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Fragments of a chapter on Encryption Schemes

(revised, third posted version)

Extracts from a working draft
for Volume 2 of Foundations of Cryptography

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Preface

The current manuscript is a preliminary draft of the chapter on encryption schemes (Chapter 5) of the second volume of the work *Foundations of Cryptography*. This manuscript subsumes previous versions posted in Dec. 1999 and June 2001, respectively.

The bigger picture. The current manuscript is part of a working draft of Part 2 of the three-part work *Foundations of Cryptography* (see Figure 0.1). The three parts of this work are *Basic Tools*, *Basic Applications*, and *Beyond the Basics*. The first part (containing Chapters 1–4) has been published by Cambridge University Press (in June 2001). The second part, consists of Chapters 5–7 (regarding Encryptioni Schemes, Signatures Schemes, and General Cryptographic Protocols, respectively). We hope to publish the second part with Cambridge University Press within a couple of years.

Part 1: Introduction and Basic Tools
- Chapter 1: Introduction
- Chapter 2: Computational Difficulty (One-Way Functions)
- Chapter 3: Pseudorandom Generators
- Chapter 4: Zero-Knowledge Proofs

Part 2: Basic Applications
- Chapter 5: Encryption Schemes
- Chapter 6: Signature Schemes
- Chapter 7: General Cryptographic Protocols

Part 3: Beyond the Basics
- ...

Figure 0.1: Organization of this work

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The partition of the work into three parts is a logical one. Furthermore, it offers the advantage of publishing the first part without waiting for the completion of the other parts. Similarly, we hope to complete the second part within a couple of years, and publish it without waiting for the third part.

Prerequisites. The most relevant background for this text is provided by basic knowledge of algorithms (including randomized ones), computability and elementary probability theory. Background on (computational) number theory, which is required for specific implementations of certain constructs, is not really required here.

Using this text. The text is intended as part of a work that is aimed to serve both as a textbook and a reference text. That is, it is aimed at serving both the beginner and the expert. In order to achieve this aim, the presentation of the basic material is very detailed so to allow a typical CS-undergraduate to follow it. An advanced student (and certainly an expert) will find the pace (in these parts) way too slow. However, an attempt was made to allow the latter reader to easily skip details obvious to him/her. In particular, proofs are typically presented in a modular way. We start with a high-level sketch of the main ideas, and only later pass to the technical details. Passage from high-level descriptions to lower level details is typically marked by phrases such as *details follow*.

> In a few places, we provide straightforward but tedious details in indented paragraphs as this one. In some other (even fewer) places such paragraphs provide technical proofs of claims that are of marginal relevance to the topic of the book.

More advanced material is typically presented at a faster pace and with less details. Thus, we hope that the attempt to satisfy a wide range of readers will not harm any of them.

Teaching. The material presented in the full (three-volume) work is, on one hand, way beyond what one may want to cover in a course, and on the other hand falls very short of what one may want to know about Cryptography in general. To assist these conflicting needs we make a distinction between *basic* and *advanced* material, and provide suggestions for further reading (in the last section of each chapter). In particular, sections, subsections, and subsubsections marked by an asterisk (*) are intended for advanced reading.

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Table of Contents

Extracted from a working draft of Goldreich's FOUNDATIONS OF CRYPTOGRAPHY. See copyright notice.

Extracted from a working draft of Goldreich's FOUNDATIONS OF CRYPTOGRAPHY. See copyright notice.

Part II

Basic Applications

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Chapter 5

Encryption Schemes

Up-to the 1970's, Cryptography was understood as the art of building encryption schemes; that is, the art of constructing schemes allowing secret data exchange over insecure channels. Since the 1970's, other tasks (e.g., signature schemes) have been recognized as falling within the domain of Cryptography (and even as being at least as central to Cryptography). Yet, the construction of encryption schemes remains, and is likely to remain, a central enterprise of Cryptography.

In this chapter we review the well-known notions of private-key and public-key encryption schemes. More importantly, we define what is meant by saying that such schemes are secure. It turns out that using randomness throughout the encryption process (i.e., not only at the key-generation phase) is essential to security. We present some basic constructions of secure (private-key and public-key) encryption schemes. Finally, we discuss "dynamic" notions of security culminating in robustness against chosen ciphertext attacks.

Teaching Tip: We assume that the reader is familiar with the material in previous chapters (and specifically with Sections 2.2, 2.4, 2.5, 3.2–3.4, and 3.6). This familiarity is important not only because we use some of the notions and results presented in these sections, but rather because we use similar proof techniques (and do it while assuming that this is *not* the reader's first encounter with these techniques).

5.1 The Basic Setting

Loosely speaking, encryption schemes are supposed to enable private communication between parties that communicate over an insecure channel. Thus, the basic setting consists of a *sender*, a *receiver*, and an *insecure channel* that may be tapped by an *adversary*. The goal is to allow the sender to transfer information to the receiver, over the insecure channel, without letting the adversary figure out this information. Thus, we distinguish between the actual (secret) information that the receiver wishes to transmit and the messages sent over the

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insecure communication channel. The former is called the *plaintext*, whereas the latter is called the *ciphertext*. Clearly, the ciphertext must differ from the plaintext or else the adversary can easily obtain the plaintext by tapping the channel. Thus, the sender must transform the plaintext into a ciphertext so that the receiver can retrieve the plaintext from the ciphertext, but the adversary cannot do so. Clearly, something must distinguish the receiver (who is able to retrieve the plaintext from the corresponding ciphertext) from the adversary (who cannot do so). Specifically, the receiver know something that the adversary does not know. This thing is called a *key*.

An encryption scheme consists of a method of transforming plaintexts to ciphertexts and vice versa, using adequate keys. These keys are essential to the ability to effect these transformations. We stress that the encryption scheme itself (i.e., the encryption/decryption algorithms) may be known to the adversary, and its security relies on the hypothesis that the adversary does not know the keys.[1] Formally, we need to consider a third algorithm; namely, a probabilistic algorithm used to generate keys. This algorithm must be probabilistic (or else, by invoking it the adversary obtains the very same key used by the receiver).

In accordance with the above, an encryption scheme consists of three algorithms. These algorithms are public (i.e., known to all parties). The obvious algorithms are the *encryption algorithm*, which transforms plaintexts to ciphertexts, and the *decryption algorithm*, which transforms ciphertexts to plaintexts. By the discussion above, it is clear that the description algorithm must employ a *key* that is known to the receiver but is not known to the adversary. This key is generated using a third algorithm, called the *key-generator*. Furthermore, it is not hard to see that the encryption process must also depend on the key (or else messages sent to one party can be read by a different party who is also a potential receiver). Thus, the key-generation algorithm is used to produce a pair of (related) keys, one for encryption and one for decryption. The encryption algorithm, given an encryption-key and a plaintext, produces a plaintext that when fed to the decryption algorithm, with the corresponding decryption-key, returns the original plaintext. We stress that knowledge of the decryption-key is essential for the latter transformation.

5.1.1 Private-Key versus Public-Key Schemes

A fundamental distinction between encryption schemes refers to the relation between the two keys (mentioned above). The simpler (and older) notion assumes that the encryption-key equals the decryption-key. Such schemes are called private-key (or *symmetric*). To use a private-key scheme, the legitimate parties must first agree on the secret key. This can be done by having one party generate the key at random and send it to the other party using a (secondary) channel that (unlike the main channel) is assumed to be secure (i.e., it can not

[1] In fact, in many cases, the legitimate interest may be served best by publicizing the scheme itself. In our opinion, this is the best way to obtain an (unbiased) expert evaluation of the security of the scheme.

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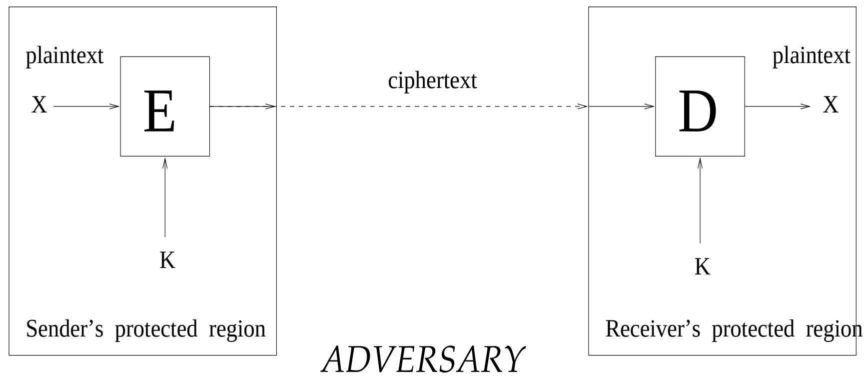

The key K is known to both receiver and sender, but is unknown to the adversary. For example, the receiver may generate K at random and pass it to the sender via a perfectly-private secondary channel (not shown here).

