Group-Invariant Percolation on Graphs

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Abstract. Let G be a closed group of automorphisms of a graph X. We relate geometric properties of G and X, such as amenability and unimodularity, to properties of G-invariant percolation processes on X, such as the number of infinite components, the expected degree, and the topology of the components. For example, G is amenable iff for all $\alpha < 1$, there is a G-invariant site percolation ω on X with $\mathbf{P}[x \in \omega] > \alpha$ for all vertices x and with no infinite components. When G is not amenable, a threshold $\alpha < 1$ appears. We obtain an inequality for the threshold in terms of the isoperimetric constant, extending an inequality of Häggström for regular trees.

If G acts transitively on X, we show that G is unimodular iff the expected degree is at least 2 in any G-invariant bond percolation on X with all components infinite.

The investigation of dependent percolation yields results on automorphism groups of graphs that do not involve percolation, as well as new insights concerning Bernoulli percolation. For instance, if X is a transitive graph with a nonamenable automorphism group and p-Bernoulli percolation on X produces a unique infinite component a.s., then $p > p_c(X)$.

§1. Introduction.

There are intimate connections between the geometry of a space or a group and the behavior of stochastic processes in it. A well-known example is Kesten's (1959a, b) theorem that a countable group G is amenable iff some (or every) symmetric group-invariant random walk with support generating G has spectral radius 1. Here, we study the relations between a group G acting on a graph X and G-invariant percolation processes on X.

Let G be a finitely generated group and let $X(G) = (V, E)$ be its right Cayley graph with respect to a (finite) set of generators. Note that left multiplication by G gives an action on X by automorphisms. We call a random subset ω of V a **site percolation**. If the distribution of ω is invariant under the action of G, we call the percolation **G-invariant**.

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The usual case of **Bernoulli** percolation, in which the events $x \in \omega$ are independent and have the same probability for all vertices x, is, of course, an example of invariant percolation. A **component** of ω is a connected component for the subgraph of X gotten by adding to ω all edges in E whose endpoints are in ω. Denote by o the vertex in X corresponding to, say, the identity in G.

The following basic result is an analogue for group-invariant percolation of Kesten's spectral radius theorem.

THEOREM 1.1. *Let $X = X(G)$ be the Cayley graph of a finitely generated group G. Then G is amenable iff for all $\alpha < 1$, there is a G-invariant site percolation on X with $\mathbf{P}[o \in \omega] > \alpha$ and with no infinite components.*

The "only if" part of Theorem 1.1 follows from the Rokhlin lemma of Ornstein and Weiss (1987), and the method of Adams and Lyons (1991) can be used to prove the other direction. We will give another proof and a generalization in Theorem 5.1.

A quantitative form of Kesten's equivalence was provided by Cheeger (1970) in the continuous setting; he gave a lower bound for the bottom of the spectrum of the Laplacian in terms of the isoperimetric (or "Cheeger") constant. In the discrete case, such a bound is equivalent to an upper bound on the spectral radius of the transition operator P, since the Laplacian is $I - P$. Cheeger's result was transferred to the discrete setting in various contexts by Dodziuk (1984), Dodziuk and Kendall (1986), Varopoulos (1985), Ancona (1988), Gerl (1988), Biggs, Mohar, and Shawe-Taylor (1988), and Kaimanovich (1992).

We have found a quantitative version of Theorem 1.1, which we now present. Given $K \subset V$, set $\partial_V K := \{y \notin K\,;\ \exists x \in K,\ x \sim y\}$ and $\partial_E K := \{(x, y)\,;\ x \sim y, x \in K, y \notin K\}$. Define the **vertex-isoperimetric constant** of X by

$$\iota_V(X) := \inf\left\{\frac{|\partial_V K|}{|K|}\,;\ K \subset V \text{ is finite}\right\},$$

and let the **edge-isoperimetric constant** of X be

$$\iota_E(X) := \inf\left\{\frac{|\partial_E K|}{|K|}\,;\ K \subset V \text{ is finite}\right\}.$$

Let d be the degree of vertices in the Cayley graph $X(G)$, i.e., the number of generators of G (and their inverses) used to define the Cayley graph. We also consider **bond percolation**, i.e., random subsets of E. (Note that every site percolation induces a bond percolation consisting of the edges joining the sites that are present.)

THEOREM 1.2. *If* $\mathbf{P}$ *is a* G*-invariant site percolation on* $X(G)$ *and* $\mathbf{P}[o \in \omega] \ge d/(d + \iota_V(X))$*, then there is an infinite component with positive probability. For* G*-invariant bond percolation, the same conclusion holds if the expected degree of each vertex is at least* $d - \iota_E(X)$*.*

(See Theorems 2.10 and 2.3.)

The key tool in proving Theorem 1.2, and many other results below, is a "mass-transport" technique due to Häggström (1996), which is a method for averaging on nonamenable graphs. He proved Theorem 1.2 for trees X and showed that the thresholds are sharp there. Our Mass-Transport Principle is given in Section 3.

For an example of invariant but not Bernoulli percolation, consider a Cayley graph X and a positive integer r. Independently color each site red with probability 1/2. Now let ω consist of those sites x such that the proportion of red sites within distance r of x is between 40% and 60%. It follows from Theorem 1.2 that if G is nonamenable, then there is an easily estimable r such that there is an infinite component of ω with positive probability. For example, if $d/(d + \iota_V(X)) \le 0.95$, then it suffices to take r so that the number of group elements within distance r of the identity is at least 100.

Other examples of group-invariant percolation include the connected components of plus spins in Ising models; random spanning forests of various kinds (see, e.g., Pemantle (1991), Alexander (1995), Häggström (1997), and Benjamini, Lyons, Peres, and Schramm (1997)); and the random cluster model (see, e.g., Grimmett (1995)).

A graph X that does not necessarily arise as a Cayley graph is called **amenable** if $\iota_V(X) = 0$. (All graphs are assumed without further comment to be countable and locally finite.) We say that a group $G \subseteq \mathrm{Aut}(X)$ is **transitive** or **acts transitively** if $|G\backslash V| = 1$; we say that G is **quasi-transitive** or **acts quasi-transitively** if $|G\backslash V| < \infty$. (The set of vertex G-orbits is denoted $G\backslash V$.) In these cases, one is often interested in G-invariant random walks on X. Invariant random walks can have spectral radius 1 even if X is not amenable. This phenomenon has been studied by Soardi and Woess (1990), Salvatori (1992), and Saloff-Coste and Woess (1996). Here, unimodularity of G plays an essential role. Likewise, Theorems 1.1 and 1.2 extend to the quasi-transitive case and unimodularity plays a key role; see Theorems 4.1 and 4.4. A review of unimodularity and amenability for automorphism groups is given in Section 3.

One result we use, which is interesting in itself, is that no finite subset achieves the isoperimetric constant in transitive graphs:

PROPOSITION 1.3. *If* X *is an infinite transitive graph, then for all finite* $K \subset V$*, we have* $|\partial_E K|/|K| > \iota_E(X)$ *and* $|\partial_V K|/|K| > \iota_V(X)$*.*

(See Corollaries 2.2 and 2.9.)

We also give the following characterization of amenability that involves the type of spanning subgraphs on which a group-invariant distribution can be carried. (A subgraph is **spanning** if it is connected and includes all vertices.)

THEOREM 1.4. *Let G be a closed unimodular subgroup of* $\mathrm{Aut}(X)$ *that acts transitively on X. Then the following are equivalent:*

(i) X is amenable;

(ii) there is a G-invariant random spanning tree of X with at most 2 ends a.s.;

(iii) there is a G-invariant random nonempty connected subgraph of X with $p_c = 1$ with positive probability.

(See Theorem 5.3.) It is still unsettled whether Theorem 1.4 holds without the assumption that G is unimodular.

In Section 6, we characterize unimodularity in terms of the expected degree of vertices in infinite components. Since any connected finite graph has average degree at least $2 - 2/|V|$, one might expect that for G-invariant percolation on a transitive graph X with all components infinite a.s., the expected degree of a vertex is at least 2. This inequality is true when the group G is unimodular, but surprisingly, whenever G is not unimodular, there is a G-invariant percolation where the inequality fails. (See Theorem 6.1.)

In Section 7, we find a relationship between the number of ends of the components in invariant percolation and the expected degree of vertices (where "degree" refers to the degree in the percolation). Suppose that G acts transitively on X and is unimodular. Let ω be a G-invariant bond percolation on X. In Corollary 7.3, we show that if a.s. every component of ω is an infinite tree with finitely many ends, then the expected degree of every vertex is 2.

In Section 8, we consider a vertex-transitive tiling X of the hyperbolic plane, and show that the number of infinite components of any automorphism-invariant bond percolation on X is either 0, 1 or ∞. Consequently, in any automorphism-invariant 2-coloring of the vertices of X, if both colors have infinite components, then a.s. at least one color must have infinitely many infinite clusters. For Bernoulli percolation on tilings of large genus, this was proved by Lalley (1996).

The history of the study of group-invariant percolation is as follows. That the usual independence assumption in percolation theory can be profitably relaxed to group invariance was demonstrated in the remarkable proof of uniqueness of the infinite cluster, discovered by Burton and Keane (1989). Although they considered only the case of Euclidean lattices, their proof is valid in any amenable transitive graph, provided that cubes are replaced by

Følner sets. Topological consequences of invariance for percolation in $\mathbb{Z}^2$, were explored by Burton and Keane (1991). The first analysis of percolation on a nonamenable graph that is not a tree was that of Bernoulli percolation on the Cartesian product of a regular tree and the integers, studied by Grimmett and Newman (1990). They proved that for some $p > p_c$, multiple infinite clusters coexist. As a consequence of a method for studying random walks, Lyons (1995) gave a threshold for Bernoulli percolation on groups of exponential growth: if $p > \lim |B_n|^{-1/n}$, then there is an infinite component with positive probability in both Bernoulli site and bond percolation with survival parameter p; here B_n is the ball of radius n. Benjamini and Schramm (1996a) gave a threshold for Bernoulli percolation on general nonamenable graphs: if $p > 1/(1 + \iota_V(X))$, then there is an infinite component with positive probability in Bernoulli site percolation with parameter p. Likewise, if $p > 1/(1+\iota_E(X))$, then there is an infinite component with positive probability in Bernoulli bond percolation with parameter p. As we already mentioned, Häggström (1996) proved Theorem 1.2 for regular trees. His paper was the original impetus for the present work. Other studies of percolation in nonamenable settings, particularly in the hyperbolic plane, were carried out by Lalley (1996) and by Benjamini and Schramm (1996c).

