Ham Sandwich is Equivalent to Borsuk-Ulam

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Joint work with Arpan Saha
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Ham Sandwich Theorem

Borsuk-Ulam Theorem

Brouwer’s Fixed-Point Theorem

Ham Sandwich is Equivalent to Borsuk-Ulam
Fixed-Point Theorems and Computation

- Ham Sandwich Theorem
- Borsuk-Ulam Theorem
- Brouwer’s Fixed-Point Theorem

Ham Sandwich is Equivalent to Borsuk-Ulam
Ham Sandwich Theorem

Borsuk-Ulam Theorem

Brouwer’s Fixed-Point Theorem

Ham Sandwich is Equivalent to Borsuk-Ulam
Ham Sandwich Theorem

Borsuk-Ulam Theorem

Brouwer’s Fixed-Point Theorem

Ham Sandwich is Equivalent to Borsuk-Ulam
Theorem (Borsuk, 1933)

Let $S^n$ denote the set of all points on the unit $n$-dimensional sphere. For any odd continuous mapping $f : S^n \rightarrow \mathbb{R}^n$ there is a point $x \in S^n$ for which $f(x) = 0$. 

Ham Sandwich is Equivalent to Borsuk-Ulam
Borsuk-Ulam Theorem

Theorem (Borsuk, 1933)

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$n = 1$
Borsuk-Ulam Theorem

**Theorem (Borsuk, 1933)**

Let $S^n$ denote the set of all points on the unit $n$-dimensional sphere. For any **odd** continuous mapping $f : S^n \rightarrow \mathbb{R}^n$ there is a point $x \in S^n$ for which $f(x) = \vec{0}$.

$n = 1$

![Diagram](image-url)
Ham Sandwich Theorem

Theorem (Stone and Tukey, 1942)

Given $n$ compact sets in $\mathbb{R}^n$ there is a $(n - 1)$-dimensional hyperplane which bisects each set into two sets of equal measure.
Ham Sandwich Theorem

Theorem (Stone and Tukey, 1942)

Given \( n \) compact sets in \( \mathbb{R}^n \) there is a \((n - 1)\)-dimensional hyperplane which bisects each set into two sets of equal measure.
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Ham Sandwich is Equivalent to Borsuk-Ulam
Borsuk-Ulam $\iff$ Ham Sandwich

Theorem (Our Result)

Ham Sandwich theorem is equivalent to Borsuk-Ulam theorem.
Query Model

\( A \)  \( B \)

\textbf{Input} \\
\textbf{Output} \\

Specific Queries \\
Specific Answers
Query Model

Input

Specific Queries

Specific Answers

Output

$A$

$B$
Query Model

Input → \( A \) → Output

Ham Sandwich is Equivalent to Borsuk-Ulam
Query Model

Input $A$ $\rightarrow$ $\rightarrow$ $\rightarrow$ $\rightarrow$ $B$ $\leftarrow$ $\leftarrow$ $\leftarrow$ $\leftarrow$ Output

Specific Queries $\rightarrow$ Specific Answers

$QC_p$: Number of queries to find correct answer with probability $p$. 

Ham Sandwich is Equivalent to Borsuk-Ulam
Query Model

\[ QC_p : \text{Number of queries to find correct answer with probability } p. \]
ABH(\(n, k, \varepsilon\)) Problem:

• Input: \(n\) compact sets \(A_1, \ldots, A_n \subseteq [−nk, nk]\).

• Output: \((n − 1)\)-dimensional hyperplane \(H\) such that:

\[\forall i \in [n], |\text{vol} (A_i \cap H^+ − \text{vol} (A_i \cap H^-)| \leq \varepsilon.\]

• Queries: Each query is an oriented hyperplane \(H\) and the answer is \(\text{vol} (A_i \cap H^+) − \text{vol} (A_i \cap H^-)\), for every \(i \in [n]\).

From Rubinstein (2016), Su’s construction (1997), and Ham Sandwich theorem ⇒ Borsuk-Ulam theorem:

Theorem (Our Result)

For large \(n\), \(\varepsilon \leq \frac{1}{\text{poly}(n)}\), and for \(p = 2^{−Ω(n)}\) and \(k \geq 5\) we have:

\[\text{QC}_p(\text{ABH}(n, k, \varepsilon)) = 2^{Ω(n)}.\]
Ham Sandwich Problem

\( \text{ABH}(n, k, \varepsilon) \) Problem:

- **Input:** \( n \) compact sets \( A_1, \ldots, A_n \subseteq [-n^k, n^k]^n \).

\( \text{From Rubinstein (2016), Su's construction (1997), and Ham Sandwich theorem} \Rightarrow \text{Borsuk-Ulam theorem:} \)

**Theorem (Our Result)**

For large \( n \), \( \varepsilon \leq \frac{1}{\text{poly}(n)} \), and for \( p = 2 - \Omega(n) \) and \( k \geq 5 \) we have:

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Ham Sandwich Problem

\textbf{ABH}(n, k, \varepsilon) \text{ Problem:}

- **Input:** \( n \) compact sets \( A_1, \ldots, A_n \subseteq [-n^k, n^k]^n \).