Figure 5.1: Private-key encryption schemes – an illustration.

be tapped by the adversary). A crucial point is that the key is generated independently of the plaintext, and so it can be generated and exchanged prior to the plaintext even being determined. Thus, private-key encryption is a way of extending a private channel over time: If the parties can use a private channel today (e.g., they are currently in the same physical location) but not tomorrow, then they can use the private channel today to exchange a secret key that they may use tomorrow for secret communication. A simple example of a private-key encryption scheme is the *one-time pad*. The secret key is merely a uniformly chosen sequence of n bits, and an n-bit long ciphertext is produced by XORing the plaintext, bit-by-bit, with the key. The plaintext is recovered from the ciphertext in the same way. Clearly, the one-time pad provides absolute security. However, its usage of the key is inefficient; or, put in other words, it requires keys of length comparable to the total length of data communicated. In the rest of this chapter we will only discuss encryption schemes where n-bit long keys allow to securely communicated data of length *greater than* n (but still polynomial in n).

A new type of encryption schemes has emerged in the 1970's. In these schemes, called public-key (or *asymmetric*), the decryption-key differs from the encryption-key. Furthermore, it is infeasible to find the decryption-key, given the encryption-key. These schemes enable secure communication without ever using a secure channel. Instead, each party applies the key-generation algorithm to produce a pair of keys. The party, called P, keeps the decryption-key, denoted d_P, secret and publishes the encryption-key, denoted e_P. Now, any party can send P private messages by encrypting them using the encryption-key e_P. Party P can decrypt these messages by using the decryption-key d_P, but nobody else

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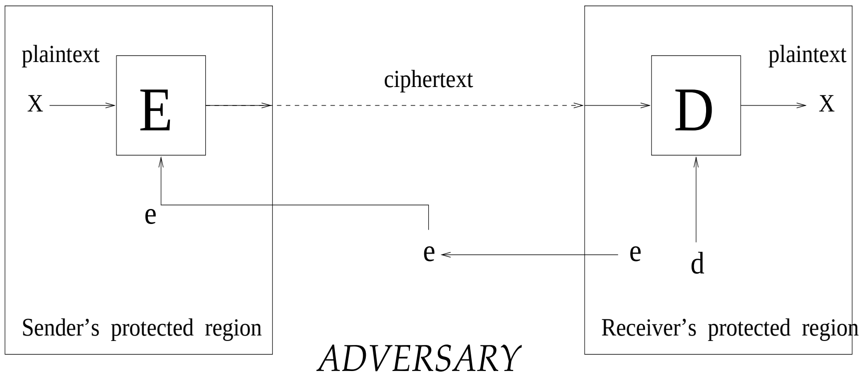

The key-pair (e, d) is generated by the receiver, who posts the encryption-key e on a public media, while keeping the decryption-key d secret.

Figure 5.2: Public-key encryption schemes – an illustration.

can do so.

5.1.2 The Syntax of Encryption Schemes

We start by defining the basic *mechanism of encryption schemes.* This definition says nothing about the security of the scheme (which is the subject of the next section).

Definition 5.1.1 (encryption scheme): *An* encryption scheme *is a triple, (G, E, D), of probabilistic polynomial-time algorithms satisfying the following two conditions*

1. *On input 1^n, algorithm G (called the* key-generator*) outputs a pair of bit strings.*

2. *For every pair (e, d) in the range of $G(1^n)$, and for every $\alpha \in \{0,1\}^*$, algorithms E* (encryption) *and D* (decryption) *satisfy*

$$\Pr[D(d, E(e, \alpha)) = \alpha] = 1$$

   where the probability is taken over the internal coin tosses of algorithms E and D.

The integer n serves as the security parameter *of the scheme. Each (e, d) in the range of $G(1^n)$ constitutes a pair of corresponding* encryption/decryption keys. *The string $E(e, \alpha)$ is the* encryption of the plaintext $\alpha \in \{0,1\}^*$ *using the encryption-key e, whereas $D(d, \beta)$ is the* decryption of the ciphertext β *using the decryption-key d.*

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We stress that Definition 5.1.1 says nothing about security, and so trivial (insecure) algorithms may satisfy it (e.g., $E(e,\alpha) \stackrel{\rm def}{=} \alpha$ and $D(d,\beta) \stackrel{\rm def}{=} \beta$). Furthermore, Definition 5.1.1 does not distinguish private-key encryption schemes from public-key ones. The difference between the two types is introduced in the security definitions: In a public-key scheme the "breaking algorithm" gets the encryption-key (i.e., e) as an additional input (and thus $e \neq d$ follows); while in private-key schemes e is not given to the "breaking algorithm" (and thus one may assume, without loss of generality, that $e = d$).

We stress that the above definition requires the scheme to operate for every plaintext, and specifically for plaintext of length exceeding the length of the encryption-key. (This rules out the information theoretic secure "one-time pad" scheme mentioned above.)

Notation: In the rest of this text, we write $E_e(\alpha)$ instead of $E(e,\alpha)$ and $D_d(\beta)$ instead of $D(d,\beta)$. Sometimes, when there is little risk of confusion, we drop these subscripts. Also, we let $G_1(1^n)$ (resp., $G_2(1^n)$) denote the first (resp., second) element in the pair $G(1^n)$. That is, $G(1^n) = (G_1(1^n), G_2(1^n))$. Without loss of generality, we may assume that $|G_1(1^n)|$ and $|G_2(1^n)|$ are polynomially related to n, and that each of these integers can be efficiently computed from the other. (In fact, we may even assume that $|G_1(1^n)| = |G_2(1^n)| = n$; see Exercise 5.)

Comments: Definition 5.1.1 may be relaxed in several ways without significantly harming its usefulness. For example, we may relax Condition (2) and allow a negligible decryption error (e.g., $\mathsf{Pr}[D_d(E_e(\alpha)) \neq \alpha] < 2^{-n}$). Alternatively, one may postulate that Condition (2) holds for all but a negligible measure of the key-pairs generated by $G(1^n)$. At least one of these relaxations is essential for each of the popular suggestions of encryption schemes.

Another relaxation consists of restricting the domain of possible plaintexts (and ciphertexts). For example, one may restrict Condition (2) to α's of length $\ell(n)$, where $\ell : \mathbb{N} \to \mathbb{N}$ is some fixed function. Given a scheme of the latter type (with plaintext length ℓ), we may construct a scheme as in Definition 5.1.1 by breaking plaintexts into blocks of length $\ell(n)$ and applying the restricted scheme separately to each block. For more details see Sections 5.2.4 and 5.3.2.

5.2 Definitions of Security

In this section we present two fundamental definitions of security and prove their equivalence. The first definition, called *semantic security*, is the most natural one. Semantic security is a computational complexity analogue of Shannon's definition of perfect privacy (which requires that the ciphertext yields no information regarding the plaintext). Loosely speaking, an encryption scheme is semantically secure if it is *infeasible* to learn anything about the plaintext from the ciphertext (i.e., impossibility is replaced by infeasibility). The second def-

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inition has a more technical flavor. It interprets security as the infeasibility of distinguishing between encryptions of a given pair of messages. This definition is useful in demonstrating the security of a proposed encryption scheme, and for the analysis of cryptographic protocols that utilize an encryption scheme.

We stress that the definitions presented below go way beyond saying that it is infeasible to recover the plaintext from the ciphertext. The latter statement is indeed a minimal requirement from a secure encryption scheme, but we claim that it is way too weak a requirement: An encryption scheme is typically used in applications where obtaining specific partial information on the plaintext endangers the security of the application. When designing an application-independent encryption scheme, we do not know which partial information endangers the application and which does not. Furthermore, even if one wants to design an encryption scheme tailored to one's own specific applications, it is rare (to say the least) that one has a precise characterization of all possible partial information that endanger these applications. Thus, we require that it is infeasible to obtain any information about the plaintext from the ciphertext. Furthermore, in most applications the plaintext may not be uniformly distributed and some a-priori information regarding it may be available to the adversary. We require that the secrecy of all partial information is preserved also in such a case. That is, even in presence of a-priori information on the plaintext, it is infeasible to obtain any (new) information about the plaintext from the ciphertext (beyond what is feasible to obtain from the a-priori information on the plaintext). The definition of semantic security postulates all of this.

Security of multiple plaintexts. In continuation to the above discussion, the definitions are presented first in terms of the security of a single encrypted plaintext. However, in many cases, it is desirable to encrypt many plaintexts using the same encryption-key, and security needs to be preserved in these cases too. Adequate definitions and discussions are deferred to Section 5.2.4.

