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DISTINGUISHING SCENERIES BY OBSERVING THE SCENERY ALONG A RANDOM WALK PATH

Itai Benjamini and Harry Kesten*

Abstract. Let $\mathcal{G}$ be an infinite connected graph with vertex set $\mathcal{V}$. A *scenery* on $\mathcal{G}$ is a map $\xi : \mathcal{V} \to \{0,1\}$ (equivalently, an assignment of zeroes and ones to the vertices of $\mathcal{G}$). Let $\{S_n\}_{n\geq 0}$ be a simple random walk on $\mathcal{G}$, starting at some distinguished vertex v_0. Now let ξ and η be two *known* sceneries and assume that we observe one of the two sequences $\{\xi(S_n)\}_{n\geq 0}$ or $\{\eta(S_n)\}_{n\geq 0}$, but we do not know which of the two sequences is observed. Can we decide, with a zero probability of error, which of the two sequences is observed? We show that if $\mathcal{G} = \mathbb{Z}$ or $\mathcal{G} = \mathbb{Z}^2$, then the answer is "yes" for each fixed ξ and "almost all" η. We also give some examples of graphs $\mathcal{G}$ for which almost all pairs (ξ, η) are not distinguishable, and discuss some variants of this problem.

1. Introduction. Let $\mathcal{G}$ be an infinite connected graph with vertex set $\mathcal{V}$. Let v_0 be a fixed vertex. Throughout we assume that all vertices have finite degree. We define $\{S_n\}_{n\geq 0}$ as the Markov chain on $\mathcal{V}$ which starts at $S_0 = v_0$ and for which

$$P\{S_{n+1} = w \mid S_0, \dots, S_n\} = (\text{degree of } S_n)^{-1}$$

if w is adjacent to S_n, and $= 0$ otherwise. $\{S_n\}$ is called a *simple random walk* on $\mathcal{G}$.

A *scenery* ξ on $\mathcal{G}$ is a map from $\mathcal{V}$ to $\{0,1\}^{\mathcal{V}}$. We now give a precise statement of the question raised in the abstract. To explain why we chose the formulation below, we observe that many pairs of sceneries (ξ, η) can be distinguished by a few observations, with a zero probability of error. For instance if $\xi(v_0) \neq \eta(v_0)$, then the first of our observations already tells us whether it comes from ξ or from η. Similarly if $\xi(v) = 1$ and $\eta(v) = 0$ for all neighbors v of v_0, then the second observation determines which scenery is being used. The following definitions are chosen to exclude the distinguishing of two sceneries by such trivial means. Let $\mathcal{X} = \{0,1\}^{\mathbb{N}}$ and denote a generic point of $\mathcal{X}$ by $x = (x_0, x_1, \dots)$. Also write $\Xi = \{0,1\}^{\mathcal{V}}$. The sceneries are elements of Ξ and the sequence of observations is an element of $\mathcal{X}$. We denote by $\mathcal{F}$ and $\mathcal{F}_\ell^m$ the σ–fields in $\mathcal{X}$ generated by all coordinate functions x_k, and by the x_k with $\ell \leq k \leq m$, respectively. For given $\xi \in \Xi$, denote by Q^ξ the measure induced on $\mathcal{F}$ by $\{\xi(S_n)\}_{n\geq 0}$ (i.e., Q^ξ is the

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distribution of $\{\xi(S_n)\}_{n\geq 0}$). Similarly Q^ξ_ℓ denotes the restriction of Q^ξ to $\mathcal{F}^\infty_\ell$ (i.e., the distribution of $\{\xi(S_n)\}_{n\geq \ell}$). We call two sceneries ξ and η *distinguishable* if Q^ξ_ℓ and Q^η_ℓ are mutually singular for all ℓ. (Thus we want to distinguish ξ and η by using only "asymptotic information"; this term was introduced by Howard (1995c), but his formal definition of distinguishability differs from ours.)

¿From the arguments of Howard (1995c), Lemma 2.1, it is easy to see that if two sceneries are distinguishable for one choice of the starting vertex then they are distinguishable for every choice of v_0. This is so, because the random walk can move from a given v_0 to any other vertex. (This comment does not apply to situations where orientation plays a role, such as in Theorem 2.) When dealing with $\mathcal{G} = \mathbb{Z}^d$ we shall always take $v_0 = \mathbf{0} =$ the origin.

Our problem then is to determine which pairs (ξ, η) are distinguishable. This problem is of course reminiscent of the simplest statistics problem of distinguishing two distribution functions by an infinite sequence of independent observations with one of these distribution functions. Unfortunately, in our problem the dependence of the observations on the basic data (i.e., the scenery) is rather messy, and this makes the problem difficult. The problem of distinguishing two sceneries was raised independently by one of us (I.B.) and by den Hollander and Keane, some 10 years ago. Part of that motivation was the similarity between this problem and some ergodic theory problems. Further motivation came from investigations on a variety of aspects of $\{\xi(S_n)\}_{n\geq 0}$ when ξ is also random (see for instance Kesten and Spitzer (1979), Keane and den Hollander (1986), den Hollander (1988)). For the case $\mathcal{G} = \mathbb{Z}$, it was conjectured that ξ and η are distinguishable whenever they cannot be obtained from one another by a combination of a reflection and an even translation, that is when there is no even $k \in \mathbb{Z}$ such that $\xi(k+n) = \eta(n)$ for all n or $\xi(k+n) = \eta(-n)$ for all n. (See Howard (1995c) for the necessity of excluding such pairs (ξ, η).) We now have doubts about the validity of this conjecture, but it was the motivation for our first theorem which shows that for $\mathcal{G} = \mathbb{Z}$ or $\mathbb{Z}^2$ "most pairs" are distinguishable.

Theorem 1. *Let ρ be the product measure on Ξ given by*

$$\rho = \prod_{v\in\mathcal{V}} \rho_v \tag{1.1}$$

with

$$\rho_v(\{0\}) = \rho_v(\{1\}) = \frac{1}{2}. \tag{1.2}$$

Let $\eta \in \Xi$ be chosen according to this measure, so that the $\eta(v)$, $v \in \mathcal{V}$, are i.i.d. Then for $\mathcal{G} = \mathbb{Z}$ or $\mathbb{Z}^2$ and any fixed scenery $\xi \in \Xi$, ξ is distinguishable from almost all $\eta \in \Xi$ $[\rho]$.

The answer to the following question may give us good insight into which pairs (ξ, η) are distinguishable on $\mathcal{G} = \mathbb{Z}$.

Problem. *Is it true that for $\mathcal{G} = \mathbb{Z}$ and almost all η $[\rho]$, η is distinguishable from the scenery $\widehat{\eta}$ obtained by flipping η only at 0? (That is $\widehat{\eta}(k) = \eta(k)$ for $k \neq 0$ and $\widehat{\eta}(0) = 1 - \eta(0)$.)*

Howard (1995b) proves that η and $\widehat{\eta}$ are distinguishable when η is a periodic scenery on $\mathbb{Z}$.

One intuitively expects that the "more transient" $\{S_n\}$ is, the harder it is to obtain information about ξ from $\{\xi(S_n)\}_{n\geq 0}$. We therefore make the following conjecture.

Conjecture. *Let $\mathcal{G} = \mathbb{Z}^d$ and let ξ and η be independent sceneries, both chosen according to the measure ρ of (1.1), (1.2). Then for sufficiently large d, almost all pairs (ξ, η) $[\rho \times \rho]$ are not distinguishable.*

We have made no progress towards this conjecture, but we do have such a result on the oriented graph $(\mathbb{Z}_+)^d$ for d large enough. In this situation the random walk is only allowed to move in the "positive" direction, that is $\{S_n\}_{n\geq 0}$ is replaced by the Markov chain $\{S_n^{or}\}_{n\geq 0}$, which starts at $\mathbf{0}$ and has the transition probabilities

$$P\{S_{n+1}^{or} = S_n^{or} + e_i \mid S_0^{or}, \dots, S_n^{or}\} = \frac{1}{d}, \ i = 1, \dots, d, \tag{1.3}$$

where e_i is the i–th coordinate vector. Distinguishability in this situation is defined by merely replacing $\{S_n\}$ by $\{S_n^{or}\}$ in the preceeding definition. The measure ρ is still defined by (1.1), (1.2).

Theorem 2. *Let $\mathcal{G} = (\mathbb{Z}_+)^d$ and replace $\{S_n\}$ by $\{S_n^{or}\}$. Let ξ and η be two independent sceneries, both chosen according to the measure ρ. Then for $d \geq$ some d_0 almost all pairs (ξ, η) $[\rho \times \rho]$ are not distinguishable.*

In the following theorems we again take $\{S_n\}$ unoriented. A result similar to Theorem 2 is valid on regular trees. We denote by $\mathbb{T}_b$ and $\mathbb{T}_{b,r}$ the regular b–ary tree and rooted regular b–ary tree. In the former all vertices have degree $b+1$, while in the latter all vertices but the root have degree $b+1$, and the root has degree b. We shall skip the proof of the following result.

Theorem 3. *There exists a b_0 such that for $\mathcal{G} = \mathbb{T}_b$ or $\mathbb{T}_{b,r}$ with $b \geq b_0$, almost all pairs (ξ, η) $[\rho \times \rho]$ are not distinguishable.*

We end with a variant of our problem which shows that even on $\mathbb{Z}^d$ with d large, almost all pairs of sceneries are distinguishable if a little more information is made available. That is, instead of only observing the scenery on the random walk path, we now observe the scenery in some "sausage" around the random walk path. Specifically, for a fixed k, and a given scenery ξ, let

$$X_n(\xi, k) = \{\xi(S_n + w) : \|w\|_1 \leq k\}, \tag{1.4}$$

where $\|w\|_1 = \sum_1^d |w(i)|$ for $w = (w(1), \dots, w(d)) \in \mathbb{Z}^d$. Let $Q_\ell^{\xi,k}$ be the distribution of $\{X_n(\xi, k)\}_{n\geq \ell}$ on

$$\mathcal{X}^{(k)} = \Big(\prod_{i=1}^d [-k, k]\Big)^{\mathbb{N}}.$$

Call ξ and η k*–distinguishable* if $Q_\ell^{\xi,k}$ and $Q_\ell^{\eta,k}$ are mutually singular for every ℓ.

Theorem 4. *Let $\mathcal{G} = \mathbb{Z}^d$. Then there exists a $k_0 = k_0(d)$ such that for $k \geq k_0$ and each fixed $\xi \in \Xi$, ξ and η are k–distingusihable for almost all $\eta \in \Xi$ $[\rho]$ (ρ is given by (1.1), (1.2)).*

A slightly easier version is obtained by observing only $\{\xi(S_n)\}_{n\geq 0}$ as before, but by taking the scenery from $\Xi^{(k)} := \{0, \dots, k-1\}^{\mathbb{N}}$, that is by assigning to each vertex an integer $j \in [0, k-1]$ instead of only in $\{0, 1\}$. In this case ρ should be replaced by $\rho^{(k)} := \prod_{v \in \mathcal{V}} \rho_v^{(k)}$ with $\rho_v^{(k)}(\{j\}) = 1/k$ for $j = 0, 1, \dots, k-1$. Again for $k \geq$ some $k_1(d)$, almost all pairs $(\xi, \eta) \in \Xi^{(k)} \times \Xi^{(k)}$ $[\rho^{(k)} \times \rho^{(k)}]$ are distinguishable (in the obvious definition).

We conclude this introduction with some references and comments about notation. For $\mathcal{G} = \mathbb{Z}$, Kahane, Peyrière, Wen and Wu (1988) used Fourier analytic methods to prove a result from which one can conclude the distinguishability of two periodic sceneries which cannot be obtained from each other by a translation and/or a reflection. Howard (1995b) also treats periodic sceneries, while Howard (1995a) shows that one can distinguish two sceneries ξ and η when $\xi(v) = 0$ for all but finitely many v, unless η is obtainable from ξ by a translation and/or a reflection. Matthew Harris (1995) considers singularity of the distributions of the observations for two sceneries which also vary randomly in time in a way which depends on the (unobservable) random walk path. Finally we mention Burdzy (1993) who solves a problem of the same flavor as ours, even though his methods seem inapplicable in our situation. Burdzy shows that if $\{B_1(t)\}_{-\infty<t<\infty}$ and $\{B_2(t)\}_{t\geq 0}$ are two independent Brownian motions then one can reconstruct B_1 and B_2 (up to a sign) almost surely from observing $\{B_1(B_2(t))\}_{t\geq 0}$.

Notation. With the exception of the proof of Lemma 1, P is used for the probability measure which governs $\{S_n\}_{n\geq 0}$ only. The measure on Ξ for the sceneries is ρ. C_i, $i \geq 1$, will denote various *strictly positive and finite* constants, whose precise value is of no importance to us and which may change from appearance to appearance. $I(E)$ denotes the indicator function of E, and $\mathbf{0}$ is the origin in $\mathbb{Z}^d$ or $(\mathbb{Z}_+)^d$.

2. Random walk on $\mathbb{Z}$ and on $\mathbb{Z}^2$. In this section we prove Theorem 1. When the graph $\mathcal{G}$ is $\mathbb{Z}$ or $\mathbb{Z}^2$ two sceneries can be distinguished by observations of

$$W_n(\xi) = \sum_0^n \xi(S_n) \tag{2.1}$$

on a sparse sequence of n's only. We take the sequence of times n at which one observes W_n thin enough to make successive observations almost independent. On the other hand, sufficient observations must be taken to obtain enough information about the scenery. Our first lemma provides us with a sufficient condition for two measures to be mutually singular; this will guarantee that we can gain enough information from observing W_n along a suitable subsequence. The lemma is a variant of Kakutani's dichotomy (see Durrett (1991), Sect. 4.3), and most likely it is not new. However, we have no reference. The set–up is as follows: $(\Omega, \mathcal{B})$ is some measurable space and $E_n \in \mathcal{B}$, $n \geq 1$.

$$\mathcal{B}_\ell^m = \sigma\text{–field generated by } \{E_k : \ell \leq k \leq m\}$$

(if $m < \ell$, then $\mathcal{B}_\ell^m$ is the trival σ–field).