All three thresholds that we have mentioned (of Lyons, of Benjamini and Schramm, and of Häggström) are sharp for trees. We do not know whether the threshold for bond percolation in Theorem 1.2 is sharp for every nonamenable Cayley graph.

§2. Nonamenable Cayley Graphs.

We shall treat first the case of bond percolation on Cayley graphs, as the mass-transport method is particularly transparent there.

For an edge $e = (u, v)$, write $\check{e} := (v, u)$. We call a function θ on E **antisymmetric** if $\theta(u, v) = -\theta(v, u)$ for all (oriented) edges $e = (u, v)$. Set $d^*\theta(v) := \sum_{u \sim v} \theta(v, u)$.

The following proposition, modelled on Proposition 2.8 below from Benjamini and Schramm (1996b), dualizes the inf in $\iota_E(X)$ not only to a sup, but to a max.

PROPOSITION 2.1. *For any graph X, we have*

$$\iota_E(X) = \max\{\alpha \geq 0\,;\ \exists\theta\ \forall e\ |\theta(e)| \leq 1 \text{ and } \forall v\ d^*\theta(v) = \alpha\}\,,$$

where θ runs over the antisymmetric functions on E.

For the proof, we need some of the standard notions for flows in networks. A **network** is a *directed* graph X, a pair of distinguished vertices of X, s_0 and s_1, called the **source** and the **sink**, respectively, and a **capacity** function on the oriented edges $C : E \to [0, \infty]$.

A **network flow** is a function $\theta : E \to \mathbb{R}$ satisfying $0 \le \theta(e) \le C(e)$ and $d^*\theta(v) = 0$ for all $v \ne s_0, s_1$. The **strength** of the network flow is $d^*\theta(s_0)$. A **cut** of the network is a set Π of oriented edges such that there is no oriented path from s_0 to s_1 in $X \setminus \Pi$. Ford and Fulkerson's (1962) Max-Flow Min-Cut Theorem says that in a finite network, the maximum strength of a network flow is equal to $\min_\Pi \sum_{e\in\Pi} C(e)$, where Π ranges over all cuts of the network.

Proof of Proposition 2.1. Suppose that $\theta : E \to \mathbb{R}$ is antisymmetric and satisfies $\forall e\ |\theta(e)| \le 1$ and $\forall v\ d^*\theta(v) = \alpha$. Let K be finite. Then

$$\alpha|K| = \sum_{v\in K} d^*\theta(v) = \sum_{e\in\partial_E K} \theta(e) \le |\partial_E K| \,.$$

This shows that $\alpha \le \iota_E(X)$ and (for later use) that $\theta(e) = 1$ for all $e \in \partial_E K$ if $\iota_E(X) = |\partial_E K|/|K|$. Consequently,

$$\iota_E(X) \ge \max\{\alpha \ge 0 \,;\ \exists\theta\ \forall e\ |\theta(e)| \le 1 \text{ and } \forall v\ d^*\theta(v) = \alpha\} \,.$$

Now suppose that $\alpha \le \iota_E(X)$. For any finite $K \subset V$, define the network X_K with vertices K and two extra vertices, the source, s_0, and the sink, s_1. The edges of X_K are as follows: For each edge in X with both endpoints x, y in K, put both oriented edges (x, y) and (y, x) in X_K; for each edge $(x, y) \in \partial_E K$, put the edge (x, s_1) in X_K; and for each $x \in K$, put the edge (s_0, x) in X_K. Give all edges e incident to s_0 capacity $C(e) := \alpha$ and all other edges capacity $C(e) := 1$.

We claim that there is a network flow from s_0 to s_1 in X_K with strength $\alpha|K|$. Consider any cut Π separating s_0 from s_1. Let K' be the vertices in K that Π separates from s_1. Then $\partial_E K' \subseteq \Pi$ and $(s_0, v) \in \Pi$ for all $v \in K \setminus K'$. Therefore,

$$\sum_{e\in\Pi} C(e) \ge |\partial_E K'| + \alpha|K \setminus K'| \ge \iota(X)|K'| + \alpha|K \setminus K'| \ge \alpha|K| \,.$$

Thus, the claim follows from the Max-Flow Min-Cut Theorem. Note that the flow along (s_0, v) is α for every $v \in K$.

Now let K_n be finite sets increasing to V and let θ_n be the corresponding flows of strength $\alpha|K_n|$ on X_{K_n}. There is a subsequence $\langle n_i\rangle$ such that for all edges $e \in E$, the limit $\theta(e) := \theta_{n_i}(e)$ exists; clearly, $\forall e\ |\theta(e)| \le 1$ and $\forall v\ d^*\theta(v) = \alpha$. ∎

The following consequence appears to be new:

Corollary 2.2. *If X is an infinite transitive graph, then for all finite $K \subset V$, we have $|\partial_E K|/|K| > \iota_E(X)$.*

Proof. Let θ be an antisymmetric function on E with $|\theta(e)| \le 1$ for all edges e and $d^*\theta(v) = \iota_E(X)$ for all vertices v. Let $K \subset V$ be finite. Suppose for a contradiction that $|\partial_E K|/|K| = \iota_E(X)$. We may assume that θ is acyclic on K, i.e., that there is no cycle of oriented edges connecting vertices in K on each of which $\theta > 0$. (Otherwise, we modify θ by subtracting appropriate cycles.) The proof of Proposition 2.1 shows that for all $e \in \partial_E K$, we have $\theta(e) = 1$; in particular, $\theta(e) > 0$. Let $(v_1, v_0) \in \partial_E K$ and let g be an automorphism of X that carries v_0 to v_1. Write v_2 for the image of v_1 and gK for the image of K. Since we also have $|\partial_E(gK)|/|gK| = \iota_E(X)$ and $(v_2, v_1) \in \partial_E(gK)$, it follows that $\theta(v_2, v_1) > 0$. We may similarly find v_3 such that $\theta(v_3, v_2) > 0$ and so on, until we arrive at some v_k that equals some previous v_j or is outside K. Both lead to a contradiction, the former contradicting the acyclicity of θ, the latter contradicting the fact that on all edges leading out of K, we have $\theta > 0$. ∎

For a finite subgraph $K \subset X$, set

$$\alpha_K := \frac{1}{|K|} \sum_{x \in K} \deg_K(x),$$

where $\deg_K(x)$ refers to the degree of x as a vertex in the graph K. Let

$$\alpha(X) := \sup\{\alpha_K\,;\ K \subset X \text{ is finite}\}.$$

If X is a regular graph of degree d, then

$$\alpha(X) + \iota_E(X) = d.$$

Let d be the number of generators of G, i.e., the degree of vertices in its Cayley graph X. Let o denote the identity of G as a vertex in X. Let $D(x)$ be the random degree of x in the percolation. Here, we consider a G-invariant bond percolation, so that all $D(x)$ are identically distributed; let D be a random variable with this same distribution. Write $K(x)$ for the component of x in the percolation.

For a connected graph X, its critical probability $p_c(X)$ for Bernoulli (site or bond) percolation is defined to be the supremum of those p for which percolation with survival parameter p has a.s. no infinite components.

We are now ready to prove our first threshold theorem:

THEOREM 2.3. *If $\mathbf{P}$ is a bond percolation on a Cayley graph $X = X(G)$ whose distribution is G-invariant and $\mathbf{E}D \ge \alpha(X)$, then there is an infinite component with positive probability. Moreover, if $\mathbf{E}D > \alpha(X)$, then with positive probability, there is a component with (bond) $p_c < 1$.*

REMARK 2.4. The intuition is that if $\mathbf{P}$ gives a high mean degree, then it cannot be supported only on finite components with smaller average internal degree.

REMARK 2.5. Functions other than the degree can be used for thresholds, such as square degree.

REMARK 2.6. If the percolation is such that the components are guaranteed to belong to a class of subgraphs $\mathcal{C}$, then we can use the possibly better threshold $\alpha(X\,;\ \mathcal{C}) := \sup\{\alpha_K\,;\ K \in \mathcal{C} \text{ is finite}\}$. For example, if $\mathcal{C}$ contains only trees, then it is enough that $\mathbf{E}D \ge 2$.

Proof of Theorem 2.3. We put mass $D(x)$ at each $x \in V$. All masses of vertices in infinite components are left alone, but the mass of a vertex in a finite component is redistributed equally among the vertices in that component (including itself). Thus, after redistribution, all vertices in a finite component K have mass α_K. Because $\mathbf{P}$ is G-invariant, this is a G-invariant mass transport, so that if $f(x, y)$ denotes the expected mass taken from x and transported to y, then for all $g \in G$, we have $f(x, y) = f(gx, gy)$. The mass at o before transport has mean $\mathbf{E}D$, whence

$$\mathbf{E}D = \sum_{x\in X} f(o, x)\,.$$

On the other hand, the mass at o after transport has mean

$$\mathbf{E}[D(o)\mathbf{1}_{\{|K(o)|=\infty\}} + \alpha_{K(o)}\mathbf{1}_{\{|K(o)|<\infty\}}] = \sum_{y\in X} f(y, o)\,.$$

By G-invariance of f, we have

$$\sum_{x\in X} f(o, x) = \sum_{g\in G} f(o, go) = \sum_{g\in G} f(g^{-1}o, o) = \sum_{y\in X} f(y, o)\,, \tag{2.1}$$

whence (using Corollary 2.2)

$$\begin{aligned}\mathbf{E}D = \mathbf{E}[D(o)\mathbf{1}_{\{|K(o)|=\infty\}} + \alpha_{K(o)}\mathbf{1}_{\{|K(o)|<\infty\}}] &< \mathbf{E}[d\mathbf{1}_{\{|K(o)|=\infty\}} + \alpha(X)\mathbf{1}_{\{|K(o)|<\infty\}}]\\ &= (d - \alpha(X))p_\infty + \alpha(X)\,,\end{aligned}$$

where $p_\infty := \mathbf{P}[|K(o)| = \infty]$. [Actually, the strict inequality fails in the trivial case that $K(o) = X$ a.s.] Since $\alpha(X) < d$, the result follows, together with the estimate

$$p_\infty > \frac{\mathbf{E}D - \alpha(X)}{d - \alpha(X)}\,.$$

If $\mathbf{E}D > \alpha(X)$, then we can intersect with an independent Bernoulli percolation with survival parameter $p' > \alpha(X)/\mathbf{E}D$ and apply the result just proved to obtain that there is still an infinite component with positive probability. This means that $p_c \le \alpha(X)/\mathbf{E}D < 1$ with positive probability. ∎

REMARK 2.7. Although Bernoulli percolation at the thresholds given in Benjamini and Schramm (1996a) can have the property that there are no infinite components a.s. (for example, on a tree), this is not the case with the threshold of Theorem 2.3.