A technical comment: non-uniform complexity formulation. To simplify the exposition, we adopt a non-uniform formulation. Namely, in the security definitions we expand the domain of efficient adversaries/algorithms to include (explicitly or implicitly) non-uniform polynomial-size circuits, rather than only probabilistic polynomial-time machines. Likewise, we make no computation restriction regarding the probability distribution from which messages are taken, nor regarding the a-priori information available on these messages. We note that employing such a non-uniform complexity formulation (rather than a uniform one) may only strengthen the definitions; yet, it does weaken the implications proven between the definitions, since these (simpler) proofs make free usage of non-uniformity. A uniform-complexity treatment is provided in Section 5.2.5.

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5.2.1 Semantic Security

Loosely speaking, semantic security means that whatever can be efficiently computed from the ciphertext, can be efficiently computed also without the ciphertext. Thus, an adversary gains nothing by intercepting ciphertexts sent between communicating parties who use a semantically secure encryption scheme, since it could have obtained the same without intercepting these ciphertexts. Indeed, this formulation follows the simulation paradigm: "lack of gain" is captured by asserting that whatever is learned from the ciphertext can be learned within related complexity also without the ciphertext.

5.2.1.1 The actual definitions

To be somewhat more accurate, semantic security means that whatever can be efficiently computed from the ciphertext, can be efficiently computed when *given only the length of the plaintext*. Note that this formulation does not rule out the possibility that the length of the plaintext can be inferred from the ciphertext. Indeed, some information about the length of the plaintext must be revealed by the ciphertext (see Exercise 3). We stress that other than information about the length of the plaintext, the ciphertext is required to yield nothing about the plaintext.

In the actual definitions, we consider only information regarding the plaintext (rather than regarding something else like the ciphertext) that can be obtained from the ciphertext. Furthermore, we restrict our attention to functions (rather than to randomized processes) applied to the plaintext. We do so because of the intuitive appeal of this special case, and are comfortable doing so because this special case implies the general one (cf. Exercise 12). We augment this formulation by requiring that the above remains valid even in presence of auxiliary partial information about the plaintext. Namely, whatever can be efficiently computed from the ciphertext and additional partial information about the plaintext, can be efficiently computed given only the length of the plaintext and the same partial information. In the definition that follows, the information regarding the plaintext that the adversary tries to obtain is captured by the function f, whereas the a-priori partial information about the plaintext is captured by the function h. The above is required to hold for any distribution of plaintexts, captured by the probability ensemble $\{X_n\}_{n\in\mathbb{N}}$.

Security holds only for plaintexts of length polynomial in the security parameter. This is captured below by the restriction $|X_n| = \text{poly}(n)$. Note that we cannot hope to provide computational security for plaintexts of unbounded length in the security parameter (see Exercise 2). Likewise, we restrict the functions f and h to be *polynomially-bounded*; that is, $|f(x)|, |h(x)| = \text{poly}(|x|)$.

The difference between private-key and public-key encryption schemes is manifested in the definition of security. In the latter case, the adversary (which is trying to obtain information on the plaintext) is given the encryption-key, whereas in the former case it is not. Thus, the difference between these schemes amounts to a difference in the adversary model (considered in the definition

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of security). We start by presenting the definition for private-key encryption schemes.

Definition 5.2.1 (semantic security – private-key): *An encryption scheme, (G,E,D), is* semantically secure (in the private-key model) *if for every probabilistic polynomial-time algorithm A there exists a probabilistic polynomial-time algorithm A' so that for every ensemble $\{X_n\}_{n\in\mathbb{N}}$, with $|X_n| = \text{poly}(n)$, every pair of polynomially-bounded functions $f,h : \{0,1\}^* \to \{0,1\}^*$, every polynomial $p(\cdot)$ and all sufficiently large n*

$$\begin{aligned} & \Pr\left[A(1^n, E_{G_1(1^n)}(X_n), 1^{|X_n|}, h(X_n)) = f(X_n)\right] \\ & < \Pr\left[A'(1^n, 1^{|X_n|}, h(X_n)) = f(X_n)\right] + \frac{1}{p(n)} \end{aligned}$$

(The probability in the above terms is taken over X_n as well as over the internal coin tosses of algorithms either G, E and A or A'.)

The input 1^n is given to both algorithms for technical reasons.[2] The function h provides both algorithms with partial information regarding the plaintext X_n. Furthermore, h also makes the definition implicitly non-uniform; see further discussion below. In addition, both algorithms get the length of X_n. These algorithms then try to guess the value $f(X_n)$; namely, they try to infer information about the plaintext X_n. Loosely speaking, in semantically secure encryption scheme the ciphertext does not help in this inference task. That is, the success probability of any efficient algorithm (i.e., algorithm A) that is given the ciphertext, can be matched, up-to a negligible fraction, by the success probability of an efficient algorithm (i.e., algorithm A') that is not given the ciphertext at all.

Definition 5.2.1 refers to private-key encryption schemes. To derive a definition of security for public-key encryption schemes, the encryption-key (i.e., $G_1(1^n)$) should be given to the adversary as an additional input. That is,

Definition 5.2.2 (semantic security – public-key): *An encryption scheme, (G,E,D), is* semantically secure (in the public-key model) *if for every probabilistic polynomial-time algorithm A, there exists a probabilistic polynomial-time algorithm A' such that for every $\{X_n\}_{n\in\mathbb{N}}$, f,h, $p(\cdot)$ and n as in Definition 5.2.1*

$$\begin{aligned} & \Pr\left[A(1^n, G_1(1^n), E_{G_1(1^n)}(X_n), 1^{|X_n|}, h(X_n)) = f(X_n)\right] \\ & < \Pr\left[A'(1^n, 1^{|X_n|}, h(X_n)) = f(X_n)\right] + \frac{1}{p(n)} \end{aligned}$$

We comment that it is pointless to give the random encryption-key (i.e., $G_1(1^n)$) to algorithm A' (since the task and main inputs of A' are unrelated to the encryption-key, and anyhow A' could generate a random encryption-key by itself).

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[2] The role of the auxiliary input 1^n is to allow smooth transition to fully non-uniform formulations as discussed below and as in Definition 5.2.3.

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Terminology: For sake of simplicity, we refer to an encryption scheme that is semantically secure in the private-key (resp., public-key) model as to a semantically-secure private-key (resp., public-key) encryption scheme.

The reader may note that a semantically-secure *public-key* encryption scheme cannot employ a deterministic encryption algorithm; that is, $E_e(x)$ must be a random variable rather than a fixed string. This is more evident with respect to the equivalent Definition 5.2.4 (below). See further discussion following Definition 5.2.4.

5.2.1.2 Further discussion of some definitional choices

We discuss several secondary issues regarding Definitions 5.2.1 and 5.2.2. The interested reader is also referred to Exercises 16, 15 and 17 that present additional variants of the definition of semantic security.

Implicit non-uniformity of the definitions. The fact that h is not required to be computable, makes the above definitions non-uniform. This is the case because both algorithms are given $h(X_n)$ as auxiliary input, and this may account for arbitrary (polynomially-bounded) advise. For example, letting $h(x) = a_{|x|}$, means that both algorithms are supplied with (non-uniform) advice (as in one of the possible formulations of non-uniform polynomial-time; see Section 1.3.3). In general, the function h can code both information regarding its input and non-uniform advice depending on its input length (i.e., $h(x) = (h'(x), a_{|x|})$). Thus, the above definitions are equivalent to allowing A and A' be *related* families of non-uniform circuits, where by 'related' we mean that the circuits in the family $A' = \{A'_n\}_{n\in\mathbb{N}}$ can be efficiently computed from the corresponding circuits in the family $A = \{A_n\}_{n\in\mathbb{N}}$. For further discussion, see Exercise 8.

Lack of computational restrictions regarding the function f. We do not require that the function f is even computable. This seems strange at first glance, because (unlike the situation w.r.t h which codes a-priori information given to the algorithms) the algorithms are asked to guess the value of f (on a plaintext implicit in the ciphertext given only to A). However, as we shall see in the sequel (see also Exercise 12), the meaning of semantic security is essentially that the distribution ensembles $(E(X_n), 1^{|X_n|}, h(X_n))$ and $(E(1^{|X_n|}), 1^{|X_n|}, h(X_n))$ are computationally indistinguishable (and so whatever A can compute can also be computed by A').

Other modifications of no impact. Actually, inclusion of a-priori information regarding the plaintext (captured by the function h) does not affect the definition of semantic security: Definition 5.2.1 remains intact if we restrict h to only depend on the length of the plaintext (and so only provide plaintext-oblivious non-uniform advice). (This can be shown in various ways; e.g., see Exercise 13.1.) Also, the function f can be restricted to be a Boolean function having polynomial-size circuits, and the random variable X_n may be restricted

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to be very "dull" (e.g., have only two strings in its support): See proof of Theorem 5.2.5. On the other hand, Definition 5.2.1 implies stronger forms discussed in Exercises 12, 17 and 18.