Lemma 1. *Assume that μ_1 and μ_2 are two probability measures on $\mathcal{B}$ and that $\{\alpha_n\}_{n\geq 1}$, $\{\beta_n\}_{n\geq 1}$ are two sequences of numbers of $[0,1]$ such that*

$$\sum_1^\infty |\mu_1\{E_n \mid \mathcal{B}_1^{n-1}\} - \alpha_n| < \infty \text{ a.e. } [\mu_1], \tag{2.2}$$

$$\sum_1^\infty |\mu_2\{E_n \mid \mathcal{B}_1^{n-1}\} - \beta_n| < \infty \text{ a.e. } [\mu_2], \tag{2.3}$$

$$\sum_1^\infty \frac{|\alpha_n - \beta_n|^2}{(\alpha_n \wedge (1-\alpha_n)) + (\beta_n \wedge (1-\beta_n))} = \infty. \tag{2.4}$$

Then the restrictions of μ_1 and μ_2 to $\mathcal{B}_\ell^\infty$ are mutually singular, for each ℓ. (The summand in (2.4) is interpreted as 0 if $\alpha_n = \beta_n$.)

Proof: We define the probability measure ν_i on $\mathcal{B}_1^\infty$ as the measure under which $E_1, E_2, \dots$ are independent with

$$\nu_i(E_k) = \begin{cases} \alpha_k & \text{if} \quad i = 1, \\ \beta_k & \text{if} \quad i = 2. \end{cases} \tag{2.5}$$

First we show that the restrictions of ν_1 and ν_2 to $\mathcal{B}_\ell^\infty$ are mutually singular for each ℓ. The argument for this is the same for each ℓ, so we restrict ourselves to $\ell = 1$. Then by Kakutani's dichotomy, ν_1 and ν_2 are mutually singular if and only if

$$\sum_1^\infty \left[1 - (\alpha_n\beta_n)^{1/2} - (1-\alpha_n)^{1/2}(1-\beta_n)^{1/2}\right] = \infty \tag{2.6}$$

(see Durrett (1991), Theorem 4.3.6). We therefore only have to show that (2.4) implies (2.6). In fact, we shall prove that there exists a constant $C_1 < \infty$ such that for all $0 \leq x, y \leq 1$

$$1 \wedge \frac{|x-y|^2}{(x \wedge (1-x)) + (y \wedge (1-y))} \leq C_1\left[1 - (xy)^{1/2} - (1-x)^{1/2}(1-y)^{1/2}\right]. \tag{2.7}$$

It is clear from (2.7) that (2.4) does imply (2.6).

To prove (2.7) we note that by the geometric–arithmetic mean inequality

$$(1-x)^{1/2}(1-y)^{1/2} \leq \frac{1}{2}(1-x+1-y),$$

so that it suffices to prove that

$$1 \wedge \frac{(\sqrt{x}-\sqrt{y})^2(\sqrt{x}+\sqrt{y})^2}{(x \wedge (1-x)) + (y \wedge (1-y))} \leq \frac{C_1}{2}(\sqrt{x}-\sqrt{y})^2. \tag{2.8}$$

It is clear that (2.8) holds if $|x-y| \geq \varepsilon$ for some fixed $\varepsilon > 0$, and we only have to verify (2.8) in some small neighborhood of the diagonal. Similarly (2.8) is clearly valid for $\varepsilon \leq x, y \leq 1-\varepsilon$, for fixed $\varepsilon > 0$. It follows that we only have to check that

(2.8) holds as $x, y \to 0$ and as $x, y \to 1$. Both these cases can be handled in the same way, so without loss of generality we take $0 < x \le y$ and let $x, y \to 0$. But then (2.8) and (2.7) follow from

$$\limsup_{\substack{x \downarrow 0, y \downarrow 0 \\ 0 \le x \le y}} \frac{(\sqrt{x} + \sqrt{y})^2}{y} \le \limsup_{y \downarrow 0} \frac{(2\sqrt{y})^2}{y} = 4.$$

We have now established that (2.4) implies (2.6), and hence that the restrictions of ν_1 and ν_2 to any $\mathcal{B}_\ell^\infty$ are mutually singular. In particular, there exist events $A_\ell \in \mathcal{B}_\ell^\infty$ such that

$$\nu_1(A_\ell) = 0, \quad \nu_2(A_\ell) = 1, \quad \ell \ge 1.$$

To complete the proof we shall show that

$$|\mu_i(A_\ell) - \nu_i(A_\ell)| \to 0, \ (\ell \to \infty, \ i = 1, 2). \tag{2.9}$$

This will indeed prove the lemma, for (2.9) implies that there exists a subsequence $\ell_1 < \ell_2 < \cdots$ with

$$|\mu_i(A_{\ell_j}) - \nu_i(A_{\ell_j})| \le \frac{1}{j^2}, \quad j \ge 1, \ i = 1, 2$$

and then, by the Borel–Cantelli lemma,

$$\mu_1\Big(\liminf_{j \to \infty} A_{\ell_j}\Big) = 0, \quad \mu_2\Big(\liminf_{j \to \infty} A_{\ell_j}\Big) = 1.$$

We next prove (2.9), for $i = 1$ say, by coupling μ_1 and ν_1. More precisely we construct a probability space $(S, \mathcal{S}, P)$ with two sequences of random variables $\{I_n\}_{n \ge 1}$, $\{\widetilde{I}_n\}_{n \ge 1}$ such that $\{I_n\}_{n \ge 1}$ and $\{\widetilde{I}_n\}_{n \ge 1}$ have the distribution of $\{I(E_n)\}_{n \ge 1}$ under ν_1 and μ_1, respectively, and such that

$$P\{I_n \ne \widetilde{I}_n \text{ infinitely often}\} = 0. \tag{2.10}$$

Then, since $A_\ell \in \mathcal{F}_\ell^\infty$, $A_\ell = \{\{I(E_n)\}_{n \ge \ell} \in B_\ell\}$ for some Borel set B_ℓ in $\{0,1\}^{\mathbb{N}}$, and consequently

$$\begin{aligned} |\mu_1(A_\ell) - \nu_1(A_\ell)| &= |P\{\{\widetilde{I}_n\}_{n \ge \ell} \in B_\ell\} - P\{\{I_n\}_{n \ge \ell} \in B_\ell\}| \\ &\le P\{\widetilde{I}_n \ne I_n \text{ for some } n \ge \ell\} \to 0 \quad (\ell \to \infty). \end{aligned}$$

The construction of the required $(S, \mathcal{S}, P)$ and I_n, $\widetilde{I}_n$ is rather easy from (2.2). We take $S = \prod_1^\infty \{0,1\}^2$ and $\mathcal{S}$ the σ–field generated by the cylinder sets. For $s = \{(s_n, \widetilde{s}_n)\}_{n \ge 1}$ a general point of S, we define $I_n(s) = s_n$, $\widetilde{I}_n(s) = \widetilde{s}_n$. Finally the measure P is chosen as follows. Let t_j, $\widetilde{t}_j \in \{0,1\}$ for $1 \le j \le n$. Then, if

$$\mu_1\{E_n \mid I(E_j) = \widetilde{t}_j, \ 1 \le j < n\} \ge \alpha_n,$$

we take

$$P\{(s_n,\widetilde{s}_n)=(t_n,\widetilde{t}_n)\mid(s_j,\widetilde{s}_j)=(t_j,\widetilde{t}_j),1\le j<n\}$$
$$=\begin{cases}\alpha_n & \text{if}\quad (t_n,\widetilde{t}_n)=(1,1),\\ \mu_1\{E_n\mid I(E_j)=\widetilde{t}_j,1\le j<n\}-\alpha_n & \text{if}\quad (t_n,\widetilde{t}_j)=(0,1),\\ 1-\mu_1\{E_n\mid I(E_j)=\widetilde{t}_j,1\le j<n\} & \text{if}\quad (t_n,\widetilde{t}_n)=(0,0).\end{cases}$$

If, on the other hand,

$$\mu_1\{E_n\mid I(E_j)=\widetilde{t}_j,1\le j<n\}<\alpha_n,$$

then we take

$$P\{(s_n,\widetilde{s}_n)=(t_n,\widetilde{t}_n)\mid(s_j,\widetilde{s}_j)=(t_j,\widetilde{t}_j),1\le j<n\}$$
$$=\begin{cases}\mu_1\{E_n\mid I(E_j)=\widetilde{t}_j,1\le j<n\} & \text{if}\quad (t_n,\widetilde{t}_n)=(1,1),\\ \alpha_n-\mu_1\{E_n\mid I(E_j)=\widetilde{t}_j,1\le j<n\} & \text{if}\quad (t_n,\widetilde{t}_n)=(1,0),\\ 1-\alpha_n & \text{if}\quad (t_n,\widetilde{t}_n)=(0,0).\end{cases}$$

It is easy to see that

$$\begin{aligned}&P\{s_n=1\mid(s_j,\widetilde{s}_j)=(t_j,\widetilde{t}_j),1\le j<n\}=\alpha_n,\\ &P\{\widetilde{s}_n=1\mid(s_j,\widetilde{s}_j)=(t_j,\widetilde{t}_j),1\le j<n\}\\ &\qquad\qquad=\mu_1\{E_n\mid I(E_j)=\widetilde{t}_j,1\le j<n\},\end{aligned}$$

so that $\{s_n\}_{n\ge1}=\{I_n(s)\}_{n\ge1}$ and $\{\widetilde{s}_n\}_{n\ge1}=\{\widetilde{I}_n(s)\}_{n\ge1}$ indeed have the distributions of $\{I(E_n)\}$, under ν_1 and μ_1, respectively, as promised. Finally,

$$P\{I_n\ne\widetilde{I}_n\mid(s_j,\widetilde{s}_j)=(t_j,\widetilde{t}_j),1\le j<n\}=|\mu_1\{E_n\mid I(E_j)=\widetilde{t}_j,1\le j<n\}-\alpha_n|,$$

and by (2.2),

$$\sum|\mu_1\{E_n\mid I(E_j)=\widetilde{t}_j,1\le j<n\}-\alpha_n|<\infty$$

outside a set of sequences $\{\widetilde{t}_j\}$ of μ_1–measure zero, or equivalently, of P–measure zero. Thus, by virtue of Lévy's generalization of the Borel–Cantelli lemma (see Durrett (1991), Corollary 4.3.2) also (2.10) holds, and P has all the desired properties. ■

We shall apply this lemma with

$$\Omega=\prod_1^\infty\mathbb{R}\tag{2.11}$$

and $\mathcal{B}$ its Borel field. If $\omega=(\omega_1,\omega_2,\dots)$ denotes a generic point of Ω, then μ_1 and μ_2 will be the measures under which $\{\omega_n\}$ has the same distribution as $\{W_n(\xi)\}$ and $\{W_n(\eta)\}$, respectively. For E_k we take the event

$$E_k=\{\omega:\omega_{r_k}-\omega_{s_k}\le x_k\}\tag{2.12}$$

with

$$r_k = [k!]^8, \quad s_k = k^4[(k-1)!]^8, \tag{2.13}$$

and some x_k, which will depend on ξ, η, and which will be determined below. We note that

$$\begin{aligned} \mu_1(E_k) &= P\{W_{r_k}(\xi) - W_{s_k}(\xi) \le x_k\}, \\ \mu_2(E_k) &= P\{W_{r_k}(\eta) - W_{s_k}(\eta) \le x_k\}. \end{aligned} \tag{2.14}$$

We take

$$\begin{aligned} \alpha_k = \alpha_k(\xi, x_k) = \mu_1(E_k) &= P\{W_{r_k}(\xi) - W_{s_k}(\xi) \le x_k\}, \\ \beta_k = \beta_k(\eta, x_k) = \mu_2(E_k) &= P\{W_{r_k}(\eta) - W_{s_k}(\eta) \le x_k\}. \end{aligned} \tag{2.15}$$

Our task is to verify (2.2)–(2.4) for a suitable choice of the x_k. The basic tool for showing that $|\alpha_k - \beta_k|$ is not too small will be the following simple estimate which is valid for any two random variables U_1 and U_2.

