Basics $(AB)_{ij} = \sum a_{is} b_{sj}$; $A[\underline{b_1}\,\underline{b_2}\,\underline{b_3}] = [A\underline{b_1}\,A\underline{b_2}\,A\underline{b_3}]$;

$[\underline{a_1}\,\underline{a_2}\,\underline{a_3}]\begin{bmatrix}\vec{b_1}\\ \vec{b_2}\\ \vec{b_3}\end{bmatrix} = \Sigma \underline{a_i}\,\vec{b_i}$

$A : p\times q, B : q\times p, AB = I \Rightarrow q \geq p, \#pivots = p, R_A = F^p$; $BA = I \Rightarrow p \geq q, \#pivots = q, N_A = \{0\}$

Schur's complement $\begin{bmatrix} A & B\\ C & D\end{bmatrix} =$

$\begin{bmatrix} I_p & 0\\ CA^{-1} & I_q\end{bmatrix}\cdot\begin{bmatrix} A & 0\\ 0 & D-CA^{-1}B\end{bmatrix}\cdot\begin{bmatrix} I_p & A^{-1}B\\ 0 & I_q\end{bmatrix} =$

$\begin{bmatrix} I_p & BD^{-1}\\ 0 & I_q\end{bmatrix}\cdot\begin{bmatrix} A-BD^{-1}C & 0\\ 0 & D\end{bmatrix}\cdot\begin{bmatrix} I_p & 0\\ D^{-1}C & I_q\end{bmatrix}$

$\begin{bmatrix} A & B\\ C & D\end{bmatrix}$ invertible $\Leftrightarrow A$ and $D - CA^{-1}B$ invertible.

Conservation of dim $A \in F^{p\times q} \Rightarrow q = dimR_A + dimN_A$

Eigenvalues Every $A \in C^{n\times n}$ has at least one eigenvalue, and $\leq n$ distinct ones. A is diagonalizable $\Leftrightarrow$ it has n lin. indep. eig'vecs.

Jordan: $\gamma_j = dimN_{A-\lambda_j I}$ = num of cells in B_{λ_j} in J = geom mult

$\alpha_j = dimN_{(A-\lambda_j I)^n}$ = total num of rows in B_{λ_j} in J =alg mult

$\mu_j = min\{\nu : dimN_{(A-\lambda_j I)^\nu} = dimN_{(A-\lambda_j I)^n}\}$ = max cell size.

$dimN_{(A-\lambda_j I)^l} - dimN_{(A-\lambda_j I)^{l-1}}$ = num cells of size l or more.

Direct sum $Y = U + V, U\cap V = \{0\} \Rightarrow Y = U\dot{+}V$

$F^n = N_{(A-\lambda_1 I)^n}\dot{+}R_{(A-\lambda_1 I)^n}$; $R_{(A-\lambda_1 I)^n}\cap\cdots\cap R_{(A-\lambda_k I)^n} = \{0\}$; $C^n = N_{(A-\lambda_1 I)^n}\dot{+}\cdots\dot{+}N_{(A-\lambda_k I)^n}$; $B_{n\times n} \Rightarrow C^n = N_{B^n}\dot{+}R_{B^n}$

If $n = \sum dimN_{A-\lambda_i I}$ then $C^n = N_{A-\lambda_1 I}\dot{+}\cdots\dot{+}N_{A-\lambda_k I}$, else subspace of C^n

Determinants (1) $d(I) = 1$. (2) $d(PA) = -d(A)$ for simple permutation P. (3) $d(A)$ is linear in rows. props: $det(AB) = det(BA); det(A^T) = det(A)$;

A inv $\Rightarrow det(A^{-1}) = 1/det(A)$; $A^{-1} = 1/det(A) * \begin{bmatrix} A_{11} & -A_{21} & A_{31}\\ -A_{12} & A_{22} & -A_{32}\\ A_{13} & -A_{23} & A_{33}\end{bmatrix}$

eigvals μ is a root of $p(\lambda) = det(\lambda I - A) \Leftrightarrow \mu$ is eigval of A.