The analogue of Proposition 2.1 for the vertex-isoperimetric constant involves the amount flowing along edges into vertices,

$$\mathsf{flow}_+(\theta, v) := \sum_{u \sim v} \bigl(\theta(u, v) \vee 0\bigr)\,.$$

PROPOSITION 2.8. (BENJAMINI AND SCHRAMM (1996B)) *For any graph X, we have*

$$\iota_V(X) = \max\{\alpha \ge 0\,;\ \exists\theta\ \forall v\ \mathsf{flow}_+(\theta, v) \le 1 \textit{ and } d^*\theta(v) = \alpha\}\,,$$

where θ runs over the antisymmetric functions on E.

COROLLARY 2.9. *If X is an infinite transitive graph, then for all finite $K \subset V$, we have $|\partial_V K|/|K| > \iota_V(X)$.*

Proof. Let θ be an antisymmetric function on E with $\mathsf{flow}_+(\theta, v) \le 1$ and $d^*\theta(v) = \iota_V(X)$ for all vertices v. Let $K \subset V$ be finite. Suppose for a contradiction that $|\partial_V K|/|K| = \iota_V(X)$. It is easy to see that for all $v \in \partial_V K$, we have $\mathsf{flow}_+(\theta, v) = 1$ and $\mathsf{flow}_+(-\theta, v) = \iota_V(X) + 1$; in particular, there is some e leading to v with $\theta(e) \ge 1/d$ and some e leading away from v with $\theta(e) \ge (\iota_V(X) + 1)/d \ge 1/d$, where d is the degree in X. Since X is transitive, the same is true for all $v \in V$. Therefore, we may find either a cycle or a bi-infinite path with all edges e having the property that $\theta(e) \ge 1/d$. We may then subtract $1/d$ from θ along these edges, yielding another function θ' that satisfies $\mathsf{flow}_+(\theta', v) \le 1$ and $d^*\theta'(v) = \iota_V(X)$ for all vertices v. But $\mathsf{flow}_+(\theta', v) < 1$ for some v, a contradiction. ∎

Now we consider site percolation on X.

THEOREM 2.10. *If* $\mathbf{P}$ *is a site percolation on a Cayley graph* $X = X(G)$ *whose distribution is* G*-invariant and* $\mathbf{P}[o \in \omega] \geq d/(d + \iota_V(X))$*, then there is an infinite component with positive probability. Moreover, if* $\mathbf{P}[o \in \omega] > d/(d + \iota_V(X))$*, then with positive probability, there is a component with* $p_c < 1$*.*

REMARK 2.11. If there are triangles in X, then the threshold $d/(d + \iota_V(X))$ may be replaced by $d'/(d' + \iota_V(X))$, where d' is the maximum number of neighbors of o that are pairwise nonadjacent.

Proof. At each $x \in \omega$, begin with mass 1. The masses in infinite components are left alone, but those in a finite component are redistributed equally among those in the vertex boundary of that component. That is, we use the G-invariant mass transport corresponding to the expected mass transfer from x to y of

$$f(x, y) := \mathbf{E}\left[\mathbf{1}_{\{|K(x)|=\infty, y=x\}} + \frac{\mathbf{1}_{\{|K(x)|<\infty, y \in \partial_V K\}}}{|\partial_V K(x)|}\right].$$

The mass at o before transport has mean $\mathbf{P}[o \in \omega]$, which must equal its mean after transport, namely

$$\mathbf{E}\left[\mathbf{1}_{\{|K(o)|=\infty\}} + \sum_{x \sim o} \frac{\mathbf{1}_{\{o \notin \omega, |K(x)|<\infty\}}|K(x)|}{|\partial_V K(x)|}\right] < \mathbf{P}[|K(o)| = \infty] + \frac{d(1 - \mathbf{P}[o \in \omega])}{\iota_V(X)}.$$

Therefore, we get the estimate

$$\mathbf{P}[|K(o)| = \infty] > \mathbf{P}[o \in \omega]\left(1 + \frac{d}{\iota_V(X)}\right) - \frac{d}{\iota_V(X)}.$$

This proves the first assertion; the second follows as in the proof of Theorem 2.3. ∎

§3. Basic Properties of Automorphism Groups.

We start by reviewing the modular function. A compact group has a unique left-invariant Radon probability measure, called Haar measure. It is also the unique right-invariant Radon probability measure. A locally compact group G has a left-invariant σ-finite Radon measure $|\bullet|$; it is unique up to a multiplicative constant. For every $g \in G$, the measure $A \mapsto |Ag|$ is left invariant, whence there is a positive number $m(g)$ such that $|Ag| = m(g)|A|$ for all measurable A. The map $g \mapsto m(g)$ is a homomorphism from G to the multiplicative group of the positive reals and is called the **modular function** of G. If $m(g) = 1$ for every $g \in G$, then G is called **unimodular**. In particular, this is the case if G is countable, where Haar measure is counting measure.

The automorphism group $\mathrm{Aut}(X)$ of a (locally finite) graph X is given the topology of pointwise convergence. If G is a closed subgroup of $\mathrm{Aut}(X)$, then G is locally compact and the stabilizer

$$S(x) := \{g \in G\,;\ gx = g\}$$

of any vertex x is compact.

Note that if $gu = y$, then $S(y) = gS(u)g^{-1}$, whence

$$|S(y)| = |S(u)g^{-1}| = m(g)^{-1}|S(u)|\,. \tag{3.1}$$

Thus, G is unimodular iff for all x and y in the same orbit, $|S(x)| = |S(y)|$.

Unimodularity of a group of graph automorphisms is a simple and natural combinatorial property, as shown by Schlichting (1979) and Trofimov (1985). Namely, if $G \subset \mathrm{Aut}(X)$ is a closed group of automorphisms of a graph X and $|\bullet|$ denotes cardinality (for subsets of X) as well as Haar measure (for subsets of G), then for any vertices $x, y \in X$,

$$|S(x)y|/|S(y)x| = |S(x)|/|S(y)|\,; \tag{3.2}$$

thus, G is unimodular iff for all x and y in the same orbit,

$$|S(x)y| = |S(y)x|\,. \tag{3.3}$$

Of course, when X is the Cayley graph of G, unimodularity is trivial. We note that (3.2) is proved by showing that the number of cosets of $S(x) \cap S(y)$ in $S(x)$ is $|S(x)y|$, whence $|S(x)| = |S(x)y| \cdot |S(x) \cap S(y)|$. See Trofimov (1985) for details.

Here is an example of a transitive graph X whose full automorphism group is unimodular but which has a closed group $G \subset \mathrm{Aut}(X)$ which is not unimodular: Let X be the 3-regular tree. Fix an end ξ of X and let G be the automorphisms preserving ξ.

For an example of a transitive graph X whose full automorphism group is not unimodular, add to the above tree, for each vertex x, the edge between x and its ξ-grandparent. These examples were described by Trofimov (1985).

For an example of a quasi-transitive tree whose full automorphism group is not unimodular, let X be the 3-regular tree again and ξ an end of X. Let the horocycles with respect to ξ be X_n in their natural order. Add a vertex to each edge joining v to w when $v \in X_n$, $w \in X_{n+1}$, and n is even. If X' denotes the resulting tree, then $\mathrm{Aut}(X')$ is nonunimodular and has a finite number of orbits.

Next, we review amenability. Let G be any locally compact group and $L^\infty(G)$ be the Banach space of measurable essentially bounded real-valued functions on G with respect to Haar measure. A linear functional on $L^\infty(G)$ is called a **mean** if it maps the constant function $\mathbf{1}$ to 1 and nonnegative functions to nonnegative numbers. If $f \in L^\infty(G)$ and $g \in G$, we write $L_g f(h) := f(gh)$. We call a mean μ **invariant** if $\mu(L_g f) = \mu(f)$ for all $f \in L^\infty(G)$ and $g \in G$. Finally, we say that G is **amenable** if there is an invariant mean on $L^\infty(G)$. A more concrete yet necessary and sufficient condition for amenability was given by Følner, namely, that for every nonempty compact $B \subset G$ and $\epsilon > 0$, there is a nonempty compact set $A \subset G$ such that $|BA \triangle A| \le \epsilon |A|$; see Paterson (1988), Theorem 4.13. In this case, one often refers informally to A as a Følner set.

For automorphism groups of graphs, amenability has another interpretation:

Lemma 3.1. *Let X be a graph and G be a closed subgroup of* $\mathrm{Aut}(X)$. *Then G is amenable iff X has a G-invariant mean.*

Here, a mean on $\ell^\infty(V)$ is called **G-invariant** if every $f \in \ell^\infty(V)$ has the same mean as does $L_g f$ (defined as taking $x \mapsto f(gx)$).

Proof. This follows from, e.g., Adams and Lyons (1991), Prop. 3.1. We outline a more direct proof here. Fix $o \in V$. Suppose that μ is an invariant mean on $L^\infty(G)$. Given $f \in \ell^\infty(V)$, define $\widehat{f} \in L^\infty(G)$ by $\widehat{f}(h) := f(ho)$. It is straightforward to check that $\widehat{L_g f} = L_g \widehat{f}$ for all $g \in G$, whence the mean $f \mapsto \mu(\widehat{f})$ is G-invariant.