Lemma 2. *Assume that $E|U^{(i)}| < \infty$, $i = 1, 2$, and that*

$$EU_2 > EU_1. \tag{2.16}$$

Let Ψ_i be the distribution function of U_i. Then

$$\sup_y [\Psi_1(y) - \Psi_2(y)] \ge \frac{1}{4} \, \frac{(EU_2 - EU_1)^2}{E[U_1]^2 + E[U_2]^2}. \tag{2.17}$$

Moreover, for any $p > 2$, there exists a $C_2(p) > 0$ such that

$$\begin{aligned} &\sup_j \frac{\Psi_1(y) - \Psi_2(y)}{[(\Psi_1(y) \wedge (1 - \Psi_1(y)) + (\Psi_2(y) \wedge (1 - \Psi_2(y))]^{1/2}} \\ &\qquad \ge C_2(p) \frac{EU_2 - EU_1}{(E|U_1|^p + E|U_2|^p)^{1/p}}. \end{aligned} \tag{2.18}$$

Proof: Integration by parts shows that (for any $R \ge 0$)

$$\begin{aligned} EU_2 - EU_1 &= \int_{-\infty}^{+\infty} y d[\Psi_2(y) - \Psi_1(y)] \\ &= \int_{-\infty}^{+\infty} [\Psi_1(y) - \Psi_2(y)] dy \le 2R \sup_{|y| \le R} [\Psi_1(y) - \Psi_2(y)] \\ &\quad + \int_R^{\infty} [1 - \Psi_1(y) + 1 - \Psi_2(y)] dy + \int_{-\infty}^{-R} [\Psi_1(y) + \Psi_2(y)] dy \\ &\le 2R \sup_{|y| \le R} [\Psi_1(y) - \Psi_2(y)] + \frac{1}{R} \int_R^{\infty} y[1 - \Psi_1(y) + 1 - \Psi_2(y)] dy \\ &\quad - \frac{1}{R} \int_{-\infty}^{-R} y[\Psi_1(y) + \Psi_2(y)] dy \\ &\le 2R \sup_y [\Psi_1(y) - \Psi_2(y)] + \frac{1}{2R} (E[U_1]^2 + E[U_2]^2). \end{aligned} \tag{2.19}$$

(2.17) now follows by taking

$$R = \frac{E[U_1]^2 + E[U_2]^2}{EU_2 - EU_1}.$$

To prove (2.18), we again begin with

$$EU_2 - EU_1 = \int_{-\infty}^{+\infty} [\Psi_1(y) - \Psi_2(y)] dy. \tag{2.20}$$

Now we take

$$R = (E|U_1|^p + E|U_2|^p)^{1/p}. \tag{2.21}$$

If

$$\sup_{|y| \le R} [\Psi_1(y) - \Psi_2(y)] \ge \frac{1}{4R} [EU_2 - EU_1], \tag{2.22}$$

then (2.18) holds with $C_2(p) = 1/4\sqrt{2}$ (because the denominator of the left hand side of (2.18) is at most $\sqrt{2}$). We may therefore assume that (2.22) fails. Then (2.20) implies that

$$\int_{-\infty}^{-R} [\Psi_1(y) - \Psi_2(y)] dy \ge \frac{1}{4} [EU_2 - EU_1] \tag{2.23}$$

or

$$\int_{R}^{\infty} [\Psi_1(y) - \Psi_2(y)] dy \ge \frac{1}{4} [EU_2 - EU_1]. \tag{2.24}$$

For the sake of argument let (2.23) hold. Then we use that for $y < 0$

$$\Psi_i(y) \le \frac{1}{|y|^p} \int_{-\infty}^{y} |z|^p d\Psi_i(z) \le \frac{1}{|y|^p} E|U_i|^p.$$

If $-\overline{R} = \inf\{y : \Psi_1(y) + \Psi_2(y) > 0\}$, then

$$\begin{aligned}
\frac{1}{4}[EU_2 - EU_1] &\le \int_{-\overline{R} \le y \le -R} \frac{\Psi_1(y) - \Psi_2(y)}{[\Psi_1(y) + \Psi_2(y)]^{1/2}} [\Psi_1(y) + \Psi_2(y)]^{1/2} dy \\
&\le \sup_{y \le -R} \frac{\Psi_1(y) - \Psi_2(y)}{[\Psi_1(y) + \Psi_2(y)]^{1/2}} \Big\{ \int_{-\infty}^{-R} \Big(\frac{E|U_1|^p}{|y|^p}\Big)^{1/2} dy \\
&\qquad\qquad + \int_{-\infty}^{-R} \Big(\frac{E|U_2|^p}{|y|^p}\Big)^{1/2} dy \Big\} \\
&= \frac{2}{p-2} R^{1-p/2} \{(E|U_1|^p)^{1/2} + (E|U_2|^p)^{1/2}\} \sup_{y \le -R} \frac{\Psi_1(y) - \Psi_2(y)}{[\Psi_1(y) + \Psi_2(y)]^{1/2}}.
\end{aligned}$$

Thus, for some $C_2(p) > 0$,

$$\sup_{y \le -R} \frac{\Psi_1(y) - \Psi_2(y)}{[\Psi_1(y) + \Psi_2(y)]^{1/2}} \ge C_2(p) R^{p/2-1} \frac{EU_2 - EU_1}{(E|U_1|^p + E|U_2|^p)^{1/2}},$$

and (2.18) again follows by substituting the value (2.21) of R.

If (2.24) holds instead of (2.23), then we multiply and divide the integrand in the left hand side of (2.24) by $[1-\Psi_1(y)+1-\Psi_2(y)]^{1/2}$, and proceed as before. ■

As this lemma indicates, the strategy for proving a lower bound for $|\alpha_k(\xi, x_k) - \beta_k(\eta, x_k)|$ should be to find a lower bound for

$$|EW_{r_k}(\xi) - EW_{s_k}(\xi) - EW_{r_k}(\eta) + EW_{s_k}(\eta)|$$

and an upper bound for

$$E|W_{r_k}(\xi) - W_{s_k}(\xi)|^p + E|W_{r_k}(\eta) - W_{s_k}(\eta)|^p.$$

Unfortunately, when η is randomly chosen in Ξ according to the distribution ρ, then the random variables $W_{r_k}(\eta) - W_{s_k}(\eta)$ are dependent, and it is necessary to work with certain approximating random variables which are independent. To this end we define,

$$\ell(v,(s,r]) = \text{number of } s < j \le r \text{ with } S_j = v \ (v \in \mathcal{V}), \tag{2.25}$$

$$\begin{aligned} Y_k(\eta) &= \sum_{j=s_k+1}^{r_k} \Big[\eta(S_j) - \frac{1}{2}\Big] I[a_{k-1} < |S_j| \le a_k] \\ &= \sum_{a_{k-1}<|v|\le a_k} \Big[\eta(v) - \frac{1}{2}\Big] \ell(v,(s_k,r_k]), \end{aligned} \tag{2.26}$$

where

$$a_k = k[r_k]^{1/2} \sim k^{-3}[r_{k+1}]^{1/2}, \tag{2.27}$$

and

$$\begin{aligned} Z_k(\eta) &= \sum_{j=s_k+1}^{r_k} \Big[\eta(S_j) - \frac{1}{2}\Big] \\ &= \sum_v \Big[\eta(v) - \frac{1}{2}\Big] \ell(v,(s_k,r_k]) \\ &= W_{r_k}(\eta) - W_{s_k}(\eta) - \frac{1}{2}(r_k - s_k). \end{aligned} \tag{2.28}$$

We begin with proving that Y_k is a good approximation to Z_k. We remind the reader that P denotes the probability measure governing the random walk $\{S_n\}$ only; E denotes expectation with respect to P. η has distribution ρ (and $\rho(U)$ will also be used to denote the expectation of a random variable U with respect to ρ). Finally the joint distribution of η and $\{S_n\}$ is denoted by $\mathbb{P}$, and expectation with respect to this measure by $\mathbb{E}$. For $u = (u(1), u(2)) \in \mathbb{Z}^2$, $|u|$ denotes $(u^2(1) + u^2(2))^{1/2}$.

Lemma 3. *For almost all η $[\rho]$ the following estimates hold:*

$$P\{|Y_k(\eta) - Z_k(\eta)| > r_k^{3/4}\} \le \Big(\frac{1}{\log r_k}\Big)^2 \textit{ for all large } k, \textit{ in case } \mathcal{G} = \mathbb{Z}, \tag{2.29}$$

$$P\{|Y_k(\eta) - Z_k(\eta)| > r_k^{1/2}\} \le \Big(\frac{1}{\log r_k}\Big)^2 \textit{ for all large } k, \textit{ in case } \mathcal{G} = \mathbb{Z}^2. \tag{2.30}$$

Proof:

$$Y_k(\eta) - Z_k(\eta) = \Big\{ \sum_{|v| \le a_{k-1}} + \sum_{|v| > a_k} \Big\} \Big[\eta(v) - \frac{1}{2}\Big] \ell(v, (s_k, r_k]). \tag{2.31}$$

Now, for every η,

$$\begin{aligned} P\Big\{ \sum_{|v| > a_k} &\Big[\eta(v) - \frac{1}{2}\Big] \ell(v, (s_k, r_k]) \ne 0 \Big\} \\ &\le P\{\ell(v, (s_k, r_k]) \ne 0 \text{ for some } |v| > a_k\} \\ &\le P\Big\{ \max_{j \le r_k} |S_j| \ge a_k \Big\} \le C_3 \exp\Big(- C_4 \frac{a_k^2}{r_k} \Big) \\ &\le C_3 \exp(-C_4 k^2) \qquad \text{(by (2.27))} \end{aligned} \tag{2.32}$$

(the one but last inequality follows from the reflection principle (e.g., Durrett (1991), Theorem 5.2.5) and standard inequalities (Rényi (1970), Theorem VII.4.2). To estimate the contribution of the sum over v with $|v| \le a_{k-1}$ we need some facts about the moments of ℓ which are basically special cases of Lemma 1 in Kesten and Spitzer (1979). Let

$$\tau(v) = \inf\{n \ge 0 : S_n = v\}. \tag{2.33}$$

Then for $p \ge 0$

$$E\ell^p(v, (s, r]) = \sum_{k=0}^{r} P\{\tau(v) = k\} E\ell^p(0, (s-k, r-k]). \tag{2.34}$$

This gives the upper bound

$$\begin{aligned} E\ell^p(v, (s, r]) &\le P\{\tau(v) \le r\}[1 + E\ell^p(v, (0, r])] \\ &\le \begin{cases} C_5(p) P\{\tau(v) \le r\} r^{p/2} & \text{if } \quad \mathcal{G} = \mathbb{Z}, \\ C_5(p) P\{\tau(v) \le r\} (\log r)^p & \text{if } \quad \mathcal{G} = \mathbb{Z}^2. \end{cases} \end{aligned} \tag{2.35}$$

The last inequality is in Lemma 1 of Kesten and Spitzer (1979) if $\mathcal{G} = \mathbb{Z}$. For $\mathcal{G} = \mathbb{Z}^2$, we have

$$P\{S_{2n+1} = 0\} = 0, \; P\{S_{2n} = 0\} \sim \frac{1}{\pi n} \tag{2.36}$$

(see Révész (1990), Lemma 16.2). One can use Darling and Kac (1957), to deduce from this that for $\mathcal{G} = \mathbb{Z}^2$,

$$E\ell^p(0, (0, r]) \sim C_4(p)(\log r)^p,$$

which leads to (2.35); alternatively one can use Theorem 19.1 and its proof in Révész (1990).

We now use that for $p = 1, 2, \dots,$

$$P\Big\{\Big| \sum_{|v| \le a_{k-1}} \Big[\eta(v) - \frac{1}{2}\Big] \ell(v, (s_k, v_k])\Big| \ge \lambda\Big\}$$

$$\le \lambda^{-2p} \sum_{\substack{v_1, \dots, v_{2p} \\ \text{such that} \\ |v_i| \le a_{k-1}}} \prod_{i=1}^{2p} \Big[\eta(v_i) - \frac{1}{2}\Big] E \prod_{i=1}^{2p} \ell(v_i, (s_k, r_k]).$$

Consequently,

$$\rho\Big\{\eta : P\Big\{\Big| \sum_{|v| \le a_{k-1}} \Big[\eta(v) - \frac{1}{2}\Big] \ell(v, (s_k, r_k]\Big| \ge \lambda\Big\} \ge \sigma\Big\}$$

$$\le \frac{1}{\sigma \lambda^{2p}} \sum_{\substack{v_1, \dots, v_{2p} \\ \text{such that} \\ |v_i| \le a_{k-1}}} \rho\Big\{\prod_{i=1}^{2p} \Big[\eta(v_i) - \frac{1}{2}\Big]\Big\} E \prod_{i=1}^{2p} \ell(v_i, (s_k, r_k]). \tag{2.37}$$

Because the $\eta(v) - \frac{1}{2}$ are independent with mean 0, $\rho\big\{\prod[\eta(v_i) - \frac{1}{2}]\big\} = 0$, unless each v which occurs among the v_i occurs at least twice. Moreover,

$$E \prod_{i=1}^{2p} \ell(v_i, (s_k, r_k]) \le \prod_{i=1}^{2p} \{E \ell^{2p}(v_i, (s_k, r_k])\}^{1/(2p)}$$

$$\le E[1 + \ell(\mathbf{0}, (0, r_k])]^{2p} \le \begin{cases} C_5(2p) r_k^p & \text{if } \mathcal{G} = \mathbb{Z}, \\ C_5(2p)(\log r_k)^{2p} & \text{if } \mathcal{G} = \mathbb{Z}^2. \end{cases} \tag{2.38}$$

For the sake of argument take $\mathcal{G} = \mathbb{Z}^2$ for the remainder of the proof. Then we obtain from (2.37) with

$$\lambda = r_k^{1/2}, \quad \sigma = \frac{1}{2}\Big(\frac{1}{\log r_k}\Big)^2$$

that

$$\rho\Big\{\eta : P\Big\{\Big| \sum_{|v| \le a_{k-1}} \Big[\eta(v) - \frac{1}{2}\Big] \ell(v, (s_k, r_k])\Big| \ge r_k^{1/2}\Big\} \ge \frac{1}{2}\Big(\frac{1}{\log r_k}\Big)^2\Big\}$$

$$\le C_6 \frac{(\log r_k)^2 a_{k-1}^{2p}}{r_k^p} (\log r_k)^{2p}$$

$$\sim C_6 k^{-4p+2} (\log k)^{2p+2} \quad \text{(see (2.27) and (2.13))}.$$

Together with (2.31) and (2.32) this shows that

$$\sum_k \rho\Big\{\eta : P\{|Y_k(\eta) - Z_k(\eta)| \ge r_k^{1/2}\} \ge \Big(\frac{1}{\log r_k}\Big)^2\Big\} < \infty.$$

By the Borel–Cantelli lemma we therefore have (2.30) for almost all η $[\rho]$. A similar argument gives (2.29). ∎

Lemma 4. *Let ξ be fixed, and let $\alpha_k = \alpha_k(\xi, x_k)$ and $\beta_k = \beta_k(\eta, x_k)$ be defined by (2.15). Then for a suitable choice of x_k, (2.4) holds for almost all η $[\rho]$.*

Proof: For $\beta', \beta'' \in [0,1]$ introduce the abbreviation

$$d(\beta', \beta'') = 1 \wedge \frac{|\beta' - \beta''|}{[(\beta' \wedge (1-\beta')) + (\beta'' \wedge (1-\beta''))]^{1/2}}; \tag{2.39}$$

this is some kind of distance between β' and β''. We shall also use the following slight extension of definition (2.15):

$$\begin{aligned} \alpha_k(\xi, x) &= P\{W_{r_k}(\xi) - W_{s_k}(\xi) \le x\}, \\ \beta_k(\eta, x) &= P\{W_{r_k}(\eta) - W_{s_k}(\eta) \le x\}. \end{aligned} \tag{2.40}$$

Roughly speaking we shall prove the lemma by showing that there exist two sequences, $\{F_k\}$ and $\{G_k\}$ of subsets of Ξ, such that for all large k

$$\sup_{x \in \mathbb{R}} d(\beta_k(\eta', x), \beta_k(\eta'', x)) \ge C_7[\log r_k]^{-1/2} \text{ for all } \eta' \in F_k, \eta'' \in G_k. \tag{2.41}$$

A triangle inequality of sorts will then show that

$$\sup_x d(\alpha_k(\xi, x), \beta_k(\eta, x)) \ge C_8[\log r_k]^{-1/2} \sim C_8[8k \log k]^{-1/2} \tag{2.42}$$

for all η in F_k or for all η in G_k. Of course it is determined by ξ for which of the sets F_k and G_k this holds. Write H_k for such a set. Thus $H_k = F_k$ or $H_k = G_k$ for each k; if (2.41) holds for all η in $F_k \cup G_k$, then make an arbitrary choice, say $H_k = F_k$, for such k. Finally we show that almost every η $[\rho]$ lies in H_k for a sequence $\mathcal{K}$ of k's of strictly positive lower density. Since $\sum_1^\infty C_8^2/(\log r_k) = \infty$, and $C_8^2/\log r_k$ is decreasing, we then also have

$$\begin{aligned} \sum_{k=1}^{\infty} \sup_x d^2(\alpha_k(\xi, x), \beta_k(\eta, x)) &\ge \sum_{k \in \mathcal{K}} \sup_x d^2(\alpha_k(\xi, x), \beta_k(\eta, x)) \\ &\ge \sum_{k \in \mathcal{K}} C_8^2[\log r_k]^{-1} = \infty \text{ for almost all } \eta \ [\rho]. \end{aligned} \tag{2.43}$$

But then, for some choice of $x_k = x_k(\xi, \eta)$

$$\sum_1^\infty d^2(\alpha_k(\xi, x_k), \beta_k(\eta, x_k)) = \infty \text{ for almost all } \eta \ [\rho].$$

This is the desired conclusion.