5.2.2 Indistinguishability of Encryptions

The following technical interpretation of security states that it is infeasible to distinguish the encryptions of two plaintexts (of the same length). That is, such ciphertexts are computationally indistinguishable as defined in Definition 3.2.7. Again, we start with the private-key variant.

Definition 5.2.3 (indistinguishability of encryptions – private-key): *An encryption scheme, (G,E,D), has* indistinguishable encryptions (in the private-key model) *if for every polynomial-size circuit family $\{C_n\}$, every polynomial p, all sufficiently large n and every $x,y \in \{0,1\}^{\text{poly}(n)}$* (i.e., $|x| = |y|$),

$$|\Pr\left[C_n(E_{G_1(1^n)}(x))=1\right] - \Pr\left[C_n(E_{G_1(1^n)}(y))=1\right]| < \frac{1}{p(n)}$$

The probability in the above terms is taken over the internal coin tosses of algorithms G and E.

Note that the potential plaintexts to be distinguished can be incorporated into the circuit C_n. Thus, the circuit models both the adversary's strategy and its a-priori information: See Exercise 10.

Again, the security definition for public-key encryption schemes can be derived by adding the encryption-key (i.e., $G_1(1^n)$) as an additional input to the algorithm. That is,

Definition 5.2.4 (indistinguishability of encryptions – public-key): *An encryption scheme, (G,E,D), has* indistinguishable encryptions (in the public-key model) *if for every polynomial-size circuit family $\{C_n\}$, and every $p(\cdot)$, n, x and y as in Definition 5.2.3*

$$|\Pr\left[C_n(G_1(1^n), E_{G_1(1^n)}(x))=1\right] - \Pr\left[C_n(G_1(1^n), E_{G_1(1^n)}(y))=1\right]| < \frac{1}{p(n)}$$

Terminology: For sake of simplicity, we refer to an encryption scheme that has indistinguishable encryptions in the private-key (resp., public-key) model as to a ciphertext-indistinguishable private-key (resp., public-key) encryption scheme.

Failure of deterministic encryption algorithms: A ciphertext-indistinguishable *public-key* encryption scheme cannot employ a deterministic encryption algorithm (i.e., $E_e(x)$ cannot be a fixed string). For a public-key encryption scheme with a *deterministic* encryption algorithm E, given an encryption-key e and a pair of candidate plaintexts (x,y), one can easily distinguish $E_e(x)$ from $E_e(y)$ (by merely applying E_e to x and comparing the result to the given ciphertext). In contrast, in case the encryption algorithm itself is randomized, the

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same plaintext can be encrypted in exponentially many different ways, under the same encryption-key. Furthermore, the probability that applying E_e twice to the same message (while using independent randomization in E_e) results in the same ciphertext may be exponentially vanishing. (Indeed, as shown below, public-key encryption scheme having indistinguishable encryptions can be constructed based on any trapdoor permutations, and these schemes employ randomized encryption algorithms.)

5.2.3 Equivalence of the Security Definitions

The following theorem is stated and proven for private-key encryption schemes. A similar result holds for public-key encryption schemes (see Exercise 11).

Theorem 5.2.5 (equivalence of definitions – private-key): *A private-key encryption scheme is semantically secure if and only if it has indistinguishable encryptions.*

Let (G, E, D) be an encryption scheme. We formulate a proposition for each of the two directions of the above theorem. Each proposition is in fact stronger than the corresponding direction stated in Theorem 5.2.5. The more useful direction is stated first: it asserts that the technical interpretation of security, in terms of ciphertext-indistinguishability, implies the natural notion of semantic security. Thus, the following proposition yields a methodology for designing semantically secure encryption schemes: design and prove your scheme to be ciphertext-indistinguishable, and conclude (by applying the proposition) that it is semantically secure. The opposite direction (of Theorem 5.2.5) establish the "completeness" of the latter methodology, and more generally assert that requiring an encryption scheme to be ciphertext-indistinguishable does not rule out schemes that are semantically secure.

Proposition 5.2.6 (useful direction – "indistinguishability" implies "security"): *Suppose that (G, E, D) is a ciphertext-indistinguishable private-key encryption scheme. Then (G, E, D) is semantically-secure. Furthermore, the simulating algorithm A' (which is used to establish semantic-security) captures the computation of a probabilistic polynomial-time oracle machine that is given oracle access to original adversary algorithm A.*

Proposition 5.2.7 (opposite direction – "security" implies "indistinguishability"): *Suppose that (G, E, D) is a semantically secure private-key encryption scheme. Then (G, E, D) has indistinguishable encryptions. Furthermore, the conclusion holds even if the definition of semantic security is restricted to the special case satisfying the following four conditions:*

1. *the random variable X_n is uniformly distributed over a set containing two strings;*

2. *the value of h depends only on the length of its input (i.e., $h(x) = h'(|x|)$);*

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3. *the function f is Boolean and is computable by a polynomial-size circuit;*

4. *the algorithm A is deterministic.*

In addition, no computational restrictions are placed on algorithm A' and it can be replaced by any function, which may depend on $\{X_n\}_{n\in\mathbb{N}}$, h, f and A.

Observe that the above four itemized conditions limit the scope of the four universal quantifiers in Definition 5.2.1, whereas the last sentence removes a restriction on the existential quantifier (i.e., removes the complexity bound on A') and allows the latter to depend on all universal quantifiers. Each of these modifications makes the resulting definition potentially weaker. Still, combining Propositions 5.2.7 and 5.2.6 it follows that a weak version of Definition 5.2.1 implies (an even stronger version than) the one stated in Definition 5.2.1.

5.2.3.1 Proof of Proposition 5.2.6.

Suppose that (G, E, D) has indistinguishable encryptions. We will show that (G, E, D) is semantically secure by *constructing*, for every probabilistic polynomial-time algorithm A, a probabilistic polynomial-time algorithm A' such that the following holds: *for every $\{X_n\}_{n\in\mathbb{N}}$, f and h, algorithm A' guesses $f(X_n)$ from $(1^n, 1^{|X_n|}, h(X_n))$ essentially as good as A guesses $f(X_n)$ from $(1^n, E(X_n), 1^{|X_n|}, h(X_n))$.* Specifically, A' merely invokes A on input $(E(1^{|X_n|}), 1^{|X_n|}, h(X_n))$, and returns whatever A does. That is, A' invokes A with a dummy encryption rather than with an encryption of X_n (which A expects to get, but A' does not have). Intuitively, the indistinguishability of encryptions implies that A behaves as well when invoked by A' (and given a dummy encryption) as when given the encryption of X_n, and this establishes the desired claim. Below, we merely implement the above plan, where the main issue in the implementation is to who that the specific formulation of indistinguishability of encryptions suffices to establish the above eluded "similar behavior" clause (which refers in success in guessing the value of $f(X_n)$).

Let A be an algorithm that tries to infer partial information (i.e., the value $f(X_n)$) from the encryption of the message X_n (when also given $1^n, 1^{|X_n|}$ and a-priori information $h(X_n)$). Intuitively, on input $E(\alpha)$ and $(1^{|\alpha|}, h(\alpha))$, algorithm A tries to guess $f(\alpha)$. We construct a new algorithm, A', that performs as well without getting the input $E(\alpha)$. The new algorithm consists of invoking A on input $E_{G_1(1^n)}(1^{|\alpha|})$ and $(1^n, 1^{|\alpha|}, h(\alpha))$, and outputting whatever A does. That is, on input $(1^n, 1^{|\alpha|}, h(\alpha))$, algorithm A' proceeds as follows:

1. A' invokes the key-generator G (on input 1^n), and obtains an encryption-key $e \leftarrow G_1(1^n)$.
2. A' invokes the encryption algorithm with key e and ("dummy") plaintext $1^{|\alpha|}$, obtaining a ciphertext $\beta \leftarrow E_e(1^{|\alpha|})$.
3. A' invokes A on input $(\beta, 1^{|\alpha|}, h(\alpha))$, and outputs whatever A does.

Observe that A' is described in terms of an oracle machine that makes a single oracle call to (any given) A, in addition to invoking the fixed algorithms G

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and E. Furthermore, the construction of A' does not depend on the functions h and f or on the distribution of messages to be encrypted (represented by the probability ensembles $\{X_n\}_{n\in\mathbb{N}}$). Thus, A' is probabilistic polynomial-time whenever A is probabilistic polynomial-time (and regardless of the complexity of h, f and $\{X_n\}_{n\in\mathbb{N}}$)

Indistinguishability of encryptions will be used to prove that A' performs essentially as well as A. Specifically, the proof will use a reducibility argument.

