Oded (memo): Equivalnce of Matrix Inversion and Matrix Multiplication

The easy part is reducing matrix multiplication to matrix invertion. We show this first, and then show a reduction of matrix invertion to matrix multiplication.

Reducing matrix multiplication to matrix invertion. Here we observe that the inverse of the matrix

$$M = \begin{pmatrix} I & A & 0 \\ 0 & I & B \\ 0 & 0 & I \end{pmatrix}$$

is

$$M^{-1} = \begin{pmatrix} I & -A & AB \\ 0 & I & -B \\ 0 & 0 & I \end{pmatrix}$$

Hence, AB is computed by computing M^{-1}.

Reducing matrix inversion to matrix multiplication. The reduction uses recursion. Specifically, we reduce inverting the matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

to matrix multiplication by observing that if we multiply M on the left by the matrix

$$F = \begin{pmatrix} I & -BD^{-1} \\ 0 & I \end{pmatrix}$$

then we get the lower-triangular matrix matrix

$$Q = \begin{pmatrix} A - BD^{-1}C & 0 \\ C & D \end{pmatrix}$$

which is "easy to invert" (i.e., it is easy to write a formula for Q^{-1}). Specifically, note that the inverse of the lower-triangular matrix

$$L = \begin{pmatrix} R & 0 \\ S & T \end{pmatrix}$$

is

$$L^{-1} = \begin{pmatrix} R^{-1} & 0 \\ -T^{-1}SR^{-1} & T^{-1} \end{pmatrix}$$

Applying this formula to the foregoing matrix Q, we write its inverse (i.e., Q^{-1}) as

$$\begin{pmatrix} (A - BD^{-1}C)^{-1} & 0 \\ -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1} \end{pmatrix}$$

Now, $FM = Q$ implies that $M^{-1} = Q^{-1}F$. Hence, the inverse of the matrix M is obtained by computing F, Q and Q^{-1} and the product $Q^{-1}F$. Hence, M^{-1} is computed as follows.

1. We compute the inverse of D.
2. We compute the product $R = A - BD^{-1}C$.
3. We compute the inverse of R.
4. We compute the product $-D^{-1}CR$.
   (This yields Q^{-1}, whereas F can be computed along the way for free.)
5. Lastly, we compute the product $Q^{-1}F$.

Now, letting $\mathrm{Inv}(m)$ (resp., $\mathrm{MM}(m)$) denote the complexity of performing m-by-m matrix inversion (resp., multiplication), we obtain the following recursion for the complexity of matrix inversion

$$\begin{aligned} \mathrm{Inv}(2n) &= 2 \cdot \mathrm{Inv}(n) + 4 \cdot \mathrm{MM}(n) + \mathrm{MM}(2n) + O(n^2) \\ &= 2 \cdot \mathrm{Inv}(n) + O(\mathrm{MM}(n)) \end{aligned}$$

where we use $\mathrm{MM}(2n) \leq 8 \cdot \mathrm{MM}(n)$ and $\mathrm{MM}(n) > n^2$. This implies

$$\mathrm{Inv}(2^t) = \sum_{i \in [t]} 2^i \cdot O(M(2^{t-i})) = O(M(2^t)).$$