Companion $A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \cdots & & & & \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1}\end{bmatrix} \Rightarrow det(\lambda I - A) = a_0 + a_1\lambda + \cdots + a_{n-1}\lambda^{n-1} + \lambda^n$

$A = \begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ -a_1 & -a_2 & -a_3\end{bmatrix}$ is similar to $J = \begin{bmatrix}\mu & 1 & 0\\ 0 & \mu & 1\\ 0 & 0 & \mu\end{bmatrix}$ when $\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = (\lambda-\mu)^3$

note: $detA = \lambda_1^{\alpha_1}\cdots\lambda_k^{\alpha_k}$; $trace(A) = \alpha_1\lambda_1 + \cdots + \alpha_k\lambda_k$

Inequalities (1) $s,t > 1; \frac{1}{s}+\frac{1}{t} = 1; a,b > 0 \Rightarrow ab \leq \frac{a^s}{s} + \frac{b^t}{t}$

(2) Holder $s,t > 1; \frac{1}{s}+\frac{1}{t} = 1 \Rightarrow \sum|a_j b_j| \leq (\sum|a_j|^s)^{1/s}(\sum|b_j|^t)^{1/t}$

(3) Cauchy-Schwartz $\sum|a_j b_j| \leq \sqrt{\sum|a_j|^2}\sqrt{\sum|b_j|^t}$

(4) Minkowski $s \geq 1 \Rightarrow (\sum|a_j + b_j|^s)^{1/s} \leq (\sum|a_j|^s)^{1/s} + (\sum|b_j|^s)^{1/s}$

Norms norm φ should meet: (1) $\varphi(u) \geq 0$, equal iff $u = 0$. (2) $\varphi(\alpha u) = |\alpha|\varphi(u)$. (3) $\varphi(u+v) \leq \varphi(u)+\varphi(v)$. Def $\|u\|_l = (\sum|a_j|^l)^{1/l}$

Matrix norm $\|A\|_{s,t} = max\{\|A\underline{x}\|_t : \|\underline{x}\|_s \leq 1\} = max\{\|A\underline{x}\|_t : \|\underline{x}\|_s = 1\}$. Usually $s = t = 2$. Props: (1) $\|A\underline{x}\|_t \leq \|A\|_{s,t}\|\underline{x}\|_s$; (2) $\|AB\|_{r,t} \leq \|A\|_{r,s}\|B\|_{s,t}$

Lemma: A invertible, $\|\underline{x}\| \leq 1$, then: (1) $I - \underline{x}$ invertible. (2) $(I-\underline{x})^{-1} = \sum_0^\infty \underline{x}^j$. (3) $\|(I-x)^{-1}\| \leq \frac{1}{1-\|\underline{x}\|}$

Thm: A invertible, $\|B - A\| \leq \frac{1}{\|A^{-1}\|} \Rightarrow B$ invertible.

Finding: To find max of $\|Ax\|^2/\|x\|^2$

Inner product (1) $\langle u+w, v\rangle = \langle u,v\rangle + \langle w,v\rangle$. (2) $\langle\alpha u, v\rangle = \alpha\langle u,v\rangle$. (3) $\langle v,u\rangle = \overline{\langle u,v\rangle}$. (4) $\langle u,u\rangle \geq 0$, equal iff $u = 0$. Property: $\langle Au, v\rangle = \langle u, A^*v\rangle$

Cauchy-schwartz $|\langle u,v\rangle| \leq \langle u,u\rangle^{1/2}\langle v,v\rangle^{1/2}$. Equality iff span$(u,v)$ is 1-dimensional.

Orthogonality $u_1,\ldots,u_k$ orthogonal if $\langle u_i,u_j\rangle = 0$. Orthonormal if $\langle u_i,u_i\rangle = 1$. A is orthogonal if $A^TA = I$. A is unitary if $A^*A = I$.

Projections P is a projection if $P^2 = P$. Orthogonal projection if $P^2 = P = P^*$.

Thm: If $P \in C^{n\times n}$ is a projection, then $C^n = R_P\dot{+}N_P$. If orthogonal projection, $C^n = R_P \oplus N_P$

Thm: $A \in C^{p\times q} \Rightarrow C^p = R_A \oplus N_{A^*}, C^q = R_{A^*}\oplus N_A$

Gram Matrix: $v_1,\ldots,v_k$ vectors, $g_{i,j} = \langle v_j, v_i\rangle$. G invertible $\Leftrightarrow v_1,\ldots,v_k$ lin. indep.