Claim 5.2.6.1: Let A' be as above. Then, for every $\{X_n\}_{n\in\mathbb{N}}$, f, h and p as in Definition 5.2.1, and all sufficiently large n's

$$\begin{aligned}&\mathsf{Pr}\left[A(1^n, E_{G_1(1^n)}(X_n), 1^{|X_n|}, h(X_n))=f(X_n)\right]\\ &\quad < \mathsf{Pr}\left[A'(1^n, 1^{|X_n|}, h(X_n))=f(X_n)\right] + \frac{1}{p(n)}\end{aligned}$$

Proof: To simplify the notations, let us incorporate $(1^n, 1^{|\alpha|})$ into $h(\alpha)$. Using the definition of A', we can rewritten the claim as asserting

$$\begin{aligned}&\mathsf{Pr}\left[A(E_{G_1(1^n)}(X_n), h(X_n))=f(X_n)\right]\\ &\quad < \mathsf{Pr}\left[A(E_{G_1(1^n)}(1^{|X_n|}), h(X_n))=f(X_n)\right] + \frac{1}{p(n)}\end{aligned}$$

Intuitively, this follows by the indistinguishability of encryptions, by fixing a violating value of X_n and incorporating the corresponding values of $h(X_n)$ and $f(X_n)$ in a description of a circuit (which will distinguish an encryption of this value of X_n from an encryption of $1^{|X_n|}$). Details follow.

Assume, towards the contradiction that for some polynomial p and infinitely many n's the above inequality is violated. Then, for each such n, we have $\mathsf{E}[\Delta(X_n)] > 1/p(n)$, where

$$\Delta(x) \stackrel{\text{def}}{=} \left|\mathsf{Pr}\left[A(E_{G_1(1^n)}(x), h(x))=f(x)\right] - \mathsf{Pr}\left[A(E_{G_1(1^n)}(1^{|x|}), h(x))=f(x)\right]\right|$$

We use an averaging argument to single out a string x_n in the support of X_n such that $\Delta(x_n) \geq \Delta(X_n)$: That is, let $x_n \in \{0,1\}^{\text{poly}(n)}$ be a string for which the value of $\Delta(\cdot)$ is maximum, and so $\Delta(x_n) > 1/p(n)$. Using this x_n, we introduce a circuit C_n, which incorporates the fixed values $f(x_n)$ and $h(x_n)$, and distinguishes the encryption of x_n from the encryption of $1^{|x_n|}$. The circuit C_n operates as follows. On input $\beta = E(\alpha)$, the circuit C_n invokes $A(\beta, h(x_n))$ and outputs 1 if and only if A outputs the value $f(x_n)$. Otherwise, C_n outputs 0.

The above circuit is indeed of polynomial-size because it merely incorporates strings of polynomial length (i.e., $f(x_n)$ and $h(x_n)$) and emulates a polynomial-time computation (i.e., of A). (Note that the circuit family $\{C_n\}$ is indeed non-uniform since its definition is based on a non-uniform selection of x_n's as well as on a hard-wiring of (possibly uncomputable) corresponding strings $h(x_n)$ and $f(x_n)$.) Clearly,

$$\mathsf{Pr}\left[C_n(E_{G_1(1^n)}(\alpha))=1\right] = \mathsf{Pr}\left[A(E_{G_1(1^n)}(\alpha), h(x_n))=f(x_n)\right] \tag{5.1}$$

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Combining Eq. (5.1) with the definition of $\Delta(x_n)$, we get

$$\begin{aligned}\left|\Pr\left[C_n(E_{G_1(1^n)}(x_n))=1\right] - \Pr\left[C_n(E_{G_1(1^n)}(1^{|x_n|}))=1\right]\right| &= \Delta(x_n) \\ &> \frac{1}{p(n)}\end{aligned}$$

This contradicts our hypothesis that E has indistinguishable encryptions, and the claim follows. □

We have just shown that A' performs essentially as well as A, and so Proposition 5.2.6 follows. ■

Comments: The fact that we deal with a non-uniform model of computation allows the above proof to proceed regardless of the complexity of f and h. All that our definition of C_n requires is the hardwiring of the values of f and h on a single string, and this can be done regardless of the complexity of f and h (provided that they are both polynomially-bounded; i.e., $|f(x)|, |h(x)| = \text{poly}(|x|)$).

When proving the public-key analogue of Proposition 5.2.6, algorithm A' is defined exactly as above, but its analysis is slightly different: the distinguishing circuit, considered in the analysis of the performance of A', obtains the encryption-key as part of its input, and passes it to algorithm A (upon invoking it).

5.2.3.2 Proof of Proposition 5.2.7

Intuitively, indistinguishability of encryption (i.e., of the encryptions of x_n and y_n) is a special case of semantic security in which f indicates one of the plaintexts and h does not distinguish them (i.e., $f(z) = 1$ iff $z = x_n$ and $h(x_n) = h(y_n)$). The only issue to be addressed by the actual proof is that semantic security refers to uniform (probabilistic polynomial-time) adversaries, whereas indistinguishability of encryption refers to non-uniform polynomial-size circuits. This gap is bridged by using the function h to provide the algorithms in the semantic-security formulation with adequate non-uniform advice (which may be used by the circuit in the indistinguishability of encryption formulation).

The actual proof is by a (direct) reducibility argument. We show that if (G, E, D) has distinguishable encryptions then it is not semantically secure (not even in the restricted sense mentioned in the furthermore-clause of the proposition). Towards this end, we assume that there exists a polynomial p, a polynomial-size circuit family $\{C_n\}$, such that for infinitely many n's there exists $x_n, y_n \in \{0,1\}^{\text{poly}(n)}$ so that

$$\left|\Pr\left[C_n(E_{G_1(1^n)}(x_n))=1\right] - \Pr\left[C_n(E_{G_1(1^n)}(y_n))=1\right]\right| > \frac{1}{p(n)} \tag{5.2}$$

Using this sequence of C_n's, x_n's and y_n's, we define $\{X_n\}_{n\in\mathbb{N}}$, f and h (referred to in Definition 5.2.1) as follows:

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- The probability ensembles $\{X_n\}_{n\in\mathbb{N}}$ is defined such that X_n is uniformly distributed over $\{x_n, y_n\}$.
- The function $f:\{0,1\}^*\rightarrow\{0,1\}$ is defined such that $f(x_n) = 1$ and $f(y_n) = 0$, for every n. Note that $f(X_n) = 1$ with probability $1/2$ and is 0 otherwise.
- The function h is defined such that $h(X_n)$ equals the description of the circuit C_n. Note that $h(X_n) = C_n$ with probability 1, and thus reveals no information on the value of X_n. (In the sequel, we write $h(X_n) = h'(n) = C_n$.)

(Note that X_n, f and h satisfy the restrictions stated in the furthermore-clause of the proposition.)

We will present a (deterministic) polynomial-time algorithm A that, given $C_n = h(X_n)$, guesses the value of $f(X_n)$ from the encryption of X_n, and does so significantly better that with probability $\frac{1}{2}$. This violates (even the restricted form of) semantic security, since no algorithm (regardless of its complexity) can guess $f(X_n)$ better than with probability $1/2$ when only given $1^{|X_n|}$ (because given the constant values $1^{|X_n|}$ and $h(X_n)$, the value of $f(X_n)$ is uniformly distributed over $\{0,1\}$). Details follow.

Let us assume, without loss of generality, that for infinitely many n's

$$\mathsf{Pr}\left[C_n(E_{G_1(1^n)}(x_n))=1\right] \;>\; \mathsf{Pr}\left[C_n(E_{G_1(1^n)}(y_n))=1\right] + \frac{1}{p(n)} \tag{5.3}$$

Claim 5.2.7.1: There exists a (deterministic) polynomial-time algorithm A such that for infinitely many n's

$$\mathsf{Pr}\left[A(1^n, E_{G_1(1^n)}(X_n), 1^{|X_n|}, h(X_n))=f(X_n)\right] \;>\; \frac{1}{2}+\frac{1}{2p(n)}$$

Proof: Algorithm A uses $C_n = h(X_n)$ in a straightforward manner: On input $\beta = E(\alpha)$ (where α is in the support of X_n) and $(1^n, 1^{|\alpha|}, h(\alpha))$, algorithm A recovers $C_n = h'(|X_n|) = h(\alpha)$, invokes C_n on input β, and outputs 1 if C_n outputs 1 (otherwise, C_n outputs 0).[3]

It is left to analyze the success probability of A. Letting $m = |x_n| = |y_n|$, we have

$$\begin{aligned}
&\mathsf{Pr}\left[A(1^n, E_{G_1(1^n)}(X_n), 1^{|X_n|}, h(X_n))=f(X_n)\right]\\
&= \frac{1}{2}\cdot\mathsf{Pr}\left[A(1^n, E_{G_1(1^n)}(X_n), 1^{|X_n|}, h(X_n))=f(X_n)\,|\,X_n=x_n\right]\\
&\quad+\frac{1}{2}\cdot\mathsf{Pr}\left[A(1^n, E_{G_1(1^n)}(X_n), 1^{|X_n|}, h(X_n))=f(X_n)\,|\,X_n=y_n\right]
\end{aligned}$$

[3] We comment that the '1' output by C_n is an indication that α is more likely to be x_n, whereas the output of A is a guess of $f(\alpha)$. This point may be better stressed by redefining f so that $f(x_n) \stackrel{\rm def}{=} x_n$ and $f(x) = y_n$ if $x \neq x_n$, and having A output x_n if C_n outputs 1 and output y_n otherwise.