We note in passing that there are no measurability problems; α_k and β_k are right continuous in x, so that the sup over x in (2.43) may be restricted to rational x, and x_k may be taken rational.

The actual proof is slightly more complicated; in order to obtain enough independence we work with the Y_k, rather than the Z_k. We carry out the details for

$\mathcal{G} = \mathbb{Z}^2$ only; the proof for $\mathcal{G} = \mathbb{Z}$ is actually much simpler. We define, for a constant κ to be determined below,

$$\widehat{F}_k = \{\eta \in \Xi : EY_k(\eta) \le -2r_k^{1/2} \text{ and } EY_k^4(\eta) \le \kappa[r_k \log r_k]^2\}, \tag{2.44}$$

$$\widehat{G}_k = \{\eta \in \Xi : EY_k(\eta) \ge 2r_k^{1/2} \text{ and } EY_k^4(\eta) \le \kappa[r_k \log r_k]^2\}, \tag{2.45}$$

$$K_k = \Big\{\eta \in \Xi : P\{|Y_k(\eta) - Z_k(\eta)| > r_k^{1/2}\} \le \Big(\frac{1}{\log r_k}\Big)^2\Big\}. \tag{2.46}$$

We set

$$F_k = \widehat{F}_k \cap K_k,\ G_k = \widehat{G}_k \cap K_k$$

and begin our proof by verifying (2.41). Note that (2.41) will hold for any choice of $\kappa < \infty$; only the value of C_7 will depend on κ.

If $\eta' \in \widehat{F}_k$, $\eta'' \in \widehat{G}_k''$, then we apply (2.18) with $p = 4$ to

$$U_1 = Y_k(\eta') + r_k^{1/2}, \quad U_2 = Y_k(\eta'') - r_k^{1/2}.$$

By definition of $\widehat{F}_k$, $\widehat{G}_k$ we then have

$$EU_2 - EU_1 \ge 2r_k^{1/2}$$

and

$$EU_i^4 \le 2^4(\kappa[r_k \log r_k]^2 + r_k^2) \le 2^5\kappa[r_k \log r_k]^2,\ i = 1, 2,\ k \ge k_0,$$

where k_0 depends on κ only. (2.18) therefore shows that for $\Psi_i :=$ distribution function of U_i, we have

$$\begin{aligned}
&\sup_y \frac{\Psi_1(y) - \Psi_2(y)}{[(\Psi_1(y) \wedge (1 - \Psi_1(y)) + (\Psi_2(y) \wedge (1 - \Psi_2(y))]^{1/2}} \\
&\qquad \ge C_2(4)\frac{2r_k^{1/2}}{(2^6\kappa)^{1/4}[r_k \log r_k]^{1/2}} = \frac{C_9}{\kappa^{1/4}}\ \frac{1}{[\log r_k]^{1/2}}.
\end{aligned} \tag{2.47}$$

If also $\eta' \in K_k$, then

$$P\{|Y_k(\eta') - Z_k(\eta')| > r_k^{1/2}\} \le [\log r_k]^{-2},$$

and

$$P\{Z_k(\eta') \le y\} \ge \Psi_1(y) - [\log r_k]^{-2}.$$

Since (see (2.28))

$$Z_k(\eta) = W_{r_k}(\eta) - W_{s_k}(\eta) - \frac{1}{2}(r_k - s_k),$$

this says that for $x = y + \frac{1}{2}(r_k - s_k)$

$$\beta_k(\eta', x) = P\{W_{r_k}(\eta) - W_{s_k}(\eta) \le x\} \ge \Psi_1(y) - [\log r_k]^{-2} \tag{2.48}$$

for $\eta' \in F_k$. Similarly, for $\eta'' \in G_k$,

$$\beta_k(\eta'', x) \leq \Psi_2(y) + [\log r_k]^2. \tag{2.49}$$

Now (2.47) says that for some $\overline{y}$

$$d(\Psi_1(\overline{y}), \Psi_2(\overline{y})) \geq \frac{C_9}{2\kappa^{1/4}}[\log r_k]^{-1/2}. \tag{2.50}$$

We either have

$$\Psi_1(\overline{y}) \leq \frac{1}{2} \quad \text{or} \quad \Psi_1(\overline{y}) > \frac{1}{2}.$$

For the sake of argument let $\Psi_1(\overline{y}) \leq \frac{1}{2}$ and set $\overline{x} = \overline{y} + (r_k - s_k)/2$. Then (2.50) and (2.39) give $\Psi_1(\overline{y}) > \Psi_2(\overline{y})$ and

$$\frac{1}{2} \geq \Psi_1(\overline{y}) \geq d^2(\Psi_1(\overline{y}), \Psi_2(\overline{y})) \geq \frac{C_9^2}{4\kappa^{1/2}}[\log r_k]^{-1}. \tag{2.51}$$

Moreover

$$\begin{aligned} d(\beta_k(\eta', \overline{x}), \beta_k(\eta'', \overline{x})) &\geq 1 \wedge \frac{\beta_k(\eta', \overline{x}) - \beta_k(\eta'', \overline{x})}{[\beta_k(\eta', \overline{x}) + \beta_k(\eta'', \overline{x})]^{1/2}} \\ &\geq 1 \wedge \frac{\beta_k(\eta', \overline{x}) - \beta_k(\eta'', \overline{x})}{[2\beta_k(\eta', \overline{x})]^{1/2}} \end{aligned} \tag{2.52}$$

(note that this is trivial if $\beta_k(\eta'', \overline{x}) \geq \beta_k(\eta', \overline{x})$). The right hand side here is increasing in $\beta_k(\eta', \overline{x})$ and decreasing in $\beta_k(\eta'', \overline{x})$. Therefore, (2.48)–(2.52) show that

$$d(\beta_k(\eta', \overline{x}), \beta_k(\eta'', \overline{x})) \geq 1 \wedge \frac{\Psi_1(\overline{y}) - \Psi_2(\overline{y}) - 2[\log r_k]^{-2}}{[2\Psi_1(\overline{y}) - 2[\log r_k]^{-2}]^{1/2}} \geq \frac{C_9}{4\kappa^{1/4}}[\log r_k]^{-1/2}$$

for $k \geq$ some $k_1(\kappa)$. This inequality remains valid if $\Psi_1(\overline{y}) > \frac{1}{2}$. For then (by (2.50)), $\Psi_2(\overline{y}) > \Psi_1(\overline{y}) > \frac{1}{2}$ and

$$\frac{1}{2} \geq 1 - \Psi_2(\overline{y}) \geq d^2(\Psi_1(\overline{y}), \Psi_2(\overline{y})) \geq \frac{C_9^2}{4\kappa^{1/2}}[\log r_k]^{-1},$$

and we can replace $\Psi_1(\overline{y})$ by $1 - \Psi_2(\overline{y})$ in the preceding argument. Thus we have proven (2.41) in all cases.

Next we prove that there exists some constant C_{10} such that for all $\alpha, \beta', \beta'' \in [0, 1]$,

$$d(\beta', \beta'') \leq C_{10}[d(\alpha, \beta') + d(\alpha, \beta'')]. \tag{2.53}$$

If at least one of β' or β'' lies in $[1/4, 3/4]$, then $\beta' \wedge (1 - \beta') + \beta'' \wedge (1 - \beta'') \geq 1/4$ and (2.53) is immediate from $|\beta' - \beta''| \leq |\beta' - \alpha| + |\beta'' - \alpha|$. If $\beta' \leq 1/4$ and $\beta'' \geq 3/4$ or vice versa, then $|\alpha - \beta'| \geq 1/4$ or $|\alpha - \beta''| \geq 1/4$ and again (2.53) is obvious. From this we see that we may restrict ourselves to $0 \le \beta' \le \beta'' \le 1/4$; the other remaining cases are equivalent to this case. But then

$$d(\beta', \beta'') \le 1 \wedge \frac{\beta'' - \beta'}{[\beta'']^{1/2}} \le [\beta'']^{1/2}.$$

If $\alpha \le (\beta' + \beta'')/2 \le \beta''$, then

$$d(\alpha, \beta') \ge 1 \wedge \frac{\beta'' - \alpha}{[\alpha + \beta'']^{1/2}} \ge 1 \wedge \frac{(\beta'' - \beta')/2}{[2\beta'']^{1/2}} \ ;$$

if $(\beta' + \beta'')/2 \le \alpha \le 2\beta''$, then

$$d(\alpha, \beta') \ge 1 \wedge \frac{\alpha - \beta'}{[\alpha + \beta']^{1/2}} \ge 1 \wedge \frac{(\beta'' - \beta')/2}{[3\beta'']^{1/2}} \ ;$$

if $2\beta'' \le \alpha$, then

$$\begin{aligned} d(\alpha, \beta'') &\ge 1 \wedge \frac{\alpha - \beta''}{[\alpha + \beta'']^{1/2}} \ge 1 \wedge \frac{\alpha/2}{[2\alpha]^{1/2}} \\ &\ge 1 \wedge 2^{-3/2}\alpha^{1/2} \ge \frac{1}{2}[\beta'']^{1/2}. \end{aligned}$$

Thus we proved (2.53) in all cases.

By taking $\beta' = \beta(\eta', x)$, $\beta'' = \beta(\eta'', x)$ in (2.53) we find that for all $\alpha \in [0,1]$, $\eta', \eta'' \in \Xi$

$$\begin{aligned} \sup_x d(\beta(\eta', x), \beta(\eta'', x)) &\le C_{10} \sup_x [d(\alpha, \beta(\eta', x)) + d(\alpha, \beta(\eta'', x))] \\ &\le C_{10} \Big[\sup_x d(\alpha, \beta(\eta', x)) + \sup_x d(\alpha, \beta(\eta'', x)) \Big]. \end{aligned}$$

Assume now that

$$\sup_x d(\alpha_k(\xi, x), \beta(\eta', x)) \le \frac{C_7}{2C_{10}} [\log r_k]^{-1/2} \text{ for some } \eta' \in F_k.$$

Then (2.41), together with (2.53), shows that

$$\sup_x d(\alpha_k(\xi, x), \beta(\eta'', x)) \ge \frac{C_7}{2C_{10}} [\log r_k]^{1/2} \text{ for } all \ \eta'' \in G_k.$$

The same argument works if the roles of (η', F_k) and (η'', G_k) are interchanged, so that (2.42) indeed holds for all η in F_k or for all η in G_k (or both). Thus, by taking $H_k = F_k$ or $H_k = G_k$ we have for a suitable $x_k = x_k(\xi, \eta)$

$$d(\alpha_k(\xi, x_k), \beta_k(\eta, x_k)) \ge \frac{1}{2} C_8 [\log r_k]^{-1/2} \text{ for all } \eta \in H_k. \tag{2.54}$$

We now show that almost all η $[\rho]$ lie in H_k for sufficiently many k. We already know from Lemma 3, that a.e. $[\rho]$ we have $\eta \in K_k$ eventually, so that we only have to prove that

$$\eta \in \widehat{H}_k \text{ on a sequence of } k\text{'s of strictly positive lower density for almost all } \eta\ [\rho], \tag{2.55}$$

where

$$\widehat{H}_k = \begin{cases} \widehat{F}_k & \text{if} \quad H_k = F_k, \\ \widehat{G}_k & \text{if} \quad H_k = G_k. \end{cases}$$

If we further define

$$\lambda_k = \begin{cases} +1 & \text{if} \quad H_k = F_k, \\ -1 & \text{if} \quad H_k = G_k, \end{cases}$$

then (2.54) is equivalent (via the definitions (2.44), (2.45)) to

$$EY_k^4(\eta) \le \kappa[r_k \log r_k]^2 \text{ and } \lambda_k EY_k(\eta) \le -2r_k^{1/2} \text{ on a sequence of } k\text{'s of stricIty positive lower density, for almost all } \eta\ [\rho]. \tag{2.56}$$

Finally, let $L_k = L_k(\kappa) \subset \Xi$ be the set

$$L_k = \{\eta \in \Xi : \lambda_k EY_k(\eta) \le -2r_k^{1/2} \text{ and } EY_k^4(\eta) \le \kappa[r_k \log r_k]^2\}.$$