P_V $V = span\{v_1,\ldots,v_k\}, W = \{u\in U : \forall i, \langle u,v_i\rangle = 0\} \Rightarrow$ (1) $U = V\oplus W$. (2) $P_V u = \sum c_j v_j, c_j = (G^{-1}\underline{b})_j, \underline{b} = \begin{bmatrix}\langle u,v_1\rangle\\ \cdots\\ \langle u,v_k\rangle\end{bmatrix}$. (3) $\|P_V u\|^2 = c^*Gc$. (4) $\|P_V u\|^2 \leq \|u\|^2$, equality iff $u \in V$. (5) if $U = C^n, V = [v_1\ldots v_k]$, then $P_V = V(V^*V)^{-1}V^*$.

Orthogonal expansion: $V = span\{v_1,\ldots,v_k\}, W = \{u\in U : \forall i, \langle u,v_i\rangle = 0\} \Rightarrow$ (1) $\{v_1,\ldots,v_k\}$ lin. indep. (2) $U = V\oplus W$. (3) $P_V u = \langle u,v_1\rangle v_1 + \cdots + \langle u,v_k\rangle v_k$: orthogonal projection. (4) $\|P_V u\|^2 = \sum|\langle u,v_i\rangle|^2$. (5) $\sum|\langle u,v_j\rangle|^2 \leq \|u\|^2$

Gram-Schmidt $\tilde{v}_3 = u_3 - \langle u_3,v_1\rangle v_1 - \langle u_3,v_2\rangle v_2, v_3 = \frac{\tilde{v}_3}{\|\tilde{v}_3\|}$

QR factorization $A \in R^{p\times q}$, rank$(A) = q \Rightarrow \exists$ a unique orthogonal matrix $Q \in R^{p\times q}$ and a unique upper tr. $R \in R^{q\times q}$ with positive entries on diag. s.t. $A = QR$.

Area/Volume Parallelogram a-b: $V = [ab], a,b \in R^n \Rightarrow area = (det(V^*V))^{1/2}$

Parallelopiped a-b-c: $W = [abc], a,b,c \in R^3 \Rightarrow vol = det(W)$

Hermitian A is hermitian if $A = A^*$. Properties: (1) It's eigvals are real. (2) Eigvecs of distinct eigvals $\perp$. (3) $\forall k,\lambda : N_{(A-\lambda I)^k} = N_{(A-\lambda I)}$. (4) $\exists$ unitary U, diagonal D s.t. $A = UDU^*$

Positive-Semidefinite A is positive semidefinite if $\forall x \in C^n : \langle Ax,x\rangle \geq 0$. (Positive definite if $>$). Properties: (1) A is hermitian. (2) eigvals ≥ 0. (3) If positive definite, then eigvals > 0.

SVD s is a singular value of A if s^2 is an eigval of A^*A. $A \in C^{p\times q}$ of rank r with sing vals $s_1 \geq \ldots \geq s_r \Rightarrow \exists$ unitary U,V s.t. $A = V\begin{bmatrix} D & 0\\ 0 & 0\end{bmatrix}U^*$.

Note: $\|A\| = s_1$

$A = VDU^*$. Finding D: eigvals of A^*A (not nec. all). Finding U: eigvecs of A^*A: $A^*Au_i = s_i^2 u_i$

Approximation Minimize $Ax - b$ by: $x = \sum_1^r\frac{\langle b,v_j\rangle}{s_j}u_j + \sum_{r+1}^q c_j u_j$ (v_i, u_i are cols of V, U. c_js arbitrary)

Normal matrices A normal if $A^*A = AA^*$. Props: (1) $N_{A^*} = N_A = N_{A^k}$. (2) $\forall\lambda,k : N_{(A-\lambda I)^k} = N_{(A-\lambda I)}$ (3) eigvecs of distinct eigvals $\perp$.

THM $A \in C^{n\times n}$ normal $\Leftrightarrow \exists$ orthonorm' basis of C^n consisting of eigvecs of A.

Schur Thm $A \in C^{n\times n} \Rightarrow \exists$ unitary V, upper tr. T s.t. $V^*AV = T$. Diag' of T is the eigvals of A.

Difference eq. $x_n = ax_{n-1} + bx_{n-2}$: 1. Find roots λ_1,λ_2 of $\lambda^2 = a\lambda + b$. 2. If $\lambda_1 \neq \lambda_2, x_n = \alpha\lambda_1^n + \beta\lambda_2^n$. if $\lambda_1 = \lambda_2, x_n = \alpha\lambda_1^n + \beta n\lambda_1^n$. 3. Find α,β using initial conditions.

Differential eq. $x^{(p)}(t) + a_{p-1}X^{(p-1)}(t) + \cdots + a_1 x^{(1)}(t) + a_o x(t) = 0$.