Conversely, suppose that μ is an invariant mean on $\ell^\infty(V)$. Given $f \in L^\infty(G)$, define $\overline{f} \in \ell^\infty(V)$ by $\overline{f}(x) := \int_{ho=x} f(h)\, dh/|S(o)|$, where dh denotes Haar measure. Again, it is easy to check that $\overline{L_g f} = L_g \overline{f}$, so that the mean $f \mapsto \mu(\overline{f})$ is G-invariant. ∎

Soardi and Woess (1990) established the following fundamental theorem for transitive graphs; the extension to quasi-transitive graphs is Theorem 1 of Salvatori (1992).

Theorem 3.2. *Let X be a graph and G be a closed subgroup of* $\mathrm{Aut}(X)$ *with $G \backslash X$ finite. Then X is amenable iff G is amenable and unimodular.*

We shall give a new proof of this later (see Remarks 3.9 and 6.3) and extend it in Theorem 3.7. The original proof of Theorem 3.2 used random walks. Our proof is more direct, depending on the Mass-Transport Principle.

After this review, we now present the very useful Mass-Transport Principle, which replaces the simple (2.1).

Mass-Transport Principle. *If $X = (V, E)$ is any graph, $G \subseteq$* $\mathrm{Aut}(X)$ *is closed, $f : X \times X \to [0, \infty]$ is invariant under the diagonal action of G, and $u, v \in V$, then*

$$\sum_{z \in Gv} f(u, z)|S(v)| = \sum_{y \in Gu} f(y, v)|S(y)|\,.$$

The idea of the following proof is to average over random elements from G that take a given vertex u to another given vertex y. However, the proof has been compressed and no longer shows this averaging.

Proof. For $z \in Gv$, pick any $g_{vz} \in G$ that takes v to z. Then for any u and v, G is partitioned by the sets $\langle g_{vz}S(v)\,;\ z \in Gv\rangle$ as well as by the sets $\langle S(u)g_{uy}^{-1}\,;\ y \in Gu\rangle$. Thus by (3.1),

$$|S(y)| = |S(u)g_{uy}^{-1}| = \sum_{z \in Gv} |g_{vz}S(v) \cap S(u)g_{uy}^{-1}|$$

and

$$|S(v)| = |g_{vz}S(v)| = \sum_{y \in Gu} |g_{vz}S(v) \cap S(u)g_{uy}^{-1}|\,.$$

Also note that if $|g_{vz}S(v) \cap S(u)g_{uy}^{-1}| \neq 0$, then there is some $h \in g_{vz}S(v) \cap S(u)g_{uy}^{-1}$, which means that $hv = z$ and $hy = u$, whence $f(y,v) = f(hy,hv) = f(u,z)$. Therefore,

$$\begin{aligned}
\sum_{y \in Gu} f(y,v)|S(y)| &= \sum_{y \in Gu} f(y,v) \sum_{z \in Gv} |g_{vz}S(v) \cap S(u)g_{uy}^{-1}| \\
&= \sum_{y \in Gu} \sum_{z \in Gv} f(u,z)|g_{vz}S(v) \cap S(u)g_{uy}^{-1}| \\
&= \sum_{z \in Gv} f(u,z) \sum_{y \in Gu} |g_{vz}S(v) \cap S(u)g_{uy}^{-1}| \\
&= \sum_{z \in Gv} f(u,z)|S(v)|\,.
\end{aligned}$$

∎

COROLLARY 3.3. *Let X be a graph, $G \subseteq \mathrm{Aut}(X)$ be a closed unimodular quasi-transitive subgroup, and $f : X \times X \to [0,\infty]$ be invariant under the diagonal action of G. Choose a complete set $\{o_1, \ldots, o_L\}$ of representatives in V of the orbits of G. Let $\mu_i := |S(o_i)|$. Then*

$$\sum_{i=1}^{L} \mu_i^{-1} \sum_{z \in V} f(o_i, z) = \sum_{j=1}^{L} \mu_j^{-1} \sum_{y \in V} f(y, o_j)\,. \tag{3.4}$$

Proof. The assumption that G is unimodular means that $|S(y)| = |S(o_i)|$ for $y \in Go_i$. Thus, for each i and j, the Mass-Transport Principle gives

$$\sum_{z \in Go_j} f(o_i, z)|S(o_j)| = \sum_{y \in Go_i} f(y, o_j)|S(o_i)|\,,$$

i.e.,

$$\mu_i^{-1} \sum_{z \in Go_j} f(o_i, z) = \mu_j^{-1} \sum_{y \in Go_i} f(y, o_j)\,.$$

Adding these equations over all i and j gives the desired result. ∎

In order to show why (3.4) corresponds to averaging in a natural way, we show how it corresponds to averaging on Følner sets when X is amenable. Recall that X is amenable if $\iota_V(X) = 0$. [Of course, one could use an alternative definition that $\iota_E(X) = 0$; in case X has bounded degree, these concepts coincide.]

For a quasi-transitive amenable graph, the total mass leaving vertices in a Følner set is approximately the number of vertices in the set times the left-hand side of (3.4) *if* the frequency of vertices that lie in Go_i is $1/\mu_i$. Likewise, the total mass entering vertices in a Følner set is approximately the number of vertices in the set times the right-hand side of (3.4). The following proposition makes this precise and thus shows how natural (3.4) is.

Proposition 3.4. *Let X be an amenable graph and $G \subseteq \mathrm{Aut}(X)$ be a closed quasi-transitive subgroup. Choose a complete set $\{o_1, \ldots, o_L\}$ of representatives in V of the orbits of G. Let $\mu_i := |S(o_i)|$ and normalize Haar measure so that $\sum_i \mu_i^{-1} = 1$. If K_n is any sequence of finite subsets of vertices such that $|\partial_V K_n|/|K_n| \to 0$, then for all i,*

$$\lim_{n\to\infty} \frac{|Go_i \cap K_n|}{|K_n|} = \mu_i^{-1} .$$

Proof. By Theorem 3.2, G is unimodular. Fix i and let r be the distance from o_1 to o_i. Let n_i be the number of vertices in Go_i at distance r from o_1 and let n_1 be the number of vertices in Go_1 at distance r from o_i. Begin with mass n_i at each vertex $x \in Go_1$ and redistribute it equally among those vertices in Go_i at distance r from x. When n is large, $|K_n|$ dominates the number of vertices at distance r from $\partial_V K_n$. Hence, the total mass transported from vertices in $Go_1 \cap K_n$ is asymptotically proportional to the total mass transported into vertices in $Go_i \cap K_n$; that is,

$$\lim_{n\to\infty} \frac{n_1 |Go_i \cap K_n|}{n_i |Go_1 \cap K_n|} = 1 .$$

By the Mass-Transport Principle, we have $n_i \mu_i = n_1 \mu_1$. Combined with the preceding equation, this implies the desired result. ∎

Analogues of Corollary 3.3 for nonunimodular groups can be proved the same way; we merely record the result for later use:

Corollary 3.5. *Let X be a graph, $G \subseteq \mathrm{Aut}(X)$ be a closed quasi-transitive subgroup, and $f : X \times X \to [0, \infty]$ be invariant under the diagonal action of G. Choose a complete set $\{o_1, \ldots, o_L\}$ of representatives in V of the orbits of G. Let $\mu_i := |S(o_i)|$ and $a_i \geq 0$. Then*

$$\sum_{i=1}^{L} a_i \sum_{z \in V} f(o_i, z) = \sum_{i,j=1}^{L} a_i/\mu_j \sum_{y \in Go_i} f(y, o_j) |S(y)| .$$

Given a finite set $K \subset V$, let

$$|K|_G := \sum_{x \in K} |S(x)|\,.$$

By (3.1), we have

$$|gK|_G = m(g)^{-1} |K|_G\,. \tag{3.5}$$

LEMMA 3.6. *Let G be a closed subgroup of $\mathrm{Aut}(X)$, $o \in V$, and $A \subseteq G$ be closed. If $S(o)A = A$, then*

$$|A^{-1}o|_G = |A|\,.$$

Proof. For each $x \in A^{-1}o$, let g_x be an element of A satisfying $g_x^{-1}o = x$. The hypothesis that $S(o)A = A$ means that $\{h \in A;\ h^{-1}o = x\} = S(o)g_x = g_x^{-1}S(x)$. In other words, A is partitioned as $\langle g_x^{-1}S(x)\,;\ x \in A^{-1}o\rangle$, whence

$$|A| = \sum_{x \in A^{-1}o} |g_x^{-1}S(x)| = \sum_{x \in A^{-1}o} |S(x)| = |A^{-1}o|_G\,. \qquad \blacksquare$$

We now give an extension of Theorem 3.2: Say that a transitive graph X is $|\bullet|_G$-**amenable** if for all $\epsilon > 0$, there is a finite $K \subset V$ such that $|\partial_V K|_G < \epsilon |K|_G$.

THEOREM 3.7. *Let G be a closed subgroup of $\mathrm{Aut}(X)$ that acts transitively on X. Then G is amenable iff X is $|\bullet|_G$-amenable.*

Proof. Fix a basepoint $o \in V$. Suppose first that G is amenable. Let N be the set containing o and its neighbors. Set

$$B := \{g \in G\,;\ g^{-1}o \in N\}\,.$$

Then $N = B^{-1}o$ and for all $A \subseteq G$,

$$\partial_V(A^{-1}o) = (A^{-1}N) \setminus (A^{-1}o) = (A^{-1}B^{-1}o) \setminus (A^{-1}o)\,.$$

Note that B is compact,

$$S(o)B = B = BS(o)\,,$$

and $BA \supseteq A$. Given $\epsilon > 0$, there is a compact $A \subset G$ such that

$$|BA| - |A| = |BA \setminus A| \le \epsilon |A|\,.$$

Because $BS(o) = B$, this inequality is also satisfied by $A' = S(o)A \supseteq A$. Consequently, we assume, with no loss of generality, that $S(o)A = A$. Since $S(o)B = B$ and $S(o)A = A$, Lemma 3.6 gives

$$|BA| - |A| = |A^{-1}B^{-1}o|_G - |A^{-1}o|_G = |\partial_V(A^{-1}o)|_G \,.$$

We see that X is $|\bullet|_G$-amenable by taking $K := A^{-1}o$.