Since

$$EY_k(\eta) = \sum_{a_{k-1} < |v| \le a_k} \Big[\eta(v) - \frac{1}{2}\Big] E\ell(v, (s_k, r_k]) \tag{2.57}$$

and

$$EY_k^4(\eta) = \sum_{a_{k-1} < |v_i| \le a_k} \prod_{i=1}^4 \Big[\eta(v_i) - \frac{1}{2}\Big] E \prod_{i=1}^4 \ell(v_i, (s_k, r_k]), \tag{2.58}$$

the events L_k depend on the disjoint sets $\{\eta(v) : a_{k-1} < |v| \le a_k\}$, and are therefore independent (under the measure ρ on Ξ). Consequently to prove (2.55) — which says that a.e. $[\rho]$, L_k occurs on a sequence of k's of strictly positive lower density — it suffices to prove that

$$\liminf_{k \to \infty} \rho(L_k) \ge C_{11} > 0. \tag{2.59}$$

To prove (2.59) we note that the $\eta(v) - \frac{1}{2}$, $a_{k-1} < |v| \le a_k$, are i.i.d. random variables of zero mean and variance $1/4$. Moreover, by (2.35), (2.27) and (2.13),

$$\sum_{a_{k-1} < |v| \le a_k} \{E\ell(v, (s_k, v_k])\}^3 \le C_{12} a_k^2 [\log r_k]^3 \sim C_{12} a_k^2 (k \log k)^3 = o(r_k^{3/2}). \tag{2.60}$$

On the other hand,

$$\sum_{a_{k-1} < |v| \le a_k} \{E\ell(v, (s_k, r_k])\}^2 \ge \sum_{[r_k/2]^{1/2} \le |v| \le r_k^{1/2}} \{E\ell(v, (s_k, r_k])\}^2.$$

Also, for $[r_k/2]^{1/2} \le |v| \le r_k^{1/2}$, by the local central limit theorem (compare also with Révész (1990), Lemmas 16.3–16.5),

$$E\ell(v,(s_k,r_k]) = \sum_{j=s_k+1}^{r_k} P\{S_j = v\} \ge \sum_{r_k/2\le j\le r_k} P\{S_j = v\}$$
$$\sim \sum_{r_k/2\le j\le r_k} \frac{1}{\pi j}\exp\Big(-\frac{|v|^2}{j}\Big) \ge C_{13}.$$

Consequently

$$\sum_{a_{k-1}<|v|\le a_k} \{E\ell(v,(s_k,r_k])\}^2 \ge C_{14} r_k. \tag{2.61}$$

(2.60) and (2.61) allow us to apply Lyapounov's central limit theorem (see Rényi (1970), Theorem VIII.1.2) to conclude that

$$\Big[\sum_{a_{k-1}<|v|\le a_k} \{E\ell(v,(s_k,r_k])\}^2\Big]^{-1/2} EY_k(\eta) \Rightarrow N(0,1/4).$$

In particular, for any $x \ge 0$,

$$\begin{aligned}
&\liminf_{k\to\infty}\ \rho\{\pm EY_k(\eta) \le -x r_k^{1/2}\}\\
&\quad\ge \liminf_{k\to\infty}\rho\Big\{\pm EY_k(\eta) \le -x C_{14}^{-1/2}\Big[\sum_{a_{k_1}<|v|\le n_k}\{E\ell(v,(s_k,r_k])\}^2\Big]^{1/2}\Big\}\\
&\quad= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-2xC_{14}^{-1/2}} e^{-t^2/2}dt > 0.
\end{aligned}$$

For $x = 2$ this shows

$$\liminf \rho\{\lambda_k EY_k(\eta) \le -2r_k^{1/2}\} > 0. \tag{2.62}$$

This is our first step in proving (2.59).

In order to complete the proof we need an upper bound for

$$\begin{aligned}
&\sum_{v_1,v_2} E\{\ell^2(v_1,(s_k,r_k])\ell^2(v_2,(s_k,r_k])\}\\
&\quad\le 4\sum_{v_1}\sum_{v_2} E\Big\{\sum_{\ell_1\le r_k}\ \sum_{\ell_1\le\ell_2\le r_k}\ \sum_{m_1\le r_k}\ \sum_{m_1\le m_2\le r_k}\\
&\qquad\qquad I(S_{\ell_1} = S_{\ell_2} = v_1, S_{m_1} = S_{m_2} = v_2)\Big\}.
\end{aligned}$$

To obtain such an upper bound note that

$$EI(S_{\ell_1} = S_{\ell_2} = v_1 S_{m_1} = S_{m_2} = v_2) = P\{S_{\ell_1} = S_{\ell_2} = v_1, S_{m_1} = S_{m_2} = v_2\}.$$

If $\ell_1 \le \ell_2 \le m_1 \le m_2$ we can write the last probability as

$$P\{S_{\ell_1} = v_1\}P\{S_{\ell_2-\ell_1} = 0\}P\{S_{m_1-\ell_2} = v_2 - v_1\}P\{S_{m_2-m_1} = 0\}.$$

The sum over $\ell_1 \le \ell_2 \le m_1 \le m_2 \le r_k$ and over v_1, v_2 therefore contributes at most

$$\begin{aligned}
\Big(\sum_{m \le r_k} P\{S_m = 0\}\Big)^2 &\Big(\sum_{m \le r_k} \sum_{v} P\{S_m = v\}\Big)^2 \\
&\le C_{15}[\log r_k]^2 \Big(\sum_{m \le r_k} 1\Big)^2 \quad \text{(see (2.35) or (2.36))} \\
&= C_{15}[r_k \log r_k]^2. \qquad (2.63)
\end{aligned}$$

The same estimate holds for the sum over $m_1 \le m_2 \le \ell_1 \le \ell_2 \le r_k$ and v_1, v_2. This leaves us with two sums of the form

$$\begin{aligned}
&\sum_{v_1,v_2} \sum_{\ell_1 \le m_1 \le \ell_2 \le m_2 \le r_k} P\{S_{\ell_1} = S_{\ell_2} = v_1, S_{m_1} = S_{m_2} = v_2\} \\
&\le \sum_{v_1,v_2} \sum_{\ell_1 \le r_k} P\{S_{\ell_1} = v_1\} \sum_{\ell_1 \le m_1 \le r_k} P\{S_{m_1-\ell_1} = v_2 - v_1\} \\
&\quad \cdot \sum_{m_1 \le \ell_2 \le r_k} P\{S_{\ell_2-m_1} = v_1 - v_2\} \sum_{\ell_2 \le m_2 \le r_k} P\{S_{m_2-\ell_2} = v_2 - v_1\}. \qquad (2.64)
\end{aligned}$$

But

$$\begin{aligned}
\sum_{\ell_2 \le m_2 \le r_k} P\{S_{m_2-\ell_2} = v_2 - v_1\} &\le \sum_{j=0}^{r_k} P\{S_j = v_2 - v_1\} \\
&= 1 + E\ell(v_2 - v_1, (0, r_k]) \le C_5 \log r_k \quad \text{(see (2.35), (2.36))}.
\end{aligned}$$

The same bound holds for the sum over ℓ_2 so that the right hand side of (2.64) is at most

$$(C_5 \log r_k)^2 \sum_{\ell_1 \le r_k} \sum_{v_1} P\{S_{\ell_1} = v_1\} \sum_{j=0}^{r_k} \sum_{v_2} P\{S_j = v_2 - v_1\} = (C_5 \log r_k)^2 (r_k + 1)^2.$$

Adding this estimate and the estimate (2.63), we obtain finally

$$\sum_{v_1,v_2} E\{\ell^2(v_1, (s_k, r_k])\ell^2(v_2, (s_1, r_k])\} \le C_{16}[r_k \log r_k]^2. \qquad (2.65)$$

With this estimate in hand it is easy to bound

$$\mathbb{E}\{Y_k^4\} = \int \rho(d\eta) E Y_k^4(\eta) = \int \rho(d\eta) \sum_{a_{k-1} < |v_i| \le a_k} \prod_{i=1}^4 \Big[\eta(v_i) - \frac{1}{2}\Big] E \prod_{i=1}^4 \ell(v_i, (s_k, r_k]).$$

As in the lines following (2.37), this is at most

$$\frac{1}{16}\sum_{v_1}\sum_{v_2} E\{\ell^2(v_1,(s_k,r_k])\ell^2(v_2,(s_k,r_k])\} \le C_{16}[r_k \log r_k]^2.$$

It follows that

$$\rho\{\eta\in\Xi : EY_k^4(\eta) > \kappa[r_k\log r_k]^2\} \le C_{16}\frac{1}{\kappa}.$$

We can therefore choose κ so large that

$$\begin{aligned}\liminf_{k\to\infty}\rho(L_k) &\ge \liminf_{k\to\infty}\big(\rho\{\eta\in\Xi : \lambda_k EY_k(\eta) \le -2r_k^{1/2}\}\\ &\qquad - \rho\{\eta\in\Xi : EY_k^4(\eta) > \kappa[r_k\log r_k]^2\}\big)\\ &\ge \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-4C_{14}^{-1/2}} e^{-t^2/2}dt - C_{16}\frac{1}{\kappa} > 0 \quad \text{(see (2.62))}.\end{aligned}$$

This proves (2.59) and completes the proof. ∎

We have now proven that for each ξ and almost all η $[\rho]$, $\alpha_k(\xi,x_k)$ and $\beta_n(\eta,x_k)$ satisfy (2.4) (when x_k is suitably chosen). For the remainder of this section ξ, η and x_k are fixed such that (2.4) holds. We remind the reader that we took μ_1 and μ_2 as the probability measures on the space Ω of (2.11) under which $\{\omega_n\}$ has the distribution of $\{W_n(\xi)\}$ and $\{W_n(\eta)\}$, respectively. Now that ξ and η have been fixed, so that these measures are also fixed. We wish to show that they satisfy (2.2) and (2.3) (with E_k given by (2.12)). Since we have little control over x_k, our proof must be more or less independent of the choice of x_k. Instead our proof depends on the fact that the only significant information which we can gain from observing the occurrence or nonoccurrence of $E_1,\dots,E_{n-1}$ is information about the position, $S_{r_{n-1}}$, of the random walk at time r_{n-1}. This is made precise in the next lemma.

Lemma 5. *For each ξ, μ_1 satisfies (2.2), and for each η, μ_2 satisfies (2.3).*

Proof: The proofs of (2.2) and (2.3) are the same, so we only prove (2.2). Note that η does not enter in (2.2) and that ξ is fixed; the only randomness is in $\{S_n\}$ now. For this reason it is advantageous to imbed $(\Omega,\mathcal{B})$ into a bigger probability space $(\widetilde{\Omega},\widetilde{\mathcal{B}})$ on which the full random walk $\{S_n\}$ is defined. Again we restrict ourselves to the case $\mathcal{G}=\mathbb{Z}^2$. In this case we take

$$\widetilde{\Omega} = \prod_1^\infty \{(0,1),(0,-1),(1,0),(-1,0)\}$$

and $\widetilde{\mathcal{B}}$ the σ–field generated by the cylinder functions. A typical element will be written as $\widetilde{\omega} = (\widetilde{\omega}_1,\widetilde{\omega}_2,\dots)$. $\widetilde{\omega}_i$ represents the i–th step of the random walk. Accordingly we define

$$S_0(\widetilde{\omega}) = 0,\ S_n(\widetilde{\omega}) = \sum_1^n \widetilde{\omega}_i,\ n\ge 1.$$

On $\widetilde{\mathcal{B}}$ we define the probability measure

$$P = \prod_1^\infty \widetilde{P}_i$$

with $\widetilde{P}_i(u) = 1/4$ for $u = (0, \pm 1)$, $(\pm 1, 0)$. Of course $\{S_n\}$ is simple symmetric random walk on $\mathbb{Z}^2$ under P.

Next we show how to formulate (2.2) on $\widetilde{\Omega}$. We define $W_n = W_n(\xi)(\widetilde{\omega})$ by (2.1) as a random variable on $\widetilde{\Omega}$, and consider the map $\varphi : \widetilde{\Omega} \to \Omega = \prod_1^\infty \mathbb{R}$ (see (2.11)) given by

$$\varphi(\widetilde{\omega}) = \{W_1(\xi)(\widetilde{\omega}), W_2(\xi)(\widetilde{\omega}), \dots\}.$$

Then the event E_k of (2.12) equals $\varphi(\widetilde{E}_k)$ with

$$\widetilde{E}_k = \{\widetilde{\omega} : W_{r_k}(\xi)(\widetilde{\omega}) - W_{s_k}(\xi)(\widetilde{\omega}) \le x_k\}.$$

This event belongs to $\widetilde{\mathcal{B}}_{s_k}^{r_k}$, where

$$\widetilde{\mathcal{B}}_\ell^m = \sigma\text{–field generated by } \{S_j : \ell \le j \le m\}.$$

With some abuse of notation we also use $\mathcal{B}_\ell^m$ for the sub σ–field of $\widetilde{\mathcal{B}}_1^{r_m}$ generated by the sets $\widetilde{E}_k$, $\ell \le k \le m$. μ_1 is simply the measure induced by φ on Ω, i.e., $\mu_1 = P \circ \varphi^{-1}$ and α_k is given by (2.15). (2.2) is therefore equivalent to

$$\sum_{n=1}^\infty |P\{\widetilde{E}_n \mid \mathcal{B}_1^{n-1}\} - P\{\widetilde{E}_n\}| < \infty \text{ a.e. } [P]. \tag{2.66}$$

(2.66) is not hard to prove by standard estimates for random walk. Indeed

$$P\{\widetilde{E}_n \mid \mathcal{B}_1^{n-1}\} = E\{P\{\widetilde{E}_n \mid \widetilde{\mathcal{B}}_1^{r_{n-1}}\} \mid \mathcal{B}_1^{n-1}\}$$

and

$$P\{\widetilde{E}_n\} = E\{P\{\widetilde{E}_n \mid \widetilde{\mathcal{B}}_1^{r_{n-1}}\}\}.$$