1. Find roots of $0 = \lambda^p + a_{p-1}\lambda^{p-1} + \cdots + a_0 = (\lambda-\lambda_1)^{\alpha_1}\cdots(\lambda-\lambda_k)^{\alpha_k}$.

2. $x(t) = \beta_1 e^{t\lambda_1} + \beta_2 te^{t\lambda_1} + \cdots + \beta_{\alpha_1}t^{(\alpha_1-1)}e^{t\lambda_1} + \gamma_1 e^{t\lambda_2} + \cdots$

$AB = BA \Rightarrow e^{A+B} = e^Ae^B$

Funcs Mean val (real) f continuous real valued on [a,b], f' exists. $f(b) - f(a) = f'(c)(b-a)$ for some $c \in (a,b)$. Taylor: $f(b) = f(a) + \sum\frac{f^{(j)}(a)}{j!}(b-a)^j + \frac{f^{(n)}(c)(b-a)^n}{n!}$.

Mean Val $f(x_1,\ldots,x_q)$ cont. on $a_j \leq x_j \leq b_j$, $\frac{\partial f}{\partial x_j}$ exists. $f(b) - f(a) = (\nabla f)(c)(b-a)$, for some $c = a + t_0(b-a), 0 < t_0 < 1$.

Mean val for vec valued $f(x) = \begin{bmatrix} f_1(x_1,\ldots,x_q)\\ \cdots\\ f_p(x_1,\ldots,x_q)\end{bmatrix}$. Each f_i cont. and partial derivatives exists. $f(b) - f(a) = \begin{bmatrix}(\nabla f_1)(c_1)\\ \cdots\\ (\nabla f_p)(c_p)\end{bmatrix}(b-a)$ for some set $c_j = a + t_j(b-a), 0 < t_j < 1$.

Thm $\|f(b) - f(a)\| \leq \|F'(c)\|\|b-a\|$, where $F'(c) = \begin{bmatrix}\frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_q}(x)\\ & \cdots & \\ \frac{\partial f_p}{\partial x_1}(x) & \cdots & \frac{\partial f_p}{\partial x_q}(x)\end{bmatrix}$

Fixed point $f(x)$ cont. map of E into itself, $\|f(b) - f(a)\| \leq K\|b-a\|$ for some K and all $a,b \in E$, then: (1) $\exists$ exactly one $x_* \in E$ s.t. $f(x_*) = x_*$. (2) let $x_0 \in E, x_{n+1} = f(x_n) \Rightarrow x_* = \lim x_n$. (3) $\|x_* - x_n\| \leq \frac{K^n}{1-K}\|x_1 - x_0\|$

Spectral radius $\lim_{n\to\infty}\|A^n\|^{1/n} = \max\{|\lambda| : \lambda\in\sigma(A)\} \stackrel{\text{def}}{=} r_\sigma(A)$

Implicit func Let $g(x) = \begin{bmatrix} g_1(x_1,\ldots,x_q,y_1,\ldots,y_p)\\ \cdots\\ g_p(x_1,\ldots,x_q,y_1,\ldots,y_p)\end{bmatrix}$ be a mapping from $x \in R^q$ and $y \in R^p$ into R^p s.t.: (a) g is continuous for $\|x - x_0\| < \alpha, \|y - y_0\| < \beta$. (b) $g(x_0,y_0) = 0$. (c) partial derivs by y_i exist and are continuous. (d) The $p\times p$ matrix $G_2'(x,y) = \begin{bmatrix}\frac{\partial g_1}{\partial y_1}(x,y) & \cdots & \frac{\partial g_1}{\partial y_p}(x,y)\\ & \cdots & \\ \frac{\partial g_p}{\partial y_1}(x,y) & \cdots & \frac{\partial g_p}{\partial y_p}(x,y)\end{bmatrix}$ is invertible at (x_0,y_0).

Then: $\exists\gamma,\delta$ s.t. $\forall x \in \{x : \|x - x_0\| < \gamma\}$ there exists exactly one $y = \varphi(x)$ in $\{y : \|y - y_0\| < \delta\}$ s.t. $g(x,\varphi(x)) = 0$.

Moreover, if partial derivs by x_i also exist and are cont., then $\frac{\partial\varphi_i}{\partial x_j}$ also exist and are cont.