Conversely, suppose that X is $|\bullet|_G$-amenable. Given $\epsilon > 0$ and a compact set $B \subset G$, let r be the maximum distance of any vertex in $B^{-1}o$ to o and $k := \sum_{d(x,o) \le r} |S(x)|/|S(o)|$. There is a finite set $K \subset V$ such that

$$|\partial_V K|_G \le \epsilon |K|_G / k \,.$$

Let $A := \{g \in G\,;\ g^{-1}o \in K\}$. Since $S(o)A = A$, we have

$$|S(o)BA \setminus A| = |S(o)BA| - |A| = |A^{-1}B^{-1}o|_G - |A|_G = |A^{-1}B^{-1}o \setminus A^{-1}o|_G \,. \tag{3.6}$$

Now every vertex in $A^{-1}B^{-1}o \setminus A^{-1}o$ lies within distance r of some vertex in $\partial_V(A^{-1}o) = \partial_V K$. Therefore, (3.6), (3.5), and Lemma 3.6 give

$$\begin{aligned} |BA \setminus A| &\le |S(o)BA \setminus A| = |A^{-1}B^{-1}o|_G - |A^{-1}o|_G \\ &\le \sum_{x \in \partial_V K} |\{y\,;\ d(x,y) \le r\}|_G = \sum_{x \in \partial_V K} |\{y\,;\ d(o,y) \le r\}|_G |S(x)|/|S(o)| \\ &= k|\partial_V K|_G \le \epsilon |K|_G = \epsilon |A| \,. \end{aligned}$$ ∎

The following simple principle will allow us on several occasions to reduce the study of quasi-transitive graphs to transitive graphs.

LEMMA 3.8. *Let X be a graph and G be a closed quasi-transitive subgroup of* $\mathrm{Aut}(X)$. *Let o be a vertex in X. Let r be such that every vertex in X is within distance r of some vertex in Go. Form the graph X' from the vertices Go by joining two vertices by an edge if their distance in X is at most $2r+1$. Restriction of the elements of G to X' yields a subgroup $G' \subseteq \mathrm{Aut}(X')$. Then X' is connected, G' acts transitively on X', and we have the following equivalences:*
(i) X is $|\bullet|_G$-amenable iff X' is $|\bullet|_{G'}$-amenable;
(ii) G is amenable iff G' is amenable;
(iii) G is unimodular iff G' is unimodular.

Proof. (i) is proved in the unimodular case by Salvatori (1992), Lemma 2; the modifications needed to establish the general case are straightforward.

It is also not hard to deduce (ii) from Lemma 3.1.

To see (iii), we use (3.3). When $x, y \in X'$, each side of (3.3) is the same whether interpreted in G or in G'. Hence if G is unimodular, so is G'. On the other hand, if G' is unimodular, then the G-stabilizers of the vertices in the orbit of o all have the same Haar measure (by the same reasoning). Now apply (3.1) to an arbitrary element of G to conclude that G is unimodular. ∎

REMARK 3.9. Together with Theorem 3.7, this immediately gives one direction of Theorem 3.2: namely, that if G is amenable and unimodular, then X is amenable.

The following will be useful in proving Theorem 5.1.

LEMMA 3.10. *Let G be a closed subgroup of* $\mathrm{Aut}(X)$ *that acts transitively on X. Let $K \subset V$ be finite. For each $x \in V$, let g_x be a random element of G that takes o to x. Fix some $y \in V$. Then*

$$\mathbf{E}\big|\{x \in V\,;\ y \in g_x K\}\big| = |K|_G |S(o)|^{-1}\,.$$

Proof. By the Mass-Transport Principle, we have

$$\mathbf{E}\big|\{x \in V\,;\ y \in g_x K\}\big| \cdot |S(o)| = \sum_{x \in V} \mathbf{P}[y \in g_x K]|S(o)| = \sum_{z \in V} \mathbf{P}[z \in g_o K]|S(z)| = \mathbf{E}|g_o K|_G\,.$$

By (3.1), $|g_o K|_G = m(g_o)^{-1}|K|_G = |K|_G$. ∎

§4. Thresholds for Quasi-Transitive Graphs.

We now extend the threshold for percolation on Cayley graphs to quasi-transitive graphs. However, in the quasi-transitive case the thresholds are not as natural as in the transitive case and there is room for improvement. For readability, we first treat the case of unimodular automorphism groups.

THEOREM 4.1. *Let X be a graph, $G \subseteq \mathrm{Aut}(X)$ be a closed unimodular quasi-transitive subgroup, and let $o_1, \ldots, o_L$ be a complete set of representatives in V of the orbits of G. Let $\mu_i := |S(o_i)|$. Let $\mathbf{P}$ be a bond percolation on X whose distribution is G-invariant. Let D_i denote the random degree of o_i in the percolation subgraph and let d_i denote the degree of o_i in X. Write $p_{\infty,i}$ for the probability that o_i is in an infinite component. Then*

$$\sum_{i=1}^{L} \mu_i^{-1}[d_i - \alpha(X)]p_{\infty,i} \geq \sum_{j=1}^{L} \mu_j^{-1}[\mathbf{E}D_j - \alpha(X)]\,. \tag{4.1}$$

In particular, if the right-hand side of (4.1) is positive, then there is an infinite component with positive probability.

REMARK 4.2. In case G acts transitively, then the $\geq$ in (4.1) can be replaced by $>$.

REMARK 4.3. For later use, we record Theorem 4.1 for transitive groups in the version where all components of the percolation are trees (as in Remark 2.6): Let G be a closed unimodular transitive subgroup of $\mathrm{Aut}(X)$. Let $\mathbf{P}$ be a bond percolation on X all of whose components are trees a.s. and whose distribution is G-invariant. If $\mathbf{E}D \geq 2$, then there is an infinite component with positive probability.

Proof of Theorem 4.1. We put mass $D(x)$ at each $x \in V$. All masses of vertices in infinite components are left alone, but those in a finite component are redistributed equally among the vertices in that component. Since $\mathbf{P}$ is G-invariant, so is this mass transport. Formally, we use the function

$$f(x,y) := \mathbf{E}\left[\mathbf{1}_{\{|K(x)|=\infty, y=x\}} D(x) + \mathbf{1}_{\{|K(x)|<\infty, y \in K(x)\}} \frac{D(x)}{|K(x)|}\right].$$

The mass at x before transportation has mean $\mathbf{E}D(x)$, whence

$$\sum_{i=1}^{L} \mu_i^{-1} \sum_{z \in V} f(o_i, z) = \sum_{i=1}^{L} \mu_i^{-1} \mathbf{E} D_i \,.$$

On the other hand, the mass of z after transportation has mean

$$\mathbf{E}\left[D(z)\mathbf{1}_{\{|K(z)|=\infty\}} + \alpha_{K(z)} \mathbf{1}_{\{|K(z)|<\infty\}}\right],$$

whence

$$\begin{aligned}
\sum_{i=1}^{L} \mu_i^{-1} \sum_{y \in V} f(y, o_i) &= \sum_{i=1}^{L} \mu_i^{-1} \mathbf{E}\left[D_i \mathbf{1}_{\{|K(o_i)|=\infty\}} + \alpha_{K(o_i)} \mathbf{1}_{\{|K(o_i)|<\infty\}}\right] \\
&\leq \sum_{i=1}^{L} \mu_i^{-1} [d_i p_{\infty,i} + \alpha(X)(1 - p_{\infty,i})] \,.
\end{aligned}$$

Corollary 3.3 and a little algebra complete the proof. ∎

An analogue of Theorem 4.1 in the (possibly) nonunimodular case is as follows: For a finite subgraph $K \subset X$, set

$$\alpha_K^G := \frac{1}{|K|_G} \sum_{x \in K} \deg_K(x) |S(x)|$$

and

$$\alpha^G(X) := \sup\{\alpha_K^G\,;\ K \subset X \text{ is finite}\}\,.$$

If X is a regular graph of degree d, then

$$\alpha^G(X) + \iota_E^G(X) = d\,,$$

where

$$\iota_E^G(X) := \inf\left\{\frac{\sum_{(x,y)\in\partial_E K}|S(x)|}{|K|_G}\,;\ K \subset V \text{ is finite}\right\}.$$

Define the G-**degree** of a vertex $x \in X$ as $\sum_{y\sim x}|S(y)|/|S(x)|$.

THEOREM 4.4. *Let X be a graph, $G \subseteq \mathrm{Aut}(X)$ be a closed quasi-transitive subgroup, and let $o_1, \ldots, o_L$ be a complete set of representatives in V of the orbits of G. Let $\mathbf{P}$ be a bond percolation on X whose distribution is G-invariant. Let D_i^G denote the random G-degree of o_i in the percolation subgraph and let d_i^G denote the G-degree of o_i in X. Write $p_{\infty,i}$ for the probability that o_i is in an infinite component. Then*

$$\sum_{i=1}^{L}[d_i^G - \alpha^G(X)]p_{\infty,i} \ge \sum_{j=1}^{L}[\mathbf{E}D_j^G - \alpha^G(X)]\,.$$

In particular, if the right-hand side of (4.1) is positive, then there is an infinite component with positive probability.

REMARK 4.5. Even in the unimodular case, Theorem 4.4 gives a different result than Theorem 4.1.

Proof. We merely indicate the changes necessary to the proof of Theorem 4.1. Given the percolation ω, put mass $D^G(x)$ at each $x \in V$, where $D^G(x)$ is the G-degree of x in ω. Redistribute the masses in finite components G-equally, i.e., proportionally to the weights $|S(y)|$ (for y in the component). This defines $f(x,y)$; use Corollary 3.5 with $a_i = 1$ for all i. ∎

§5. Qualitative Characterizations of Amenability.