$\widetilde{E}_n$ depends only on $\{S_j : s_n \le j \le r_n\}$ and $s_n \ge r_{n-1}$ (see (2.13)). Therefore, by the Markov property,

$$P\{\widetilde{E}_n \mid \mathcal{B}_1^{r_{n-1}}\} = P\{\widetilde{E}_n \mid S_{r_{n-1}}\},$$

and with $\gamma_n = r_{n-1}^{1/2} \log\log r_{n-1}$

$$\begin{aligned}
&|P\{\widetilde{E}_n \mid \mathcal{B}_1^{n-1}\} - P\{\widetilde{E}_n\}| \\
&= \Big| \sum_{u \in \mathbb{Z}^2} P\{S_{r_{n-1}} = u \mid \mathcal{B}_1^{n-1}\} P\{\widetilde{E}_n \mid S_{r_{n-1}} = u\} \\
&\qquad - \sum_{v \in \mathbb{Z}^2} P\{S_{r_{n-1}} = v\} P\{\widetilde{E}_n \mid S_{r_{n-1}} = v\} \Big| \\
&= \Big| \sum_{u,v} P\{S_{r_{n-1}} = u \mid \mathcal{B}_1^{n-1}\} P\{S_{r_{n-1}} = v\} \\
&\qquad \cdot [P\{\widetilde{E}_n \mid S_{r_{n-1}} = u\} - P\{\widetilde{E}_n \mid S_{r_{n-1}} = v\}] \Big| \\
&\le \max_{|u|,|v| \le \gamma_n} |P\{\widetilde{E}_n \mid S_{r_{n-1}} = u\} - P\{\widetilde{E}_n \mid S_{r_{n-1}} = v\}| \\
&\qquad + P\{|S_{r_{n-1}}| > \gamma_n \mid \mathcal{B}_1^{n-1}\} + P\{|S_{r_{n-1}}| > \gamma_n\}.
\end{aligned} \tag{2.67}$$

Here, and in the remainder of this proof u and v are restricted to have the same parity as r_{n-1}, that is $r_{n-1}+u(1)+u(2)$ and $r_{n-1}+v(1)+v(2)$ are even, when $u=(u(1),u(2))$ and $v=(v(1),v(2))$. Now (again by Rényi (1970), Theorem VII.4.2),

$$P\{|S_{r_{n-1}}|\geq \gamma_n\}\leq C_{17}\exp(-C_{18}(\log\log r_{n-1})^2)\leq C_{17}\exp(-C_{19}(\log n)^2)$$

so that

$$\sum_n P\{|S_{r_{n-1}}|\geq \gamma_n\}<\infty. \tag{2.68}$$

Also

$$\sum_n P\{|S_{r_{n-1}}|\geq \gamma_n \mid \mathcal{B}_1^{n-1}\}<\infty \text{ a.e. } P, \tag{2.69}$$

because

$$E\Big\{\sum_n P\{|S_{r_{n-1}}|\geq \gamma_n \mid \mathcal{B}_1^{n-1}\}\Big\}=\sum_n P\{|S_{r_{n-1}}|\geq \gamma_n\}<\infty.$$

Finally we must estimate

$$\max_{|u|,|v|\leq\gamma_n}|P\{\widetilde{E}_n\mid S_{r_{n-1}}=u\}-P\{\widetilde{E}_n\mid S_{r_{n-1}}=v\}|.$$

Because $\widetilde{E}_n$ depends only on S_j for $s_n\leq j\leq r_n$, we can use the Markov property once more to find

$$\begin{aligned}
&|P\{\widetilde{E}_n\mid S_{r_{n-1}}=u\}-P\{\widetilde{E}_n\mid S_{r_{n-1}}=v\}|\\
&\quad=\Big|\sum_w[P\{S_{s_n}=w\mid S_{r_{n-1}}=u\}-P\{S_{s_n}=w\mid S_{r_{n-1}}=v\}]P\{\widetilde{E}_n\mid S_{s_n}=w\}\Big|\\
&\quad\leq\sum_w|P\{S_{s_n}=w\mid S_{r_{n-1}}=u\}-P\{S_{s_n}=w\mid S_{r_{n-1}}=v\}|\\
&\quad=\sum_w|P\{S_{s_n-r_{n+1}}=w\}-P\{S_{s_n-r_{n+1}}=w+u-v\}|. \qquad (2.70)
\end{aligned}$$

However, it is fairly well known that if $z\in\{(0,\pm2),(\pm2,0),(\pm1,\pm1)\}$ (the possible values of S_2), then

$$\sum_w|P\{S_t=w\}-P\{S_t=w+z\}|\leq C_{20}t^{-1/2} \tag{2.71}$$

(see Kerstan and Matthes (1965), Lindvall (1992), Sects. II.13–17, especially p. 45). By writing $u-v$ as a sum of z's as above, one obtains from (2.70), (2.71) that for any u,v of the same parity, with $|u|,|v|\leq\gamma_n$, it holds that

$$\begin{aligned}
&|P\{\widetilde{E}_n\mid S_{r_{n-1}}=u\}-P\{\widetilde{E}_n\mid S_{r_{n-1}}=v\}|\\
&\quad\leq C_{20}|u-v|[s_n-r_{n-1}]^{-1/2}\\
&\quad\leq C_{21}\gamma_n[n^4r_{n-1}]^{-1/2}\quad\text{(see (2.13))}\\
&\quad\leq C_{22}\frac{\log n}{n^2}.
\end{aligned}$$

Together with (2.67)–(2.69) this implies (2.66). ■

It is now easy to complete the **Proof of Theorem 1**. By Lemma 1 the restrictions of μ_1 and μ_2 (introduced after Lemma 1) to the σ–field generated by $E_k,\ k \ge \ell$, are mutually singular, for almost all choices of η $[\rho]$. This shows that for almost all η, the measures governing $\{W_n(\xi) - W_{s_\ell}(\xi) : n \ge s_\ell\}$ and $\{W_n(\eta) - W_{s_\ell}(\eta) : n \ge s_\ell\}$ are mutually singular. Since $s_\ell \to \infty$ as $\ell \to \infty$, and $W_n(\xi) - W_{s_\ell}(\xi)$ and $W_n(\eta) - W_{s_\ell}(\eta)$ depend only on $\{\xi(S_j) : j \ge s_\ell\}$ and $\{\eta(S_j) : j \ge s_\ell\}$, respectively, Theorem 1 follows. ■

3. Oriented random walk on $(\mathbb{Z}_+)^d$. In this section we prove Theorem 2. We define

$$H_n = \{w \in (\mathbb{Z}_+)^d : w(1) + \cdots + w(d) = n\} \tag{3.1}$$

(here and in the remainder of this section $w(1), \dots, w(d)$ are the coordinates of w). An *oriented path* on $(\mathbb{Z}_+)^d$ is a finite or infinite sequence $\{w_n\} \subset (\mathbb{Z}_+)^d$ such that for all $w_n,\ w_{n+1}$ in the path, one has $w_{n+1} = w_n + e_i$ for some $1 \le i \le d$ (e_i is the i–th coordinate vector). Following the terminology of Benjamini and Kesten (1995) we say that $x_1, \dots, x_k$ is *seen along the path* $(w_0, w_1, \dots, w_k)$ if $\xi(w_i) = x_i$, $1 \le i \le k$. we also define

$$\begin{aligned}&K_n(x_1, \dots, x_n; \xi) = \text{number of points in } H_n \text{ which are the}\\ &\text{endpoint of an oriented path from the origin to } H_n \text{ along}\\ &\text{which } (x_1, \dots, x_n) \text{ is seen in the scenery } \xi,\end{aligned} \tag{3.2}$$

and for $\Lambda \subset H_n$

$$\begin{aligned}&N_{n,k}(x_{n+1}, \dots, x_{n+k}; \xi, \Lambda) = \text{number of oriented paths}\\ &(w_0, w_1, \dots, w_k) \text{ with } w_i \in H_{n+i}, w_0 \in \Lambda, \text{ and}\\ &\xi(w_i) = x_{n+i},\ 1 \le i \le k.\end{aligned} \tag{3.3}$$

(Thus $N_{n,k}(x_{n+1}, \dots, x_{n+k}; \xi, \Lambda)$ is the number of oriented paths from Λ to H_{n+k} along which $(x_{n+1}, \dots, x_{n+k})$ is seen in the scenery ξ.)

As before $\mathcal{X} = \{0,1\}^{\mathbb{N}}$ and $\mathcal{F}_\ell^m$ is the σ–field of subsets of $\mathcal{X}$ generated by the coordinate functions $x \to x_k,\ \ell \le k \le m$. With a slight abuse of notation we use Q_ℓ^ξ in this section to denote the distribution of $\{\xi(S_n^{or})\}_{n \ge \ell}$ on $\mathcal{F}_\ell^\infty$. Q^ξ will be the distribution of the full sequence $\{\xi(S_n^{or})\}_{n \ge 0}$.

The basis for the proof of Theorem 1 is the formula

$$\begin{aligned}&Q^\xi\{\{0,1\} \times (x_1, \dots, x_k) \times \{0,1\}^{\mathbb{N}}\}\\ &\qquad = \text{Probability of } \{\xi(S_j^{or}) = x_j,\ 1 \le j \le k\}\\ &\qquad = \frac{1}{d^k} N_{0,k}(x_1, \dots, x_k; \xi, \{\mathbf{0}\}).\end{aligned} \tag{3.4}$$

This holds because

$$P\{S_j^{or} = w_j, 1 \le j \le k\} = d^{-k}, \tag{3.5}$$

for any oriented path $w_0 = \mathbf{0}, w_1, \dots, w_k$ starting at the origin. Roughly speaking we prove that $Q^\eta\{\{0,1\} \times (x_1, \dots, x_k) \times \{0,1\}^{\mathbb{N}}\}$ is almost constant in probability, when η has the distribution ρ and x is fixed. This will show that for most pairs (ξ, η) the two measures Q^ξ and Q^η have mass in common, and are therefore not mutually singular. The line of attack then is to show that quantities analogous to $d^{-k} N_{0,k}(x_1, \dots, x_k; \eta, \{\mathbf{0}\})$ converges to a constant in ρ–measure as $k \to \infty$. This is done by means of estimates for first and second moments which are known from oriented percolation. (Actually we only prove a one–sided inequality (see Lemma 7), but that suffices for our purposes.)

We define

$$\mathcal{H}_n = \sigma\text{–field of subsets of } \Xi \text{ generated by } \{w \to \xi(w) : \|w\|_1 \le n\}, \tag{3.6}$$

and for $x = (x_0, x_1, \dots) \in \mathcal{X}$

$$\begin{aligned}\tau(x) = \tau(x, \xi) &= \max\{n : (x_1, \dots, x_{n-1}) \text{ is seen from the origin}\} \\ &= \max\{n : N_{0,n-1}(x_1, \dots, x_{n-1}; \xi, \{\mathbf{0}\}) > 0\}.\end{aligned} \tag{3.7}$$

Note that we ignore x_0 in $\tau(x)$ and in $N_{0,n-1}$. This is a matter of convenience only, since we can simply restrict ourselves to sceneries with a fixed value x_0 at the origin; in this way the formulas become simpler and match Benjamini and Kesten (1995) better. For the rest of this section $\mathcal{G} = (\mathbb{Z}_+)^d$.

Lemma 6. *For $\alpha < \frac{1}{2}$ there exists a $d_0 = d_0(\alpha) < \infty$, and for $d \ge d_0$ there exist constants $0 < C_1(d)$, $C_2(d) < \infty$ such that for all $x_1, \dots, x_n$ and $n \ge 1$*

$$\rho\{\eta \in \Xi : 1 \le K_n(x_1, \dots, x_n; \eta) \le \alpha n\} \le C_1 e^{-C_2 n}. \tag{3.8}$$

Proof: If $K_n(x_1, \dots, x_n; \eta) \ge 1$, then $\tau(x_0, x_1, \dots, x_n, 1, 1, \dots) > n$. In fact there exist $K_n(x_0, \dots, x_n; \eta)$ endpoints of paths along which $(x_1, \dots, x_n)$ is seen from the origin. Let Γ be the collection of endpoints of these paths. Of course $\Gamma \subset H_n$. If $\eta(w + e_i) = 0$ for each $w \in \Gamma$ and $1 \le i \le d$ (i.e., if $\eta = 0$ at all points to which S^{or} can move from Γ), then $\tau(x_0, \dots, x_n, 1, 1, \dots) = n + 1$. Therefore, for fixed $x_0, \dots, x_n$

$$\begin{aligned}&\rho\{\tau(x_0, \dots, x_n, 1, 1, \dots, \eta) = n + 1 \mid \mathcal{H}_n\} \\ &\qquad \ge 2^{-dK_n(x_1, \dots, x_n; \eta)} I(K_n(x_1, \dots, x_n, \eta) \ge 1).\end{aligned}$$

By virtue of equation (3.16) in Benjamini and Kesten (1995) we therefore have

$$\begin{aligned}&2^{-\alpha d n} \rho\{1 \le K_n(x_1, \dots, x_n; \eta) \le \alpha n\} \\ &\quad \le \rho\{2^{-dK_n(x_1, \dots, x_n; \eta)}; 1 \le K_n(x_1, \dots, x_n; \eta) \le \alpha n\} \\ &\quad \le \rho\{\tau(x_0, \dots, x_n, 1, 1, \dots, \eta) = n + 1\} \\ &\quad \le \frac{z^{2n+6}}{1 - z}\end{aligned} \tag{3.9}$$

for

$$z = \mu 2^{-\frac{1}{2}\lfloor\frac{d}{2}\rfloor + \frac{1}{4}} \le (2.7) 2^{-\frac{1}{2}\lfloor\frac{d}{2}\rfloor + \frac{1}{4}}$$

(μ is the so–called connective constant for $\mathbb{Z}^2$). Thus, the probability in the left hand side of (3.8) is at most

$$2^{\alpha dn}\frac{z^6}{1-z}\Big[(2.7)2^{-\frac{1}{2}\lfloor\frac{d}{2}\rfloor+\frac{1}{4}}\Big]^{2n} \le C_1[(2.7)2^{\alpha d+1/2-\lfloor d/2\rfloor}]^n.$$