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$$
\begin{aligned}
&= \frac{1}{2} \cdot \Pr\left[A(1^n, E_{G_1(1^n)}(x_n), 1^{|x_n|}, C_n)=1\right] \\
&\quad + \frac{1}{2} \cdot \Pr\left[A(1^n, E_{G_1(1^n)}(y_n), 1^{|y_n|}, C_n)=0\right] \\
&= \frac{1}{2} \cdot \left(\Pr\left[C_n(E_{G_1(1^n)}(x_n))=1\right] + 1 - \Pr\left[C_n(E_{G_1(1^n)}(y_n))=1\right]\right) \\
&> \frac{1}{2} + \frac{1}{2p(n)}
\end{aligned}
$$

where the inequality is due to Eq. (5.3). □

In contrast, as observed above, no algorithm (regardless of its complexity) can guess $f(X_n)$ with success probability above 1/2, when given only $1^{|X_n|}$ and $h(X_n)$. That is, we have

Fact 5.2.7.2: For every n and every algorithm A'

$$
\Pr\left[A'(1^n, 1^{|X_n|}, h(X_n))=f(X_n)\right] \leq \frac{1}{2} \tag{5.4}
$$

Proof: Just observe that the output of A', on its constant input values $1^n, 1^{|X_n|}$ and $h(X_n)$, is stochastically independent of the random variable $f(X_n)$, which in turn is uniformly distributed in $\{0,1\}$. Eq. (5.4) follows (and equality holds in case A' always outputs a value in $\{0,1\}$). □

Combining Claim 5.2.7.1 and Fact 5.2.7.2, we reach a contradiction to the hypothesis that the scheme is semantically secure (even in the restricted sense mentioned in the furthermore-clause of the proposition). Thus, the proposition follows. ■

Comment: When proving the public-key analogue of Proposition 5.2.7, algorithm A is defined as above except that it passes the encryption-key, given to it as part of its input, to the circuit C_n. The rest of the proof remains intact.

5.2.4 Multiple Messages

The above definitions only refer to the security of an encryption scheme that is used to encrypt a single plaintext (per a generated key). Since the plaintext may be longer than the key, these definitions are already non-trivial, and an encryption scheme satisfying them (even in the private-key model) implies the existence of one-way functions (see Exercise 1). Still, in many cases, it is desirable to encrypt many plaintexts using the same encryption-key. Loosely speaking, an encryption scheme is secure in the multiple-message setting if analogous definitions (to the above) hold also when polynomially-many plaintexts are encrypted using the same encryption-key.

We show that *in the public-key model*, security in the single-message setting (discussed above) implies security in the multiple-message setting (defined below). We stress that this is not necessarily true *for the private-key model.*

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5.2.4.1 Definitions

For a sequence of strings $\overline{x} = (x^{(1)}, ..., x^{(t)})$, we let $\overline{E}_e(\overline{x})$ denote the sequence of the t results that are obtained by applying the randomized process E_e to the t strings $x^{(1)}, ..., x^{(t)}$, respectively. That is, $\overline{E}_e(\overline{x}) = E_e(x^{(1)}), ..., E_e(x^{(t)})$. We stress that in each of these t invocations, the randomized process E_e utilizes independently chosen random coins. For sake of simplicity, we consider the encryption of (polynomially) many plaintexts of the same (polynomial) length (rather than the encryption of plaintexts of various lengths as discussed in Exercise 19). The number of plaintexts as well as their total length (in unary) are given to all algorithms either implicitly or explicitly.[4]

Definition 5.2.8 (semantic security – multiple messages):

For private-key: *An encryption scheme, (G, E, D), is* semantically secure for multiple messages in the private-key model *if for every polynomial $t(\cdot)$ and every probabilistic polynomial-time algorithm A, there exists a probabilistic polynomial-time algorithm A' such that for every ensemble $\{\overline{X}_n = (X_n^{(1)}, ..., X_n^{(t(n))})\}_{n\in\mathbb{N}}$, with $|X_n^{(i)}| = \text{poly}(n)$, every pair of functions $f, h : \{0,1\}^* \rightarrow \{0,1\}^*$, every polynomial $p(\cdot)$ and all sufficiently large n*

$$\begin{aligned}&\mathsf{Pr}\left[A(1^n, \overline{E}_{G_1(1^n)}(\overline{X}_n), 1^{|\overline{X}_n|}, h(\overline{X}_n)) = f(\overline{X}_n)\right] \\ &\quad < \mathsf{Pr}\left[A'(1^n, t(n), 1^{|\overline{X}_n|}, h(\overline{X}_n)) = f(\overline{X}_n)\right] + \frac{1}{p(n)}\end{aligned}$$

For public-key: *An encryption scheme, (G, E, D), is* semantically secure for multiple messages in the public-key model *if for $t(\cdot)$, A, A', $\{\overline{X}_n\}_{n\in\mathbb{N}}$, f, h, $p(\cdot)$ and n as above*

$$\begin{aligned}&\mathsf{Pr}\left[A(1^n, G_1(1^n), \overline{E}_{G_1(1^n)}(\overline{X}_n), 1^{|\overline{X}_n|}, h(\overline{X}_n)) = f(\overline{X}_n)\right] \\ &\quad < \mathsf{Pr}\left[A'(1^n, t(n), 1^{|\overline{X}_n|}, h(\overline{X}_n)) = f(\overline{X}_n)\right] + \frac{1}{p(n)}\end{aligned}$$

We stress that the elements of $\overline{X}_n$ are not necessarily independent; they may depend on one another. Note that the above definition also cover the case where the adversary obtains some of the plaintexts themselves. In this case it is still infeasible for him/her to obtain information about the missing plaintexts (see Exercise 21).

Definition 5.2.9 (indistinguishability of encryptions – multiple messages):

[4] For example, A can infer the number of plaintexts from the number of ciphertexts, whereas A' is given this number explicitly. Given the number of the plaintexts as well as their total length, both algorithms can infer the length of each plaintext.

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For private-key: *An encryption scheme, (G, E, D), has* indistinguishable encryptions for multiple messages in the private-key model *if for every polynomial $t(\cdot)$, every polynomial-size circuit family $\{C_n\}$, every polynomial p, all sufficiently large n and every $x_1, ..., x_{t(n)}, y_1, ..., y_{t(n)} \in \{0,1\}^{\mathrm{poly}(n)}$*

$$|\mathsf{Pr}\left[C_n(\overline{E}_{G_1(1^n)}(\bar{x}))=1\right] - \mathsf{Pr}\left[C_n(\overline{E}_{G_1(1^n)}(\bar{y}))=1\right]| < \frac{1}{p(n)}$$

where $\bar{x} = (x_1, ..., x_{t(n)})$ and $\bar{y} = (y_1, ..., y_{t(n)})$.

For public-key: *An encryption scheme, (G, E, D), has* indistinguishable encryptions for multiple messages in the public-key model *if for $t(\cdot)$, $\{C_n\}$, p, n and $x_1, ..., x_{t(n)}, y_1, ..., y_{t(n)}$ as above*

$$|\mathsf{Pr}\left[C_n(G_1(1^n), \overline{E}_{G_1(1^n)}(\bar{x}))=1\right] - \mathsf{Pr}\left[C_n(G_1(1^n), \overline{E}_{G_1(1^n)}(\bar{y}))=1\right]| < \frac{1}{p(n)}$$

The equivalence of Definitions 5.2.8 and 5.2.9 can be established analogously to the proof of Theorem 5.2.5.

Theorem 5.2.10 (equivalence of definitions – multiple messages): *A private-key (resp., public-key) encryption scheme is semantically secure for multiple messages if and only if it has indistinguishable encryptions for multiple messages.*

Thus, proving that single-message security implies multiple-message security for one definition of security, yields the same for the other. We may thus concentrate on the ciphertext-indistinguishability definitions.

5.2.4.2 The effect on the public-key model

We first consider public-key encryption schemes.

Theorem 5.2.11 (single-message security implies multiple-message security): *A* public-key *encryption scheme has indistinguishable encryptions* for multiple messages (*i.e., satisfies Definition 5.2.9 in the public-key model*) *if and only if it has indistinguishable encryptions* for a single message (*i.e., satisfies Definition 5.2.4*).

Proof: Clearly, multiple-message security implies single-message security as a special case. The other direction follows by adapting the proof of Theorem 3.2.6 to the current setting.