Besides the many characterizations of amenable groups in general, we have the following characterization for automorphism groups:

THEOREM 5.1. *Let X be a graph and G be a closed subgroup of* $\mathrm{Aut}(X)$. *Then G is amenable iff for all $\alpha < 1$, there is a G-invariant site percolation on X with $\mathbf{P}[x \in \omega] > \alpha$ for all x and with no infinite components.*

REMARK 5.2. The analogous result for bond percolation is proved similarly.

Proof. Suppose that for all $n = 1, 2, \ldots$, there is some G-invariant site percolation ω_n on X with $\mathbf{P}[o \in \omega_n] \geq 1 - \frac{1}{n}$, but with no infinite components. Then we can define a G-invariant mean on X by mapping $f \in \ell^\infty(V) \mapsto \mathrm{LIM}_{n\to\infty} \mathbf{E}_n[\sum_{x \in K(o)} f(x)/|K(o)|]$, where LIM is a Banach limit, o is any fixed vertex in X, and $\mathbf{E}_n$ denotes expectation with respect to ω_n. Since X has a G-invariant mean, G is itself amenable by Lemma 3.1.

Now we prove the converse. Suppose that G is amenable. We consider first the case that G acts transitively on X.

Fix a finite set $K \subset V$ and consider the following percolation. For each $x \in V$, choose a random $g_x \in G$ that takes o to x and let Z_x be a random bit that equals 1 with probability $|S(o)|/|K|_G$. Choose all g_x and Z_x independently. Remove the sites

$$\bigcup_{x \in X, Z_x = 1} \partial_V(g_x K),$$

that is, consider the percolation

$$\omega := \omega_K := V - \cup\{\partial_V(g_x K);\ Z_x = 1\}.$$

Then ω is a G-invariant percolation on X.

We claim that

$$\mathbf{P}[o \notin \omega] \leq |\partial_V K|_G / |K|_G \tag{5.1}$$

and

$$\mathbf{P}[|K(o)| < \infty] \geq 1 - 1/e. \tag{5.2}$$

To prove (5.1), note that the probability that $o \notin \omega$ is at most the expected number of x such that $Z_x = 1$ and $o \in \partial_V(g_x K)$. Lemma 3.10 shows that this expectation is exactly the right-hand side of (5.1). To prove (5.2), use the independence to calculate that

$$\mathbf{P}[|K(o)| < \infty] \geq \mathbf{P}[\exists x\ o \in g_x K \text{ and } Z_x = 1]$$

$$
\begin{aligned}
&= 1 - \mathbf{P}[\forall x\ \neg(o \in g_x K \text{ and } Z_x = 1)] \\
&= 1 - \prod_{x\in V} (1 - \mathbf{P}[o \in g_x K]\mathbf{P}[Z_x = 1]) \\
&= 1 - \prod_{x\in V} (1 - \mathbf{P}[o \in g_x K]|S(o)|/|K|_G) \\
&\geq 1 - \exp\left\{ - \sum_{x\in V} \mathbf{P}[o \in g_x K]|S(o)|/|K|_G \right\} \\
&= 1 - \exp\left\{ -\mathbf{E}\big|\{x \in V\,;\ o \in g_x K\}\big||S(o)|/|K|_G \right\} \\
&= 1 - 1/e
\end{aligned}
$$

by Lemma 3.10 again.

Now, since X is $|\bullet|_G$-amenable by Theorem 3.7, there is a sequence K_n of finite sets of vertices with $\sum_n |\partial_V K_n|_G/|K_n|_G < 1-\alpha$. For each n, let ω_n be the random subgraph from the percolation arising from the set K_n. Choose ω_n to be independent and consider the percolation $\omega := \cap\omega_n$. By (5.1), we have $\mathbf{P}[o \in \omega] > \alpha$ and by (5.2), we have $\mathbf{P}[|K(o)| < \infty] = 1$.

This finishes the transitive case. Next, assume that G is quasi-transitive. We reduce to the transitive case as follows. Let o be a vertex in X and form the graph X' as in Lemma 3.8. Given a G'-invariant site percolation ω' on X' with only finite components and $\mathbf{P}[o \in \omega']$ very close to 1, form the G-invariant percolation ω on X by keeping those x such that all vertices in X' within distance r of x are in ω'. Trivially, $\mathbf{P}[x \in \omega]$ is very close to 1 for all x. If there were an infinite component in ω, say, a path $\langle x_1, x_2, \ldots\rangle$, then we would be able to find an infinite component $\{y_1, y_2, \ldots\}$ in ω' by taking y_i to be any vertex in Go within distance r of x_i.

Finally, consider the general case. Fix $o \in V$ and $0 < \alpha < 1$. Choose an integer R at random in the range $1/(1-\alpha) < R < 2/(1-\alpha)$. Remove every vertex whose distance from Go is a multiple of R. This breaks X into a disconnected union of quasi-transitive graphs. On each, we may find an invariant percolation with marginal $> \alpha$ and only finite components. Putting these together does the job. ∎

For unimodular groups, the next theorem gives another characterization of amenability.

THEOREM 5.3. *Let G be a closed subgroup of* $\mathrm{Aut}(X)$ *that acts transitively on X. Then each of the following conditions implies the next one:*

(i) X is amenable;

(ii) there is a G-invariant random spanning tree of X with at most 2 ends a.s.;

(iii) there is a G-invariant random nonempty connected subgraph ω of X that satisfies $p_c(\omega) = 1$ with positive probability;

(iv) G is amenable.

If G is assumed to be unimodular, then all four conditions are equivalent.

In part (iii), to say that ω is connected means that $\omega \cap V$ is connected by edges from $\omega \cap E$.

Note that if X is a 3-regular tree and G is the group of automorphisms fixing a given end ξ, then G is amenable, but obviously the only spanning tree of X is X itself. Thus (iv) does not imply (ii) without assuming unimodularity.

The same example shows that (iv) does not imply (iii). In order to see this, let ω be any stationary connected nonempty subgraph of X. Suppose that with positive probability, ω does not contain a path to ξ. Conditioned on this event, there is a unique vertex of ω which is closest to ξ. This allows a G-invariant choice of a vertex in X, which is impossible. Consequently, ω a.s. contains a path leading to ξ. Fix any vertex $v \in V$, and let v_n be the vertex on the path from v to ξ at distance n from v. Then the probability that $v_n \in \omega$ tends to 1 as $n \to \infty$. Since the probability of a vertex to be in ω is independent of the vertex, this means that $v \in \omega$ a.s.; that is, $\omega = X$ a.s.

Question. Which, if any, of the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) can be reversed without the assumption that G is unimodular?

Proof. Assume (i). Then G is amenable by Theorem 3.2. From the proof of Theorem 5.1, we obtain a sequence of invariant percolations $\tilde{\omega}_1 \subset \tilde{\omega}_2 \subset \cdots$ with each of them having no infinite components and with the marginals tending to 1: namely, let $\tilde{\omega}_n := \bigcap_{k \geq n} \omega_k$, with ω_k as in the proof of Theorem 5.1. An invariant random spanning tree is constructed as follows. For each finite component of $\tilde{\omega}_1$, take a uniformly chosen spanning tree and let T_1 be their union (a forest). The finite components of $\tilde{\omega}_1$ partition the finite components of $\tilde{\omega}_2$ and thus induce a graph structure on the components of $\tilde{\omega}_2$. Hence a uniformly chosen spanning tree on this induced graph structure gives a spanning tree for the components of $\tilde{\omega}_2$; let T_2 be their union. Continue inductively and let $T := \bigcup T_n$ be the union of these forests. Clearly T is G-invariant. We claim that, in fact, T is a tree a.s. Since the marginals of $\tilde{\omega}_n$ tend to 1, the probability that any two given vertices lie in the same component of $\tilde{\omega}_n$ also tends to 1. Thus, with probability arbitrarily close to 1, they are in the same component of T. Thus, T is a tree a.s. The fact that T has at most 2 ends is true for any invariant spanning tree in X by the argument in Burton and Keane (1989). Thus, we have (ii).

The implication (ii) implies (iii) is trivial, so assume (iii). Let ω be a G-invariant random connected subgraph of X with $p_c(\omega) = 1$. To deduce (iv), we use the Adams-Lyons method. First, let o_ω be a random vertex in ω. For example, one may take choose o_ω uniformly among the closest vertices in ω to some fixed vertex o. Define a mean μ on X by mapping

$$f \in \ell^\infty(V) \mapsto \mu(f) := \operatorname*{LIM}_{n\to\infty} \mathbf{E} \sum_{x\in K_n(o_\omega)} f(x)/|K_n(o_\omega)|\,,$$

where LIM is a Banach limit and $K_n(o_\omega)$ denotes the cluster of o_ω in a Bernoulli bond percolation on ω with survival parameter $1 - 1/n$. Since ω is connected, the choice of o_ω does not effect the value of $\mu(f)$. Thus, μ is G-invariant. Since X has a G-invariant mean, G is itself amenable by Lemma 3.1, proving (iv).

If G is unimodular, then (iv) $\Rightarrow$ (i) by Theorem 3.2. ∎

This result raises a natural question:

Question. Is there an invariant random spanning tree in any transitive graph? In particular, is there one if the graph is the product of a tree and the integers, i.e., the Cayley graph of the direct product of a free group and the integers with the usual generating set?

Corollary 5.4. *If X is a transitive graph with a nonamenable automorphism group and Bernoulli (site or bond) percolation with survival parameter p produces a unique infinite component a.s., then $p > p_c(X)$.*

Proof. Suppose that $p = p_c(X)$ and that there is a unique infinite component a.s. Then the infinite component ω has $p_c(\omega) = 1$ a.s. Hence G is amenable. ∎

Remark 5.5. This relates to two conjectures about Bernoulli percolation on nonamenable Cayley graphs: (1) that there always is some p giving more than one infinite component; (2) that at $p = p_c(X)$, there are no infinite components a.s. The first conjecture here was made in Benjamini and Schramm (1996a), while the second is folklore.