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  \[ \forall i \in [n], \quad |\text{vol}(A_i \cap H^+) - \text{vol}(A_i \cap H^-)| \leq \varepsilon. \]
Ham Sandwich Problem

ABH($n, k, \varepsilon$) Problem:

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From Rubinstein (2016), Su's construction (1997), and Ham Sandwich theorem: 

Theorem (Our Result)

For large $n$, $\varepsilon \leq 1/poly(n)$, and for $p = 2 - \Omega(n)$ and $k \geq 5$ we have: 

$\text{QC}_p(\text{ABH}(n, k, \varepsilon)) = 2^{\Omega(n)}$. 

Ham Sandwich Problem

\( \text{ABH}(n, k, \varepsilon) \) Problem:

- **Input:** \( n \) compact sets \( A_1, \ldots, A_n \subseteq [-n^k, n^k]^n \).
- **Output:** \((n - 1)\)-dimensional hyperplane \( H \) such that:
  \[
  \forall i \in [n], \quad |\text{vol}(A_i \cap H^+) - \text{vol}(A_i \cap H^-)| \leq \varepsilon.
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From Rubinstein (2016), Su’s construction (1997), and Ham Sandwich theorem \( \Rightarrow \) Borsuk-Ulam theorem:

**Theorem (Our Result)**

*For large \( n \), \( \varepsilon \leq 1/poly(n) \), and for \( p = 2^{-\Omega(n)} \) and \( k \geq 5 \) we have:*

\[
QC_p(\text{ABH}(n, k, \varepsilon)) = 2^{\Omega(n)}.
\]
Borsuk-Ulam $\iff$ Ham Sandwich

- Given $A_1, \ldots, A_n, A_{n+1}$ compact sets in $\mathbb{R}^{n+1}$
- Build odd $f : S^n \to \mathbb{R}^n$ such that:
  
  vanishing points $\iff$ bisecting hyperplanes
Borsuk-Ulam $\iff$ Ham Sandwich

- Given $A_1, \ldots, A_n, \{\vec{0}\}$ compact sets in $\mathbb{R}^{n+1}$
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- $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$, where $f_i(x) : S^n \to \mathbb{R}$
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- Every $x \in S^n$ is the normal of unique linear hyperplane $H_x$
Borsuk-Ulam $\iff$ Ham Sandwich

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- $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$, where $f_i(x) : S^n \to \mathbb{R}$
- Every $x \in S^n$ is the normal of unique linear hyperplane $H_x$
- For every $x \in S^n$:
  \[
  f_i(x) = \text{vol}(A_i \cap H_x^+) - \text{vol}(A_i \cap H_x^-)
  \]
Observation (From Previous Proof)

Let $A$ be a compact set in $\mathbb{R}^{n+1}$. Then, there is a continuous odd function $f : S^n \to \mathbb{R}$ such that $\forall x \in S^n$, $f(x) = \text{vol}(A \cap H^+_x) - \text{vol}(A \cap H^-_x)$. 
Our Result: Borsuk-Ulam $\iff$ Ham Sandwich

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Conjecture (Wishful Thinking)

Let $f : S^n \to \mathbb{R}$ be a polynomial odd function. Then, there is a compact set $A$ in $\mathbb{R}^{n+1}$ such that $\forall x \in S^n$, $f(x) = \text{vol}(A \cap H^+_x) - \text{vol}(A \cap H^-_x)$. 
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Lemma (Our Result)

Let $f : S^n \to \mathbb{R}$ be a polynomial odd function. Then, there is a compact set $A$ in $\mathbb{R}^{n+1}$ such that $\forall x \in S^n$, $f(x) = \text{vol}(A \cap H_x^+) - \text{vol}(A \cap H_x^-)$.
Lemma (Our Result)

Let $f : S^n \rightarrow \mathbb{R}$ be a polynomial odd function. Then, there is a compact set $A$ in $\mathbb{R}^{n+1}$ such that $\forall x \in S^n$, $f(x) = \text{vol}(A \cap H^+_x) - \text{vol}(A \cap H^-_x)$.

Proof Outline.

$$f : S^n \rightarrow \mathbb{R}$$
Lemma (Our Result)

Let \( f : S^n \to \mathbb{R} \) be a polynomial odd function. Then, there is a compact set \( A \) in \( \mathbb{R}^{n+1} \) such that \( \forall x \in S^n, \ f(x) = \text{vol}(A \cap H^+_x) - \text{vol}(A \cap H^-_x) \).