■

Lemma 7. *There exists some constant* $C_3 = C_3(d)$ *such that for all* $\Lambda \subset H_n$ *with* $|\Lambda| :=$ *cardinality of* $\Lambda \ge \alpha n$ *and all* $x_{n+1},\dots,x_{n+k}$ *and* $\varepsilon > 0$

$$\rho\Big\{N_{n,k}(x_{n+1},\dots,x_{n+k};\eta,\Lambda) \le |\Lambda|\Big(\frac{d}{2}\Big)^k(1-\varepsilon) \mid \mathcal{H}_n\Big\} \le C_3\frac{1}{\varepsilon^2}(\alpha n)^{-(d-2)/(d-1)}. \tag{3.10}$$

Proof: The conditional distribution of $N_{n,k}(x_{n+1},\dots,x_{n+k};\eta,\Lambda)$ given $\mathcal{H}_n$ is the same for all $x_{n+1},\dots,x_{n+k}$, so we may restrict ourselves to $(x_{n+1},\dots,x_{n+k}) = (1,\dots,1)$. For brevity, we write $N_{n,k}(\eta,\Lambda)$ for $N_{n,k}(1,\dots,1;\eta,\Lambda)$. Then

$$\begin{aligned}\rho\{N_{n,k}(\eta,\Lambda) \mid \mathcal{H}_n\} &= 2^{-k}(\text{number of oriented path } (w_0,w_1,\dots,w_k) \\ &\qquad\qquad \text{with } w_0 \in \Lambda) \\ &= \Big(\frac{d}{2}\Big)^k|\Lambda|,\end{aligned} \tag{3.11}$$

so that the left hand side of (3.10) is bounded by (by virtue of Chebyshev's inequality)

$$\frac{\mathrm{Var}\,(N_{n,k}(\eta,\Lambda) \mid \mathcal{H}_n)}{|\Lambda|^2\varepsilon^2(d/2)^{2k}}, \tag{3.12}$$

where $\mathrm{Var}\,(N \mid \mathcal{H}_n)$ denotes the conditional variance of N, given $\mathcal{H}_n$. Actually, $N_{n,k}(\eta,\Lambda)$ is independent of $\mathcal{H}_n$. We now estimate its variance. Let $\pi'(u)$ and $\pi''(v)$ be generic oriented paths from $u \in \Lambda$ to H_{n+k} and from $v \in \Lambda$ to H_{n+k}, respectively. Then

$$\begin{aligned}&\mathrm{Var}\,(N_{n,k}(\eta,\Lambda) \mid \mathcal{H}_n) = \mathrm{Var}\,(N_{n,k}(\eta,\Lambda)) \\ &= \sum_{u,v\in\Lambda}\ \sum_{\pi'(u),\pi''(v)} [\rho\{(1,1,\dots,1) \text{ is seen along } \pi'(u) \\ &\qquad \text{and along } \pi''(v)\} - \rho\{(1,1,\dots,1) \text{ is seen along } \pi'(u)\} \\ &\qquad \rho\{(1,1,\dots,1) \text{ is seen along } \pi''(v)\}] \\ &\le \sum_{u,v\in\Lambda}\ \sum_{\pi'(u),\pi''(v)} 2^{-2k+|\pi'(u)\cap\pi''(v)|}I(\pi'(u)\cap\pi''(v) \ne \phi),\end{aligned} \tag{3.13}$$

where $|\pi'(v)\cap\pi''(v)|$ is the number of common vertices of $\pi'(u)$ and $\pi''(v)$ other than u, v. We now rewrite the sum over $\pi'(u)$, $\pi''(v)$ as an expectation. Let $P_{u,v,k}$ be the probability measure on pairs of oriented paths of $(k+1)$ vertices which start at u and v, respectively, and which asigns probability d^{-2k} to any such pair of paths. In other words, $P_{u,v,k}$ is the measure governing two independent copies of S_j^{or}, $0 \le j \le k$, with one copy starting at u and the other at v (rather than starting at the origin, as assumed until now). Then we can write (3.13) as

$$\mathrm{Var}\,(N_{n,k}(\eta,\Lambda) \mid \mathcal{H}_n) \le \Big(\frac{d}{2}\Big)^{2k} \sum_{u,v\in\Lambda} E_{u,v,k}\{2^{|\pi'(u)\cap\pi''(v)|}; \pi'(u)\cap\pi''(v) \ne \phi\}.$$

(Of course $E_{u,v,k}$ is expectation with respect to $P_{u,v,k}$.) Let

$$p(d) = P_{\mathbf{0},\mathbf{0},\infty}\{\pi'(\mathbf{0}) \text{ and } \pi''(\mathbf{0}) \text{ intersect at some time } k \ge 1\}.$$

Then it is known (see Cox and Durrett (1983) for similar estimates) that

$$p(d) = \frac{1}{d} + O\Big(\frac{1}{d^2}\Big) \quad (d \to \infty)$$

and

$$P_{\mathbf{0},\mathbf{0},\infty}\{|\pi'(\mathbf{0})\cap\pi''(\mathbf{0})| \ge r\} = [p(d)]^r, \quad r \ge 0.$$

Moreover, by the random walk interpretation

$$\begin{aligned} &E_{u,v,k}\{2^{|\pi'(u)\cap\pi''(v)|}; \pi'(u)\cap\pi''(v) \ne \phi\} \\ &\quad\le P_{u,v,\infty}\{\pi'(u) \text{ and } \pi''(v) \text{ intersect at some time} \\ &\qquad\qquad k \ge 1\} E_{\mathbf{0},\mathbf{0},\infty}\{2^{1+|\pi'(\mathbf{0})+\pi''(\mathbf{0})|}\} \\ &\quad= P_{u,v,\infty}\{\pi'(u) \text{ and } \pi''(v) \text{ intersect at some time} \\ &\qquad\qquad k \ge 1\} \sum_{r=0}^{\infty} 2^{1+r}[p(d)]^r(1-p(d)). \end{aligned}$$

Finally, if we write π'_k and π''_k for the k–th position of two paths $\pi'(\mathbf{0})$ and $\pi''(\mathbf{0})$, respectively, then if $d \ge 3$ and $\|u\|_1 = \|v\|_1$,

$$\begin{aligned} &P_{u,v,\infty}\{\pi'(u) \text{ and } \pi''(v) \text{ intersect at some time } k \ge 1\} \\ &\quad= P_{\mathbf{0},\mathbf{0},\infty}\{\pi'_k - \pi''_k = v - u \text{ for some } k \ge 1\} \\ &\quad\le \frac{C_4}{1 + \|u-v\|_1^{d-2}}, \end{aligned}$$

because under $P_{\mathbf{0},\mathbf{0},\infty}$, $\{\pi'_k - \pi''_k\}_{k\ge 0}$ is an (unoriented) random walk on $\mathbb{Z}^d$ whose steps take the values $e_i - e_j$ with probability $1/d^2$ for $i \ne j$, and the value $\mathbf{0}$ with probability $1/d$ (compare Itô and McKean (1960), Spitzer (1964), Proposition 26.1). Combining these estimates we find

$$\mathrm{Var}\,(N_{n,k}(\eta,\Lambda) \mid \mathcal{H}_n) \le 2C_4 \Big(\frac{d}{2}\Big)^{2k} \frac{1-p(d)}{1-2p(d)} \sum_{u\in\Lambda}\sum_{v\in\Lambda} \frac{1}{1+\|u-v\|_1^{d-2}}.$$

Now for fixed $u \in \Lambda$ and $d \geq 3$,

$$\begin{aligned}
&\sum_{v\in\Lambda} \frac{1}{1+\|u-v\|_1^{d-2}} \\
&\quad = \sum_{r=0}^{\infty} \frac{1}{1+r^{d-2}} \text{ (number of vertices } v \in \Lambda \text{ with } \|u-v\| = r) \\
&\quad = \sum_{r=0}^{\infty} \Big[\frac{1}{1+r^{d-2}} - \frac{1}{1+(r+1)^{d-2}}\Big] \text{ (number of vertices} \\
&\qquad\qquad v \in \Lambda \text{ with } \|u-v\|_1 \leq r).
\end{aligned}$$

The last sum is maximized by putting the vertices in Λ as close to u as possible (so as to maximize

$$\frac{1}{1+r^{d-2}} - \frac{1}{1+(r+1)^{d-2}} \sim (d-2)r^{1-d} \quad (x \to \infty)$$

with $r = \|u-v\|_1$ for as many v as possible). Since $\Lambda \subset H_n$, it contains at most $C_5(1+r^{d-1})$ vertices v with $\|u-v\|_1 \leq r$. Analogously to p. 181 of Kesten (1987) we therefore have

$$\begin{aligned}
\sum_{v\in\Lambda} \frac{1}{1+\|u-v\|_1^{d-2}} &\leq C_6 \sum_{r=0}^{\infty} (1+r^{1-d})[(1+r^{d-1}) \wedge |\Lambda|] \\
&\leq C_7(|\Lambda|^{1/(d-1)} + 1).
\end{aligned}$$

Finally, then

$$\operatorname{Var}(N_{n,k}(\eta,\Lambda) \mid \mathcal{H}_n) \leq C_8(d)\Big(\frac{d}{2}\Big)^{2k}(|\Lambda|^{d/(d-1)} + 1).$$

Together with (3.12) this implies (3.10) for all $\Lambda \in H_n$ with $|\Lambda| \geq \alpha n$. ■

Proof of Theorem 2. Let $\xi, \eta \in \Xi$ be independent and with distribution ρ. Fix n and $x_1, \dots, x_n$ for the time being. Abbreviate $(x_1, \dots, x_n)$ to $x^{(n)}$, and let $\Lambda_n(\xi)$ be the collection of endpoints in H_n of oriented paths from the origin to H_n along which $x^{(n)}$ is seen in the scenery ξ. $\Lambda_n(\eta)$ is defined similarly with ξ replaced by η. By definition $|\Lambda_n(\xi)| = K_n(x^{(n)};\xi)$, and essentially as in (3.11) we have

$$\begin{aligned}
\rho\{N_{0,n+k}(x_1,\dots,x_{n+k};\xi,\{\mathbf{0}\}) \mid \mathcal{H}_n\} &= \Big(\frac{d}{2}\Big)^k N_{0,n}(x^{(n)};\xi,\{\mathbf{0}\}) \\
&\geq \Big(\frac{d}{2}\Big)^k K_n(x^{(n)};\xi) \qquad (3.14)
\end{aligned}$$

and the same relations hold when ξ is replaced by η.

Next, we write $\mathcal{H}_n \times \mathcal{H}_n$ for the σ–field in $\Xi \times \Xi$ generated by the coordinate functions $w \to \xi(w)$, $w \to \eta(w)$ for all w with $\|w\|_1 \leq n$. We also use that

$$N_{0,n+k}(x_1,\dots,x_{n+k};\xi,\{\mathbf{0}\}) \geq N_{n,k}(x_{n+1},\dots,x_{n+k};\xi,\Lambda_n(\xi)),$$

as is evident from the definitions. Therefore, by Lemma 7, we have on the event

$$\begin{aligned}\Theta_n(\alpha) &:= \{|\Lambda_n(\xi)| \geq \alpha n, |\Lambda_n(\eta)| \geq \alpha n\}\\ &= \{K_n(x^{(n)};\xi) \geq \alpha n, K_n(x^{(n)};\eta) \geq \alpha n\}, \end{aligned} \tag{3.15}$$

that

$$\begin{aligned}&(\rho\times\rho)\Big\{N_{0,n+k}(x_1,\dots,x_{n+k};\xi,\{\mathbf{0}\}) \leq (1-\varepsilon)\Big(\frac{d}{2}\Big)^k K_n(x^{(n)};\xi)\\ &\qquad \text{or } N_{0,n+k}(x_1,\dots,x_{n+k};\eta,\{\mathbf{0}\}) \leq (1-\varepsilon)\Big(\frac{d}{2}\Big)^2 K_n(x^{(n)};\eta) \mid \mathcal{H}_n\times\mathcal{H}_n\Big\}\\ &\quad\leq 2\frac{C_3}{\varepsilon^2}(\alpha n)^{-(d-2)/(d-1)}. \end{aligned}\tag{3.16}$$

For the remainder of this section we fix $\varepsilon = \alpha = 1/4$ and then n such that the right hand side of (3.6) is at most $1/2$. Then

$$\begin{aligned}&(\rho\times\rho)\Big\{\frac{1}{d^{n+k}}\sum_{x_{n+1},\dots,x_{n+k}}|N_{0,n+k}(x_1,\dots,x_{n+k};\xi,\{\mathbf{0}\})\\ &\qquad\qquad - N_{0,n+k}(x_1,\dots,x_{n+k};\eta,\{\mathbf{0}\})| \;\Big|\; \mathcal{H}_n\times\mathcal{H}_n\Big\}\\ &= \frac{2^k}{d^{n+k}}(\rho\times\rho)\{|N_{0,n+k}(x^{(n)},1,\dots,1;\xi,\{\mathbf{0}\})\\ &\qquad\qquad - N_{0,n+k}(x^{(n)},1,\dots,1;\eta,\{\mathbf{0}\})| \;\big|\; \mathcal{H}_n\times\mathcal{H}_n\}\\ &= \frac{2^k}{d^{n+k}}(\rho\{N_{0,n+k}(x^{(n)},1,\dots,1;\xi,\{\mathbf{0}\}) \mid \mathcal{H}_n\}\\ &\qquad\qquad + \rho\{N_{0,n+k}(x^{(n)},1,\dots,1;\eta,\{\mathbf{0}\} \mid \mathcal{H}_n\}\\ &\qquad - 2(\rho\times\rho)\{N_{0,n+k}(x^{(n)},1,\dots,1;\xi,\{\mathbf{0}\})\\ &\qquad\qquad \wedge N_{0,n+k}(x^{(n)},1,\dots,1;\eta,\{\mathbf{0}\})\} \mid \mathcal{H}_n\times\mathcal{H}_n\})\\ &\leq \frac{2^k}{d^{n+k}}\Big(\Big(\frac{d}{2}\Big)^k N_{0,n}(x^{(n)};\xi,\{\mathbf{0}\}) + \Big(\frac{d}{2}\Big)^k N_{0,n}(x^{(n)};\eta,\{\mathbf{0}\})\\ &\qquad - 2\Big(\frac{d}{2}\Big)^k\frac{3}{4}K_n(x^{(1)};\xi)\wedge K_n(x^{(n)};\eta)\\ &\qquad\quad (\rho\times\rho)\{N_{0,n+k}(x^{(n)},1,\dots,1;\xi,\{\mathbf{0}\}) \geq \Big(\frac{d}{2}\Big)^k\frac{3}{4}K_n(x^{(n)};\xi)\\ &\qquad\quad \text{and } N_{0,n+k}(x^{(n)},1,\dots,1;\eta,\{\mathbf{0}\}) \geq \Big(\frac{d}{2}\Big)^k\frac{3}{4}K_n(x^{(n)};\eta) \mid \mathcal{H}_n\times\mathcal{H}_n\}\Big)\\ &\leq \frac{1}{d^n}\Big(N_{0,n}(x^{(n)};\xi,\{\mathbf{0}\}) + N_{0,n}(x^{(n)};\eta,\{\mathbf{0}\}) - 2\cdot\frac{3}{4}\cdot\frac{1}{4}n\cdot\frac{1}{2}I\Big(\Theta_n\Big(\frac{1}{4}\Big)\Big)\Big)\end{aligned}$$