The argument in Burton and Keane (1989) that we used in the proof of Theorem 5.3 implies that any invariant percolation on an amenable transitive graph has only components with at most 2 ends. The converse is true as well, namely, on any nonamenable transitive graph, there is an invariant percolation that a.s. has a component with at least 3 ends. This is proved in Benjamini, Lyons, Peres, and Schramm (1997). The proof of this characterization of amenability is related to another characterization that appears in that paper: Let X be a transitive graph. If T denotes the wired uniform spanning forest and ω_p is independent Bernoulli bond percolation with survival parameter p, then X is amenable iff for every $p > 0$, the union $T \cup \omega_p$ is connected a.s.

§6. Characterization of Unimodularity.

THEOREM 6.1. *Let G be a closed subgroup of* $\mathrm{Aut}(X)$ *that acts transitively on X. Let m be the minimum of $|S(x)|/|S(y)|$ for x, y neighbors. Then for any G-invariant percolation that has infinite components with positive probability, the expected degree of x given that x is in an infinite component is at least $1+m$. This is sharp for all pairs (G, X) in the sense that there is a G-invariant bond percolation on X with every vertex belonging to an infinite component and having expected degree $1+m$.*

Proof. Begin with mass 1 at each vertex. Transport the mass at x to y if

(1) x and y are adjacent;

(2) x and y are both in an infinite component; and

(3) if the edge (x, y) is removed, then x is isolated to a finite component.

Note that for each x in an infinite component, there is at most one vertex to which the mass at x might be transported. Also, there are at most $D(x)-1$ vertices that might transport their mass to x. Define $\theta(x, y)$ to be the mass transported from x to y and, as usual, $f(x, y) := \mathbf{E}[\theta(x, y)]$.

Fix x. By the Mass-Transport Principle, it follows that $\mathbf{E}[\Delta] = 0$, where

$$\Delta := \sum_{y\in V} \theta(x, y) - \sum_{z\in V} \theta(z, x)|S(z)|/|S(x)|\,.$$

Now $\Delta = 1-1 = 0$ if $K(x)$ is finite. If $K(x)$ is infinite, then consider three cases: (1) If $D(x) = 1$, then $\Delta = 1$. (2) If $D(x) \ge 2$ and for some $y \ne x$, we have $\theta(x, y) = 1$, then $\sum_{y\in V} \theta(x, y) = 1$ and $\sum_{z\in V} \theta(z, x)|S(z)|/|S(x)| \le (D(x)-1)/m$, whence

$$\Delta \ge 1 - (D(x)-1)/m\,. \tag{6.1}$$

(3) If $D(x) \ge 2$ and for all $y \ne x$, we have $\theta(x, y) = 0$, then $\sum_{z\in V} \theta(z, x)|S(z)|/|S(x)| \le (D(x)-2)/m$; since $m \le 1$, again (6.1) holds. Thus, in all three cases, (6.1) holds. Taking expectation, we obtain that

$$0 = \mathbf{E}[\Delta] = \mathbf{E}[\Delta\,;\ |K(x)| = \infty] \ge \mathbf{E}[1-(D(x)-1)/m\,;\ |K(x)| = \infty]\,,$$

which is the same as $\mathbf{E}\big[D(x) \bigm| |K(x)| = \infty\big] \ge 1+m$.

To show that this bound is sharp in case G is not unimodular, i.e., $m < 1$, consider the following percolation. For each vertex x, let

$$L(x) := \{y \sim x\,;\ |S(y)|/|S(x)| = m\}\,;$$

choose $y \in L(x)$ at random (uniformly and independently for different x) and put $(x, y) \in \omega$. No other edges are put in ω. Clearly every component of ω is infinite and the distribution of ω is G-invariant. To show that $\mathbf{E}[D(x)] = 1 + m$, we calculate that

$$\begin{aligned}
\mathbf{E}[D(x)] &= \mathbf{E}\left[\sum_{y \in L(x)} \mathbf{1}_{\{(x,y)\in\omega\}} + \sum_{x \in L(z)} \mathbf{1}_{\{(z,x)\in\omega\}}\right] \\
&= 1 + \sum_{z \in V} \mathbf{1}_{\{x \in L(z)\}} \mathbf{P}[(z,x) \in \omega] \\
&= 1 + \sum_{z \in V} \mathbf{1}_{\{z \in L(x)\}} \mathbf{P}[(x,z) \in \omega] |S(z)|/|S(x)| \qquad (6.2) \\
&= 1 + m\mathbf{E}\left[\sum_{z \in L(x)} \mathbf{1}_{\{(x,z)\in\omega\}}\right] = 1 + m\,,
\end{aligned}$$

where we have used the Mass-Transport Principle in deriving (6.2).

To show that this bound is sharp in case G is unimodular, we borrow a result from Benjamini, Lyons, Peres, and Schramm (1997): the wired uniform spanning forest of X is a G-invariant percolation with every component infinite and expected degree of every vertex equal to 2. ∎

COROLLARY 6.2. *For G a closed subgroup of* $\mathrm{Aut}(X)$*, let $m(G)$ be the minimum of $|S(x)|/|S(y)|$ for x, y neighbors. If G is transitive and $G \subseteq H$ with H a closed subgroup of* $\mathrm{Aut}(X)$*, then $m(G) \le m(H)$. In particular, if G is unimodular, then so is H.*

Proof. Since H must also be transitive, there is an H-invariant percolation ω on X with infinite components and expected degree $1 + m(H)$. Then ω is also G-invariant, whence $1 + m(H) \ge 1 + m(G)$. ∎

REMARK 6.3. We can also use Theorem 6.1 to complete our proof of Theorem 3.2 that we began in Remark 3.9: Assume that X is amenable and G acts quasi-transitively. Let $o \in V$ and define the G'-transitive amenable graph X' as in Lemma 3.8. Let $\mathbf{P}$ be a G'-invariant percolation on X' all of whose components are infinite. Let K_n be Følner sets in X', i.e., $|\partial_V K_n|/|K_n| \to 0$. Fix n and write $\omega \cap K_n$ as the disjoint union of L_n connected graphs $\omega_i = (V_i, E_i)$. Since each component of ω is infinite, each ω_i intersects the boundary of K_n, whence $L_n/|K_n| \to 0$ as $n \to \infty$. Therefore, we have

$$\begin{aligned}
\frac{1}{|K_n|} \sum_{x \in K_n} \deg_\omega x &\ge \frac{1}{|K_n|} \sum_{i=1}^{L_n} \sum_{x \in K_n} \deg_{\omega_i} x \\
&\ge \frac{1}{|K_n|} \sum_{i=1}^{L_n} (2|V_i| - 1) = \frac{2|K_n| - L_n}{|K_n|} \to 2\,,
\end{aligned}$$

as $n \to \infty$. Taking expectation gives

$$\mathbf{E}[D(o)] = \frac{1}{|K_n|} \sum_{x \in K_n} \mathbf{E}[\deg_\omega x] \geq \frac{2|K_n| - L_n}{|K_n|} \to 2\,,$$

whence by Theorem 6.1, G' is unimodular. By Lemma 3.8, also G is unimodular. It follows from Theorem 3.7 that G' is amenable and hence, by Lemma 3.8 again, that G is amenable.

We now extend Theorem 6.1 to the quasi-transitive case:

THEOREM 6.4. *Let X be a graph, $G \subseteq \mathrm{Aut}(X)$ be a closed quasi-transitive subgroup, $o_1, \ldots, o_L$ be a complete set of representatives in V of the orbits of G, and write $\tau(x) := i$ if $x \in Go_i$. Let $\mathbf{P}$ be a G-invariant bond percolation on X that has infinite components with positive probability. Let $\mu_i := |S(o_i)|$,*

$$m_i := \min\{\mu_{\tau(y)}/|S(y)|\,;\ y \sim o_i\}\,,$$

and $p_{\infty,i}$ be the probability that o_i is in an infinite component. Then

$$\sum_i \frac{\mathbf{E}\big[D(o_i);\ |K(o_i)| = \infty\big]}{\mu_i m_i} \geq \sum_i \frac{1 + (m_i \wedge 1)}{\mu_i m_i} p_{\infty,i}\,. \tag{6.3}$$

Proof. The argument is very similar to the proof of Theorem 6.1. In fact, we use exactly the same mass transport. Define

$$\Delta_i := \sum_{z \in V} \theta(o_i, z) - \sum_{y \in V} \theta(y, o_i) \frac{|S(y)|}{\mu_{\tau(y)}}\,.$$

Then Corollary 3.5 with $a_i := 1/\mu_i$ implies that

$$\sum_i \frac{\mathbf{E}\Delta_i}{\mu_i} = 0\,. \tag{6.4}$$

Now $\Delta_i = 0$ if $K(o_i)$ is finite. Otherwise, we again consider three cases: (1) If $D(o_i) = 1$, then $\Delta_i = 1$. (2) If $D(o_i) \geq 2$ and for some $z \neq o_i$, we have $\theta(o_i, z) = 1$, then $\sum_{z \in V} \theta(o_i, z) = 1$ and $\sum_{y \in V} \theta(y, o_i)|S(y)|/\mu_{\tau(y)} \leq (D(o_i) - 1)/m_i$, whence

$$\Delta_i \geq 1 - (D(o_i) - 1)/m_i\,.$$

(3) If $D(o_i) \geq 2$ and for all $z \neq o_i$, we have $\theta(o_i, z) = 0$, then $\sum_{y \in V} \theta(y, o_i)|S(y)|/\mu_{\tau(y)} \leq (D(o_i) - 2)/m_i$. Thus, in all three cases, we have

$$\Delta_i \geq (1 \wedge \frac{1}{m_i}) - \frac{D(o_i) - 1}{m_i}\,.$$

From (6.4), we obtain that

$$0 = \sum_i \frac{\mathbf{E}\Delta_i}{\mu_i} = \sum_i \frac{\mathbf{E}[\Delta_i\,;\ |K(o_i)| = \infty]}{\mu_i}$$
$$\geq \sum_i \frac{1}{\mu_i}\mathbf{E}[(1 \wedge m_i^{-1}) - (D(o_i) - 1)/m_i\,;\ |K(o_i)| = \infty]\,,$$

which is the same as (6.3). ∎

§7. Ends of Components.

An **end** of a graph X is an equivalence class of infinite nonself-intersecting paths in X, with two paths equivalent if for all finite $A \subset X$, the paths are eventually in the same connected component of $X \setminus A$. A **forest** is a graph all of whose components are trees.