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\begin{align*}
  f : S^n & \to \mathbb{R} \\
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  f &: S^n \to \mathbb{R} \\
  r &: S^n \to \mathbb{R}^+ \\
  &\downarrow \\
  A_r &: \mathbb{R}^{n+1}
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Proof. Given $r : S^n \to \mathbb{R}^+$. $A_r$ is a compact set given by:

$$A_r = \left\{ k \cdot r(x) \cdot \vec{x} \mid x \in S^n, 0 \leq k \leq 1 \right\}.$$
Borsuk-Ulam ⇐ Ham Sandwich: Proof

Lemma (Our Result)

Let $f : S^n \rightarrow \mathbb{R}$ be a polynomial odd function. Then, there is a compact set $A$ in $\mathbb{R}^{n+1}$ such that $\forall x \in S^n$, $f(x) = \text{vol}(A \cap H^+_x) - \text{vol}(A \cap H^-_x)$.

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\text{vol}(A_r) = \int_{y \in S^n} (r(y))^{n+1} / (n + 1) \, dy
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\text{vol}(A_r \cap H^+_x) - \text{vol}(A_r \cap H^-_x) = \int_{y \in S^n} \text{sgn}(\langle x, y \rangle) \cdot (r(y))^{n+1}/(n + 1) \ dy
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Borsuk-Ulam \iff \text{Ham Sandwich: Proof}

\textbf{Lemma (Our Result)}

Let \( f : S^n \to \mathbb{R} \) be a polynomial odd function. Then, there is a compact set \( A \) in \( \mathbb{R}^{n+1} \) such that \( \forall x \in S^n, \ f(x) = \text{vol}(A \cap H_x^+) - \text{vol}(A \cap H_x^-) \).

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\[
\text{vol}(A_r \cap H_x^+) - \text{vol}(A_r \cap H_x^-) = \int_{y \in S^n} \text{sgn}(\langle x, y \rangle) \cdot r'(y) \ dy
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Lemma (Our Result)

Let \( f : S^n \to \mathbb{R} \) be a polynomial odd function. Then, there is a compact set \( A \) in \( \mathbb{R}^{n+1} \) such that \( \forall x \in S^n, \ f(x) = vol(A \cap H^+_x) - vol(A \cap H^-_x) \).

Proof (continued). Let \( p_1, p_2, \ldots, p_{m(d)} \) be a basis of polynomials of degree \( d \) over the hypersphere.
Lemma (Our Result)

Let \( f : S^n \to \mathbb{R} \) be a polynomial odd function. Then, there is a compact set \( A \) in \( \mathbb{R}^{n+1} \) such that \( \forall x \in S^n, f(x) = \text{vol}(A \cap H_x^+) - \text{vol}(A \cap H_x^-) \).

Proof (continued). Let \( p_1, p_2, \ldots, p_{m(d)} \) be a basis of polynomials of degree \( d \) over the hypersphere. If \( r' = \sum_i \alpha_i \cdot p_i \) then we have:

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Let \( f : S^n \rightarrow \mathbb{R} \) be a polynomial odd function. Then, there is a compact set \( A \) in \( \mathbb{R}^{n+1} \) such that \( \forall x \in S^n, f(x) = \text{vol}(A \cap H^+_x) - \text{vol}(A \cap H^-_x) \).

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f = \sum_i \beta_i \cdot p_i
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Proof (continued). Let \( p_1, p_2, \ldots, p_m(d) \) be a basis of polynomials of degree \( d \) over the hypersphere. If \( r' = \sum_i \alpha_i \cdot p_i \) then we have:

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\]

\[
f = \sum_i \beta_i \cdot p_i
\]

We need to find a basis such that:

\[
\int_{y \in S^n} \text{sgn}(\langle x, y \rangle) \cdot p_i(y) \ dy = \lambda_i \cdot p_i(x)
\]
Remarkable Objects: Hyperspherical Harmonics

- Homogeneous polynomials restricted to hypersphere
Remarkable Objects: Hyperspherical Harmonics

- Homogeneous polynomials restricted to hypersphere
- For every polynomial $p$ there is a hyperspherical harmonic $h$ such that $p|_{S^n} = h$ (Folklore, 1800s)
Remarkable Objects: Hyperspherical Harmonics

- Homogenenous polynomials restricted to hypersphere
- For every polynomial $p$ there is a hyperspherical harmonic $h$ such that $p|_{S^n} = h$ (Folklore, 1800s)
- Eigen functions of the following operator $T$ (Funk and Hecke, 1917):

\[ T(f)(x) := \int_{y \in S^n} u(\langle x, y \rangle) \cdot f(y) \, dy, \]

where $u : [-1, 1] \to \mathbb{R}$ is bounded and measurable
Key Takeaways

• Borsuk-Ulam Theorem is **Equivalent** to Ham Sandwich Theorem!
Key Takeaways

- Borsuk-Ulam Theorem is **Equivalent** to Ham Sandwich Theorem!
- Ham Sandwich Problem in high dimensions is **Hard**!
Thank you!