure in the disc (the measure used in sampling the Poisson points) in a uniformly absolutely continuous way, should not effect crossing probability, as the intensity grows and showed that conformal change of the metric do not effect crossing probabilities. Here we observe that when placing an infinite measure and unbounded metric, so that as the intensity grows the local tiling geometry also converges to that of the high intensity Euclidean, conformality follows via hyperbolicity.

As in conjecture 2.2 we want: show that the limiting of the crossing probabilities are bounded away from 0 and 1 for any non trivial intervals A and C?

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