Suppose, towards the contradiction, that there exist a polynomial $t(\cdot)$, a polynomial-size circuit family $\{C_n\}$, and a polynomial p, such that for infinitely many n's, there exists $x_1, ..., x_{t(n)}, y_1, ..., y_{t(n)} \in \{0,1\}^{\mathrm{poly}(n)}$ so that

$$\left|\mathsf{Pr}\left[C_n(G_1(1^n), \overline{E}_{G_1(1^n)}(\bar{x}))=1\right] - \mathsf{Pr}\left[C_n(G_1(1^n), \overline{E}_{G_1(1^n)}(\bar{y}))=1\right]\right| > \frac{1}{p(n)}$$

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where $\bar{x} = (x_1, ..., x_{t(n)})$ and $\bar{y} = (y_1, ..., y_{t(n)})$. Let us consider such a generic n and the corresponding sequences $x_1, ..., x_{t(n)}$ and $y_1, ..., y_{t(n)}$. We use a hybrid argument: define

$$\begin{aligned} \bar{h}^{(i)} &\stackrel{\rm def}{=} (x_1, ..., x_i, y_{i+1}, ..., y_{t(n)}) \\ \text{and} \quad H_n^{(i)} &\stackrel{\rm def}{=} (G_1(1^n), \overline{E}_{G_1(1^n)}(\bar{h}^{(i)})) \end{aligned}$$

Since $H_n^{(0)} = (G_1(1^n), \overline{E}_{G_1(1^n)}(\bar{y}))$ and $H_n^{(t(n))} = (G_1(1^n), \overline{E}_{G_1(1^n)}(\bar{x}))$, it follows that there exists an $i \in \{0, ..., t(n)-1\}$ so that

$$\left| \Pr\left[C_n(H_n^{(i)})=1\right] - \Pr\left[C_n(H_n^{(i+1)})=1\right] \right| > \frac{1}{t(n) \cdot p(n)} \qquad (5.5)$$

We show that Eq. (5.5) yields a polynomial-size circuit that distinguishes the encryption of x_{i+1} from the encryption of y_{i+1}, and thus derive a contradiction to security in the single-message setting. Specifically, we construct a circuit D_n that incorporates the circuit C_n as well as the index i and the strings $x_1, ..., x_{i+1}, y_{i+1}, ..., y_{t(n)}$. On input an encryption-key e and (corresponding) ciphertext β, the circuit D_n operates as follows:

- For every $j \leq i$, the circuit D_n generates an encryption of x_j using the encryption-key e. Similarly, for every $j \geq i+2$, the circuit D_n generates an encryption of y_j using the encryption-key e.

  Let us denote the resulting ciphertexts by $\beta_1, ..., \beta_i, \beta_{i+2}, ..., \beta_{t(n)}$. That is, $\beta_j \leftarrow E_e(x_j)$ for $j \leq i$ and $\beta_j \leftarrow E_e(y_j)$ for $j \geq i+2$.

- Finally, D_n invokes C_n on input the encryption-key e and the sequence of ciphertexts $\beta_1, ..., \beta_i, \beta, \beta_{i+2}, ..., \beta_{t(n)}$, and outputs whatever C_n does.

We stress that the construction of D_n relies in an essential way on the fact that the encryption-key is given to it as input.

We now turn to the analysis of the circuit D_n. Suppose that β is a (random) encryption of x_{i+1} with key e; that is, $\beta = E_e(x_{i+1})$. Then, $D_n(e, \beta) \equiv C_n(e, E_e(\bar{h}^{(i+1)})) = C_n(H_n^{(i+1)})$, where $X \equiv Y$ means that the random variables X and Y are identically distributed. Similarly, for $\beta = E_e(y_{i+1})$, we have $D_n(e, \beta) \equiv C_n(e, E_e(\bar{h}^{(i)})) = C_n(H_n^{(i)})$. Thus, by Eq. (5.5), we have

$$\begin{aligned} &\left| \Pr\left[D_n(G_1(1^n), E_{G_1(1^n)}(y_{i+1})=1\right] \right. \\ &\quad \left. - \Pr\left[D_n(G_1(1^n), E_{G_1(1^n)}(x_{i+1})=1\right] \right| > \frac{1}{t(n) \cdot p(n)} \end{aligned}$$

in contradiction to our hypothesis that (G, E, D) is a ciphertext-indistinguishable public-key encryption scheme (in the single message sense). The theorem follows. ■

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Discussion: The fact that we are in the public-key model is essential to the above proof. It allows the circuit D_n to form encryptions relative to the same encryption-key used in the ciphertext given to it. In fact, as stated above (and proven next), the analogous result does not hold in the private-key model.

5.2.4.3 The effect on the private-key model

In contrary to Theorem 5.2.11, *in the private-key model*, ciphertext-indistinguishability for a single message does NOT necessarily imply ciphertext-indistinguishability for multiple messages.

Proposition 5.2.12 *Suppose that there exist pseudorandom generators* (robust against polynomial-size circuits). *Then, there exists a private-key encryption scheme that satisfies Definition 5.2.3 but does not satisfy Definition 5.2.9.*

Proof: We start with the construction of the private-key encryption scheme. The encryption/decryption key for security parameter n is a uniformly distributed n-bit long string, denoted s. To encrypt a ciphertext, x, the encryption algorithm uses the key s as a seed for a pseudorandom generator, denoted g, that stretches seeds of length n into sequences of length $|x|$. The ciphertext is obtained by a bit-by-bit exclusive-or of x and $g(s)$. Decryption is done in an analogous manner.

We first show that this encryption scheme satisfies Definition 5.2.3. Intuitively, this follow from the hypothesis that g is a pseudorandom generator and the fact that $x \oplus U_{|x|}$ is uniformly distributed over $\{0,1\}^{|x|}$. Specifically, suppose towards the contradiction that for some polynomial-size circuit family $\{C_n\}$, a polynomial p, and infinitely many n's

$$|\mathsf{Pr}[C_n(x \oplus g(U_n))=1] - \mathsf{Pr}[C_n(y \oplus g(U_n))=1]| > \frac{1}{p(n)}$$

where U_n is uniformly distributed over $\{0,1\}^n$ and $|x| = |y| = m = \text{poly}(n)$. On the other hand,

$$\mathsf{Pr}[C_n(x \oplus U_m)=1] = \mathsf{Pr}[C_n(y \oplus U_m)=1]$$

Thus, without loss of generality

$$|\mathsf{Pr}[C_n(x \oplus g(U_n))=1] - \mathsf{Pr}[C_n(x \oplus U_m)=1]| > \frac{1}{2 \cdot p(n)}$$

Incorporating x into the circuit C_n we obtain a circuit that distinguishes U_m from $g(U_n)$, in contradiction to our hypothesis (regarding the pseudorandomness of g).

Next, we observe that the above encryption scheme does not satisfy Definition 5.2.9. Specifically, given the ciphertexts of two plaintexts, one may easily retrieve the exclusive-or of the corresponding plaintexts. That is,

$$E_s(x_1) \oplus E_s(x_2) \;=\; (x_1 \oplus g(s)) \oplus (x_2 \oplus g(s)) \;=\; x_1 \oplus x_2$$

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This clearly violates Definition 5.2.8 (e.g., consider $f(x_1, x_2) = x_1 \oplus x_2$) as well as Definition 5.2.9 (e.g., consider any $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$ such that $x_1 \oplus x_2 \neq y_1 \oplus y_2$). Viewed in a different way, note that any plaintext-ciphertext pair yields a corresponding prefix of the pseudorandom sequence, and knowledge of this prefix violates the security of additional plaintexts. That is, given the encryption of a known plaintext x_1 along with the encryption of an unknown plaintext x_2, we can retrieve x_2. On input the ciphertexts β_1, β_2, knowing that the first plaintext is x_1, first retrieves the pseudorandom sequence (i.e., it is just $r \stackrel{\text{def}}{=} \beta_1 \oplus x_1$), and next retrieves the second plaintext (i.e., by computing $\beta_2 \oplus r$). ■

Discussion: The single-message security of the above scheme was proven by considering an ideal version of the scheme in which the pseudorandom sequence is replaced by a truly random sequence. The latter scheme is secure in an information theoretic sense, and the security of the actual scheme followed by the indistinguishability of the two sequences. As we show in Section 5.3.1 (below), the above construction can be modified to yield a private-key "stream-cipher" that is secure for multiple message encryptions. All that is needed is to make sure that (as opposed to the construction above) the same portion of the pseudorandom sequence is never used twice.

An alternative proof of Proposition 5.2.12: Given an arbitrary private-key encryption scheme (G, E, D), consider the following private-key encryption scheme (G', E', D'):

- $G'(1^n) = ((k, r), (k, r)$, where $(k, k) \leftarrow G(1^n)$ and r is uniformly selected in $\{0,1\}^{|k|}$;
- $E'_{(k,r)}(x) = (E_k(x), k \oplus r)$ with probability $1/2$ and $E'_{(k,r)}(x) = (E_k(x), r)$ otherwise;
- and $D'_{(k,r)}(y, z) = D_k(y)$.