(by (3.16) and our choice of ε, α, and n). Consequently

$$\begin{aligned}(\rho\times\rho)\Big\{\frac{1}{d^{n+k}}&\sum_{x_1,\dots,x_{n+k}}|N_{0,n+k}(x_1,\dots,x_{n+k};\xi,\{\mathbf{0}\})\\&\qquad\qquad-N_{0,n+k}(x_1,\dots,x_{n+k};\eta,\{\mathbf{0}\})|\\&\le\frac{1}{d^n}\sum_{x_1,\dots,x_n}(\rho\{N_{0,n}(x^{(1)};\xi,\{\mathbf{0}\})\}+\rho\{N_{0,n}(x^{(n)};\eta,\{\mathbf{0}\})\})\\&\qquad-\frac{3}{16}\,\frac{n}{d^n}\sum_{x_1,\dots,x_n}(\rho\times\rho)\Big\{K_n(x^{(n)};\xi)\ge\frac{1}{4}n\text{ and }K_n(x^{(n)};\eta)\ge\frac{1}{4}n\Big\}\\&=2-\frac{3}{16}\,\frac{n}{d^n}\sum_{x_1,\dots,x_n}\Big(\rho\Big\{K_n(x^{(n)};\xi)\ge\frac{1}{4}n\Big\}\Big)^2.\end{aligned}\tag{3.17}$$

Finally, by Lemma 6, we have uniformly in $x^{(n)}$

$$\begin{aligned}\rho\Big\{K_n(x^{(n)};\xi)\ge\frac{1}{4}n\Big\}&=\rho\{K_n(x^{(n)};\xi)\ge1\}-C_1e^{-C_2n}\\&=\rho\{K_n(1,1,\dots,1;\xi)\ge1\}-C_1e^{-C_2n}\\&=\rho\{\tau(1,1,\dots,\xi)>n\}-C_1e^{-C_2n}.\end{aligned}\tag{3.18}$$

But, as $n\to\infty$,

$$\begin{aligned}\rho\{\tau(1,1,\dots,\xi)>n\}-C_1e^{-C_2n}&\to\rho\{\tau(1,1,\dots,\xi)=\infty\}\\&=\rho\{(1,1,\dots,\xi)\text{ is seen along some oriented path starting at the origin}\}.\end{aligned}$$

The last probability is just the probability that oriented percolation occurs on $(\mathbb{Z}_+)^d$, and it is known that this is strictly positive when $\rho\{\xi(n)=1\}=\frac{1}{2}$ and d is large enough (see Cox and Durrett (1983)). Thus n can be increased, if necessary, so that (3.18) is $\ge$ some constant $C_9>0$, and then, uniformly in k,

$$\begin{aligned}(\rho\times\rho)\Big\{\frac{1}{d^{n+k}}&\sum_{x_1,\dots,x_{n+k}}|N_{\mathbf{0},n+k}(x_1,\dots,x_{n+k};\xi,\{\mathbf{0}\})\\&\qquad\qquad-N_{0,n+k}(x_1,\dots,x_{n+k};\eta,\{\mathbf{0}\})|\Big\}\\&\le2-\frac{3n}{16}\Big(\frac{2}{d}\Big)^nC_9.\end{aligned}\tag{3.19}$$

By means of (3.4), the left hand side of (3.19) can be written as

$$(\rho\times\rho)\Big\{\sum_{x_1,\dots,x_{n+k}}|P\{\xi(S_j^{or})=x_j,1\le j\le n+k\}-P\{\eta(S_j^{or})=x_j,1\le j\le n+k\}|\Big\}.$$

But

$$\sum_{x_1,\dots,x_{n+k}}|P\{\xi(S_j^{or})=x_j,1\le j\le n+k\}-P\{\eta(S_j^{or})=x_j,1\le j\le n+k\}|$$

is clearly increasing in k, and has limit 2 for each pair (ξ,η) for which ξ and η are distinguishable. This is because

$$\frac{P\{S^{or}(\xi_j)=x_j, 1\le j\le n\}}{P\{S^{or}(\xi_j)=x_j, 1\le j\le n\}+P\{S^{or}(\eta_j)=x_j, 1\le j\le n\}}$$
$$\to \frac{dQ_1^\xi}{d(Q_1^\xi+Q_1^\eta)} \quad \text{a.e.} \quad [Q_1^\xi+Q_1^\eta]$$

(see Doob (1953), Sect. VII.8; in any case this is probably well known). Thus it follows that

$$(\rho\times\rho)\{(\xi,\eta):\xi \text{ and } \eta \text{ are not distinguishable}\}\ge \frac{1}{2}\,\frac{3n}{16}\Big(\frac{2}{d}\Big)^n C_9>0.$$

It then follows from Kolmogorov's $0-1$ law that ξ and η are not distinguishable a.e. $[\rho\times\rho]$, because the event $\{(\xi,\eta):\xi$ and η are distinguishable$\}$ is a tail event in $\Xi\times\Xi$. In fact, since $\|S_n^{or}\|_1\ge \ell$ for $n\ge\ell$, the distribution of $\{\xi(S_n),\eta(S_n)\}_{n\ge\ell}$, (i.e., $Q_\ell^\xi\times Q_\ell^\eta$) does not depend on the values of $(\xi(w),\eta(w))$ for $\|w\|_1<\ell$. ■

4. Observations in a sausage. We give here the **proof of Theorem 4**. Throughout $\mathcal{G}=\mathbb{Z}^d$, and the observations consist of the vectors $\{X_n(\xi,k)\}_{n\ge0}$ or $\{X_n(\eta,k)\}_{n\ge0}$ (see (1.4)). We may regard $X_n(\xi,k)$ or $X_n(\eta,k)$ as an element of $\{0,1\}^{(2k+1)^d}$, since $[-k,k]^d$ contains $(2k+1)^d$ lattice points. Throughout this section we take $d\ge3$, because for $d=1,2$ Theorem 4 is contained in Theorem 1. In fact the latter states that the environments ξ and η can already be distinguished on the basis of the observations $\{X_n(\xi,0)\}_{n\ge0}$ or $\{X_n(\eta,0)\}_{n\ge0}$. They are then obviously k–distinguishable for any k.

Now note that there are only $(2d)^n$ possible paths for $S_0=\mathbf{0},S_1,\dots,S_n$. Therefore, when ξ is fixed, at most $(2d)^n$ sequences $(x_0,x_1,\dots,x_n)$ with $x_i\in\{0,1\}^{(2k+1)^d}$ are possible for the sequence of outcomes $(X_0(\xi,k),X_1(\xi,k),\dots,X_n(\xi,k))$. Denote the collection of these possible outcome sequences by $\Gamma(\xi,k,n)$. We then have

$$|\Gamma(\xi,k,n)| := \text{cardinality of } \Gamma(\xi,k,n)\le(2d)^n. \tag{4.1}$$

For $x_i\in\{0,1\}^{(2k+1)^d}$, define

$$\pi_\ell(x_0,\dots,x_n)=(x_\ell,\dots,x_n).$$

Thus π_ℓ is projection on the last $n-\ell+1$ coordinates. By definition

$$Q_\ell^{\xi,k}\{\{X_j(\xi,k)\}_{\ell\le j\le n}\in\pi_\ell\Gamma(\xi,k,n)\}=1.$$

Therefore, to show that $Q_\ell^{\xi,k}$ and $Q_\ell^{\eta,k}$ are mutually singular for almost all η, it suffices to show that for almost all η

$$P\{\{X_j(\eta,k)\}_{\ell\le j\le n}\in\pi_\ell\Gamma(\xi,k,n)\}\to0 \quad (n\to\infty). \tag{4.2}$$

In order to prove (4.2) we introduce

$$R_{\ell,n} = \bigcup_{j=\ell}^{n} \{S_j + [-k,k]^d\}.$$

$R_{\ell,n}$ is the "sausage" swept out by moving the cube $[-k,k]^d$ along the random walk path over the time interval $[\ell, n]$. Let $|R_{\ell,n}|$ be the cardinality $R_{\ell,n}$. Spitzer (1964), Problem 32.10, or (1973), proved that if $d \geq 3$, then

$$\lim_{n\to\infty} \frac{R_{\ell,n}}{n} = \lim_{n\to\infty} \frac{R_{0,n}}{n} = C([-k,k]^d) \text{ a.e. } [P], \tag{4.3}$$

where $C(A)$ denotes the capacity of the set A, and is defined in Spitzer (1964), Definition 25.3. The precise value of $C([-k,k]^d)$ is not important for us; we merely need to know that

$$\lim_{k\to\infty} C([-k,k]^d) = \infty.$$

In fact, the argument on p. 181 of Kesten (1987) shows that for some constant $C_{10} = C_{10}(d) > 0$

$$C([-k,k]^d) \geq C_{10}(k^{d-2}+1). \tag{4.4}$$

(4.3) and (4.4) show that

$$P\Big\{|R_{\ell,n}| \leq \frac{1}{2} C_{10} k^{d-2} n\Big\} \to 0 \quad (n\to\infty),$$

so that (4.2) only requires us to prove

$$P\Big\{\{X_j(\eta,k)\}_{\ell\leq j\leq n} \in \pi_\ell \Gamma(\xi,k,n), |R_{\ell,n}| > \frac{1}{2} C_{10} k^{d-2} n\Big\} \to 0 \quad \text{as} \quad n\to\infty \tag{4.5}$$

for a suitably large k and almost all η.

To prove (4.5) we take the expectation with respect to ρ of the left hand side of (4.5). This expectation is at most

$$\sum_{(x_\ell,\dots,x_n)\in\pi_\ell\Gamma} \sum_{v_\ell,\dots,v_n} \rho\{P\{S_j = v_j, X_j(\eta,k) = x_j, \ell\leq j\leq n\}\},$$

where the sum over $(v_\ell,\dots,v_n)$ is only over paths $(v_\ell,\dots,v_n)$ on $\mathbb{Z}^d$ for which

$$\Big|\bigcup_{j=\ell}^{n} \{v_j + [-k,k]^d\}\Big| > \frac{1}{2} C_{10} k^{d-2} n. \tag{4.6}$$

But if we specify that $X_j(\eta,k) = \{\eta(S_j+w) : \|w\|_1 \leq k\} = x_j$ and $S_j = v_j$, $\ell\leq j\leq n$, then we actually specify $\eta(w)$ for all w in

$$\bigcup_{j=\ell}^{n} \{v_j + [-k,k]^d\}.$$

If (4.6) holds, this proves that η is specified on at least $C_{10}k^{d-2}n/2$ vertices. Consequently

$$\begin{aligned}&\rho\{P\{S_j = v_j, X_j(\eta, k) = x_j, \ell \le j \le n\}\}\\&\quad = P\{S_j = v_j, \ell \le j \le n\}\rho\{\{\eta(v_j + w) : \|w\|_1 \le k\} = x_j, \ell \le j \le n\}\\&\quad \le P\{S_j = v_j, \ell \le j \le n\}2^{-C_{10}k^{d-2}n/2},\end{aligned}$$

whenever (4.6) holds. But then

$$\begin{aligned}&\rho\Big\{P\{X_j(\eta,k)\}_{\ell\le j\le n} \in \pi_\ell\Gamma(\xi,k,n), |R_{\ell,n}| > \frac{1}{2}C_{10}k^{d-2}n\Big\}\\&\quad\le \sum_{(x_\ell,\dots,x_n)\in\pi_\ell\Gamma} 2^{-C_{10}k^{d-2}n/2}\\&\quad\le |\Gamma(\xi,k,n)|2^{-C_{10}k^{d-2}n/2} \le (2d)^n 2^{-C_{10}k^{d-2}n/2}.\end{aligned}$$

Finally, if we take k so large that

$$(2d)2^{-C_{10}k^{d-2}/2} < 1, \tag{4.7}$$

then

$$\sum_n \rho\Big\{P\{X_j(\eta,k)\}_{\ell\le j\le n} \in \pi_\ell\Gamma(\xi,k,n), |R_{\ell,n}| > \frac{1}{2}C_{10}k^{d-2}n\Big\} < \infty$$

so that (4.5) holds for almost all $\eta[\rho]$.

As argued before we have now proven that for all ℓ, and for almost all η $[\rho]$, $Q_\ell^{\xi,k}$ and $Q_\ell^{\eta,k}$ are mutually singular (under (4.7)). This proves Theorem 4. ■

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Department of Statistics
Evans Hall
University of California
Berkeley, CA 94720

Department of Mathematics
White Hall
Cornell University
Ithaca, NY 14853