A generalization Let $g_i(u_1,\ldots,u_n), i = 1\ldots k$ be cont. with first order partials in an open set Q that contains the point u°, and suppose rank $\begin{bmatrix}\frac{\partial g_1}{\partial u_1}(u) & \cdots & \frac{\partial g_1}{\partial u_n}(u)\\ & \cdots & \\ \frac{\partial g_k}{\partial u_1}(u) & \cdots & \frac{\partial g_k}{\partial u_n}(u)\end{bmatrix} = p$ near u°, and let $q = n - p$. Then there exists a permutation σ of the indices $\{1\ldots n\}$ of the components of u $x_i = u_{\sigma(i)}, i = 1\ldots k$ and $y_i = u_{\sigma(q+i)}, i = 1\ldots p$, and a pair $\gamma,\delta > 0$ s.t.: (1) rank $\begin{bmatrix}\frac{\partial g_1}{\partial y_1}(u^\circ) & \cdots & \frac{\partial g_1}{\partial y_p}(u^\circ)\\ & \cdots & \\ \frac{\partial g_k}{\partial y_1}(u^\circ) & \cdots & \frac{\partial g_k}{\partial y_p}(u^\circ)\end{bmatrix} = p$. (2) $\forall x \in \{x \in R^q : \|x - x^0\| < \gamma\}$ there exists exactly one point $y = \varphi(x)$ in $\{y \in R^p : \|y - y^0\| < \delta\}$ s.t. $g_i(u) = 0$ for $i = 1\ldots k$ when $\begin{bmatrix}u_{\sigma(1)}\\ \cdots\\ u_{\sigma(n)}\end{bmatrix} = \begin{bmatrix}x\\ \varphi(x)\end{bmatrix}$. (3) $\varphi(x)$ is cont and has cont first order partial derivs in $\{x \in R^q : \|x - x^\circ\| < \gamma\}$

Inverse func thm $f(x) = \begin{bmatrix} f_1(x_1,\ldots,x_p)\\ \cdots\\ f_p(x_1,\ldots x_p)\end{bmatrix}$ cont in $\{x : \|x - x_0\| < \alpha\}$, partials exist and are cont, and $F'(x)$ is invertible at x_0. Let $y = f(x_0)$. Then $\exists\delta > 0$ s.t. $\forall y \in \{y \in R^p : \|y - y_0\| < \delta\}$, there exists exactly one point x in $\{x \in R^p : \|x - x_0\| < \delta\}$ s.t. $y = f(x)$. Moreover, $x = \vartheta(y)$ has cont first order partials.

Extremal If a is a local max or min for $f(x)$ the $(\nabla f)(a) = 0$

H_f Let $H_f(c) = \begin{bmatrix}\frac{\partial^2 f}{\partial x_1\partial x_1}(c) & \cdots & \frac{\partial^2 f}{\partial x_1\partial x_n}(c)\\ & \cdots & \\ \frac{\partial^2 f}{\partial x_n\partial x_1}(c) & \cdots & \frac{\partial^2 f}{\partial x_n\partial x_n}(c)\end{bmatrix}$. $(\nabla f)(a) = 0$ and $H_f(a) > 0 \Rightarrow$ a is a local min. $(\nabla f)(a) = 0$ and $H_f(a) < 0 \Rightarrow$ a is a local max. (Note: $H_f(a) > 0$ means $\langle H_f(a)u,u\rangle > 0$ for all nonzero u)

Extremal w. constraints $f(u) = f(u_1,\ldots,u_n), g_1(u) = g_1(u_1,\ldots,u_n),\ldots g_k(u) = g_k(u_1,\ldots,u_n)$ are cont with cont partials on Q, $k < n$. Let $S = \{(u_1,\ldots,u_n) \in Q : g_j(u_1,\ldots,u_n) = 0$ for $j = 1\ldots k\}$, and assume: (a) $\exists a \in S$ s.t. $f(u) \geq f(a)$ or $f(u) \leq f(a)$ for all $u \in S$ which are close enough to a. (b) rank $\begin{bmatrix}(\nabla g_1)(u)\\ \cdots\\ (\nabla g_k)(u)\end{bmatrix} = p$ for all $u \in \{u \in R^n : \|u - a\| < \alpha\}$. Then Exists k const's $\lambda_1,\ldots,\lambda_k$ s.t. $(\nabla f)(a) = \lambda_1(\nabla g_1)(a) + \cdots + \lambda_k(\nabla g_k)(a)$