THEOREM 7.1. *Let G be a closed unimodular subgroup of* $\mathrm{Aut}(X)$ *that acts transitively on X. Let ω be a G-invariant percolation on X such that ω has infinite components with positive probability. If*

(i) some component of ω has at least 3 ends with positive probability,

then

(ii) some component of ω has $p_c < 1$ with positive probability and

(iii) for every vertex x, $\mathbf{E}\big[D(x) \bigm| |K(x)| = \infty\big] > 2$.

If ω is a forest a.s., then the three conditions are equivalent.

LEMMA 7.2. *Let $G \subset \mathrm{Aut}(X)$. Let ω be a G-invariant bond percolation on X. Then (on a larger probability space) there is a G-invariant bond percolation process ω' such that $\omega' \subset \omega$ and that enjoys the following properties: For $x \in V$, let $K(x)$, $K'(x)$ denote the component of x in ω, ω', resp. Then*

(i) ω' is a forest a.s.;

(ii) $\forall x \in V$ $|K'(x)| = |K(x)|$ a.s.; and

(iii) $\forall x \in V$ and $k \in \mathbb{Z}^+$, if $K(x)$ has at least k ends with positive probability, then $K'(x)$ also has at least k ends with positive probability.

Proof. Use the so-called free minimal spanning forest of ω: Assign to the edges i.i.d. uniform $[0, 1]$ random variables. An edge $e \in \omega$ is present in ω' iff there is no cycle in ω containing e in which e is assigned the maximum value. The first two properties are straightforward and well known. To see the third, assume that $K(x)$ has at least k ends with positive probability. Choose any ball A about x so that $K(x) \setminus A$ has at least k

infinite components with positive probability. Then with positive probability, $K(x) \setminus A$ has at least k infinite components, all edges in A are assigned values less than $1/2$, and all edges in $\partial_E A$ are assigned values greater than $1/2$. On this event, ω' contains a spanning tree of $A \cap \omega$ that is part of a tree in ω' with at least k ends. ∎

Proof of Theorem 7.1. First consider the case where ω is a forest. The proof will be by throwing away a nonempty invariant collection of edges and appealing to Theorem 6.1.

Call a vertex x an **encounter point** if its removal breaks $K(x)$ into at least 3 infinite components. For each encounter point x, choose one of its incident edges e_x in ω at random (uniformly) and independently; let ζ be the set of edges chosen. Let η_1 be the set of edges $e_x \in \zeta$ such that there is a one-sided infinite path on ω starting at x that passes through e_x and does not contain any encounter point other than x. For each pair of encounter points $x \neq y$ such that the shortest path $\mathcal{P} \subset \omega$ from x to y does not go through any other encounter point and contains both e_x and e_y, choose e_x or e_y at random with equal probability (and independently of other such decisions); let $\eta_2 \subseteq \zeta$ be the resulting set of chosen edges. (Note that in case x and y are neighbors and $(x, y) \in \zeta$, then this means that $(x, y) \in \eta_2$.) Finally, set $\eta := \eta_1 \cup \eta_2 \subseteq \zeta$.

A moment's thought shows that every vertex that is in an infinite component of ω is also in an infinite component of $\omega \setminus \eta$.

For every vertex x, Theorem 6.1 implies that $\mathbf{E}\big[\deg_{\omega\setminus\eta} x \bigm| |K(x)| = \infty\big] \geq 2$. If (i) holds, then η is nonempty with positive probability. Since G acts transitively, this means that some edge incident to x lies in η with positive probability. Since $\eta \subseteq \omega$ contains only edges in infinite components of ω, it follows that $\mathbf{E}\big[D(x) \bigm| |K(x)| = \infty\big] = \mathbf{E}\big[\deg_\omega x \bigm| |K(x)| = \infty\big] > 2$, i.e., that (iii) holds.

Next, still assuming that ω is a forest, we deduce (ii) from (iii) by using Remark 4.3. Finally, that (ii) implies (i) for forests is trivial.

In case ω is not a forest a.s, choose $\omega' \subset \omega$ as in Lemma 7.2. If (i) holds for ω, then it also holds for ω', whence (ii) and (iii) hold for ω', whence also for ω. ∎

Corollary 7.3. *Let G be a closed unimodular subgroup of* $\mathrm{Aut}(X)$ *that acts transitively on X and let ω be a G-invariant percolation on X. Fix a vertex o. Let F_o be the event that $K(o)$ is an infinite tree with finitely many ends, and suppose that $P[F_o] > 0$. Then* $\mathbf{E}\big[D(o) \bigm| F_o\big] = 2$.

Proof. Assume that all components of ω are infinite trees with finitely many ends. There is no loss of generality, since we can remove from ω all components that are not of this type. Since the components have only finitely many ends, they have $p_c = 1$. Thus, Theorem 7.1 gives $\mathbf{E}\big[D(o) \bigm| F_o\big] \leq 2$. On the other hand, Theorem 6.1 gives $\mathbf{E}\big[D(o) \bigm| F_o\big] \geq 2$. ∎

REMARK 7.4. We describe a transitive nonunimodular graph with an invariant random forest where each component has 3 ends, but the expected degree is smaller than 2. Let $\mathbb{Z}_3 := \mathbb{Z}/3\mathbb{Z}$ be the group of order 3 and let T be the 3-regular tree with a distinguished end, ξ. On T, let every vertex be connected to precisely one of its offspring (as measured from ξ), each with probability 1/2. Then every component is a ray. Let ω_1 be the preimage of this configuration under the projection $T \times \mathbb{Z}_3 \to T$. For every vertex x in T that has distance 5 from the root of the ray containing x, add to ω_1 two edges at random in the $\mathbb{Z}_3$ direction, in order to connect the 3 preimages of x in $T \times \mathbb{Z}_3$. The resulting configuration is a stationary spanning forest with 3 ends per tree and expected degree $(1/2)1 + (1/2)2 + 2^{-(5+1)}\big((2/3)1 + (1/3)2\big)$.

Similarly, one can construct a stationary spanning forest where each tree has one end and the expected degree is greater than 2. To do this, for every vertex x that is a root of the ray in T, add to ω_1 two edges at random in the $\mathbb{Z}_3$ direction to connect the 3 preimages of x, and for every edge e in a ray in T that has distance 5 from the root of the ray, delete two of its preimages at random.

§8. Invariant Percolation in the Hyperbolic Plane.

THEOREM 8.1. *Let G be a closed group of isometries of the hyperbolic plane $\mathbb{H}^2$. Suppose that $\mathcal{T}$ is a tiling of $\mathbb{H}^2$ by finite-sided compact polygons, that the isometries in G are automorphisms of $\mathcal{T}$, and that G acts quasi-transitively on V, where V is the set of vertices of $\mathcal{T}$. Let ω be a G-invariant bond percolation on $\mathcal{T}$. Then the number of infinite components of ω is a.s. 0, 1 or ∞.*

Let X denote the 1-skeleton (graph) of $\mathcal{T}$.

REMARK 8.2. It follows that the total number of infinite monochromatic clusters in a stationary 2-coloring of the vertices of X is a.s. 0, 1 or ∞. This was proved for Bernoulli percolation assuming that X has sufficiently large genus by Lalley (1996), and stated without proof in the continuous Bernoulli Voronoi model setting by Benjamini and Schramm (1996c).

REMARK 8.3. Burton and Keane (1991) described Aut($\mathbb{Z}^2$)-invariant percolation on $\mathbb{Z}^2$ with any finite number of infinite components. For example, let T be the uniform spanning tree in $\mathbb{Z}^2$, and let $T^\dagger$ be the union of edges in the planar dual of $\mathbb{Z}^2$ that do not intersect edges of T. Then $T \cup T^\dagger$ can be thought of as an invariant percolation on a refined square lattice; it has 2 infinite components. An example with 3 infinite components is obtained by considering $T \cup T^\dagger \cup \Upsilon$ (in a further refined lattice), where Υ is the set of points in the plane that have the same distance from T as from $T^\dagger$.

Proof of Theorem 8.1. Suppose that with positive probability, the number of infinite components in ω is k, where $2 \le k < \infty$, and condition on this event. If ω has any finite components, absorb each finite component to an infinite component as follows. For each finite component A that is adjacent to an infinite component, choose an edge in $\partial_E A$ at random that connects A to an infinite component. Make these choices uniformly and independently. If ω still has finite components, then repeat this procedure at most countably many times until all components of ω are infinite. Since the result is still a G-invariant percolation, we may assume from the start that a.s. ω has only infinite components and that their number is k.

It follows from the assumptions that G is nonamenable. See, for example, Zimmer (1984), Propositions 4.1.8 and 4.1.11. We now define a G-invariant mean on X and appeal to Lemma 3.1 to obtain a contradiction. Consider the edges in X that join different components of ω. Their dual edges in the planar dual of X form a forest, Γ. It is not hard to verify that each component in Γ has no leaves, the number of components of Γ is finite and ≥ 1, and the number of ends in any component of Γ is finite and ≥ 2.

We claim that a.s., all of the trees in Γ have at most 2 ends. For if not, Γ would have finitely many vertices of degree at least 3. Each of these vertices is a face of X; let K be the set of vertices (in X) in these faces. Since K is finite, by taking at random a vertex from K, we could choose in an invariant way a random vertex of X, which is impossible since G has infinite orbits. It follows that each component of Γ is isomorphic to $\mathbb{Z}$.

Now, given $f \in \ell^\infty(V)$, assign each vertex v in Γ the average $\bar{f}(v)$ of the values of f on the face corresponding to v.

Fix an enumeration of V. For each component Γ_i of Γ, let v_i be the first vertex in Γ_i according to the enumeration. Let $M_n(f)$ denote the average of $\bar{f}$ on the union of the n-balls in Γ centered at $v_1, \ldots, v_k$.

Given a Banach limit LIM, define $M(f) := \mathrm{LIM}_{n\to\infty} \mathbf{E} M_n(f)$. Then $M(f)$ does not depend on the choice of enumeration and is an invariant mean on X. By Lemma 3.1, this contradicts the nonamenability of G, and completes the proof. ∎

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