If (G, E, D) is secure than so is (G', E', D') (with respect to a single message); however, (G', E', D') is not secure with respect to two messages. For further discussion see Exercise 20.

5.2.5 * A uniform-complexity treatment

As stated at the beginning of this section, the non-uniform formulation was adopted here for sake of simplicity. In this subsection we sketch a uniform-complexity definitional treatment of security. We stress that by uniform or non-uniform complexity treatment of cryptographic primitives we merely refer to the modeling of the adversary. The honest (legitimate) parties are always modeled by uniform complexity classes (most commonly probabilistic polynomial-time).

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The notion of *efficiently constructible ensembles*, defined in Section 3.2.3, is central to the uniform-complexity treatment. Recall that an ensemble, $X = \{X_n\}_{n\in\mathbb{N}}$, is said to be polynomial-time constructible if there exists a probabilistic polynomial time algorithm S so that for every n, the random variables $S(1^n)$ and X_n are identically distributed.

5.2.5.1 The definitions

We present only the definitions of security for multiple messages; the single-message variant can be easily obtained by setting the polynomial t (below) to be identically 1. Likewise, we present the public-key version, and the private-key analogous can be obtained by omitting $G_1(1^n)$ from the inputs to the various algorithms.

The uniformity of the following definitions is reflected in the complexity of the inputs given to the algorithms. Specifically, the plaintexts are taken from polynomial-time constructible ensembles and so are the auxiliary inputs given to the algorithms. For example, in the following definition we require the ensemble $\{\overline{X}_n\}$ to be polynomial-time constructible and the function h to be polynomial-time computable.

Definition 5.2.13 (semantic security – uniform-complexity version): *An encryption scheme, (G,E,D), is* uniformly semantically secure *in the public-key model if for every polynomial t, and every probabilistic polynomial-time algorithm A there exists a probabilistic polynomial-time algorithm A' such that for every polynomial-time constructible ensemble $\{\overline{X}_n = (X_n^{(1)},...,X_n^{(t(n))})\}_{n\in\mathbb{N}}$, with $|X_n^{(i)}| = \mathrm{poly}(n)$, every polynomial-time computable $h:\{0,1\}^*\to\{0,1\}^*$, every $f:\{0,1\}^*\to\{0,1\}^*$, every positive polynomial p and all sufficiently large n's*

$$\begin{aligned} &\Pr\left[A(1^n, G_1(1^n), \overline{E}_{G_1(1^n)}(\overline{X}_n), 1^{|\overline{X}_n|}, h(\overline{X}_n)) = f(\overline{X}_n)\right] \\ &\quad < \Pr\left[A'(1^n, t(n), 1^{|\overline{X}_n|}, h(\overline{X}_n)) = f(\overline{X}_n)\right] + \frac{1}{p(n)} \end{aligned}$$

where $\overline{E}_e(\overline{x}_n) \stackrel{\rm def}{=} E_e(x_n^{(1)}),...,E_e(x_n^{(t(n))})$ (for $\overline{x} = (x_n^{(1)},...,x_n^{(t(n))})$) is as in Definition 5.2.8.

Again, we stress that $\overline{X}_n$ is a sequence of random variables, which may depend on one another. Also, the encryption-key $G_1(1^n)$ was omitted from the input of A' (since the latter may generate it by itself). We stress that even here (i.e., in the uniform complexity setting) no computational limitation are placed on the function f.

Definition 5.2.14 (indistinguishability of encryptions – uniform-complexity version): *An encryption scheme, (G,E,D), has* uniformly indistinguishable encryptions *in the public-key model if for every polynomial t, every probabilistic polynomial-time algorithm D', every polynomial-time constructible ensemble $\overline{T} \stackrel{\rm def}{=} \{\overline{T}_n =$*

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$\overline{X}_n\overline{Y}_nZ_n\}_{n\in\mathbb{N}}$, *with* $\overline{X}_n = (X_n^{(1)},...,X_n^{(t(n))})$, $\overline{Y}_n = (Y_n^{(1)},...,Y_n^{(t(n))})$, *and* $|X_n^{(i)}| = |Y_n^{(i)}| = \text{poly}(n)$,

$$\begin{aligned}|\Pr\left[D'(Z_n, G_1(1^n), \overline{E}_{G_1(1^n)}(\overline{X}_n))=1\right] \\ - \Pr\left[D'(Z_n, G_1(1^n), \overline{E}_{G_1(1^n)}(\overline{Y}_n))=1\right]| < \frac{1}{p(n)}\end{aligned}$$

for every positive polynomial p and all sufficiently large n's.

The random variable Z_n captures a-priori information about the plaintexts for which encryptions should be distinguished. A special case of interest is when $Z_n = \overline{X}_n\overline{Y}_n$. Uniformity is captured in the requirement that D' is a probabilistic polynomial-time algorithm (rather than a family of polynomial-size circuits) and that the ensemble $\{\overline{T}_n = \overline{X}_n\overline{Y}_nZ_n\}_{n\in\mathbb{N}}$ be polynomial-time constructible.

5.2.5.2 Equivalence of the multiple-message definitions

We prove the equivalence of the uniform-complexity definitions (presented above) for (multiple-message) security.

Theorem 5.2.15 (equivalence of definitions – uniform treatment): *A public-key encryption scheme satisfies Definition 5.2.13 if and only if it satisfies Definition 5.2.14. Furthermore, this holds even if Definition 5.2.14 is restricted to the special case where $Z_n = \overline{X}_n\overline{Y}_n$, and even if Definition 5.2.13 is restricted to the special case where f is polynomial-time computable.*

An analogous result holds for the private-key model. The important direction of the theorem holds also for the single-message version (this is quite obvious from the proof below). In the other direction, we seem to use the multiple-message version (of semantic security) in an essential way.

Proof Sketch: Again, we start with the more important direction; that is, assuming that (G,E,D) has (uniformly) indistinguishable encryptions in the special case where $Z_n = \overline{X}_n\overline{Y}_n$, we show that it is (uniformly) semantically secure. Our construction of algorithm A' is analogous to the construction used in the non-uniform treatment. Specifically, on input $(1^{|\overline{\alpha}_n|}, h(\overline{\alpha}_n))$, algorithm A' generates a random encryption of a dummy sequence of message (i.e., $1^{|\overline{\alpha}_n|}$), feeds it to A, and outputs whatever A does.[5] That is,

$$A'(1^{|\overline{\alpha}_n|}, h(\overline{\alpha}_n)) = A(G_1(1^n), \overline{E}_{G_1(1^n)}(1^{|\overline{\alpha}_n|}), 1^{|\overline{\alpha}_n|}, h(\overline{\alpha}_n)) \tag{5.6}$$

As in the non-uniform case, the analysis of algorithm A' reduces to the following claim.

[5] The above description is slightly inaccurate. Algorithm A' is also provided with the auxiliary inputs 1^n and $t(n)$. Using $t(n)$, the algorithm breaks $1^{|\overline{\alpha}_n|}$ into a sequence of $t(n)$ equal-length (unary) strings, using 1^n it generates a random encryption-key, and using this key it generates the corresponding sequence of encryptions.

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Claim 5.2.15.1: For every polynomial-time constructible ensemble $\{\overline{X}_n\}_{n\in\mathbb{N}}$, with $\overline{X}_n = (X_n^{(1)},...,X_n^{(t(n))})$ and $|X_n^{(i)}| = \text{poly}(n)$, every polynomial-time computable h, every positive polynomial p and all sufficiently large n's

$$\begin{aligned}&\mathsf{Pr}\left[A(G_1(1^n),\overline{E}_{G_1(1^n)}(\overline{X}_n),h(\overline{X}_n))=f(\overline{X}_n)\right]\\ &\quad < \ \mathsf{Pr}\left[A(G_1(1^n),\overline{E}_{G_1(1^n)}(1^{|\overline{X}_n|}),h(\overline{X}_n))=f(\overline{X}_n)\right]+\frac{1}{p(n)}\end{aligned}$$

Proof sketch: Analogously to the non-uniform case, assuming towards the contradiction that the claim does not hold, yields an algorithm that distinguishes encryptions of $\overline{X}_n$ from encryptions of $\overline{Y}_n = 1^{|\overline{X}_n|}$, when getting auxiliary information $Z_n = \overline{X}_n\overline{Y}_n = \overline{X}_n 1^{|\overline{X}_n|}$. Thus, we derive contradiction to Definition 5.2.14 (even under the special case postulated in the theorem).

We note that the auxiliary information that is given to the distinguishing algorithm replaces the hard-wiring of auxiliary information that was used in the non-uniform case (and is not possible in the uniform complexity model). Specifically, rather than using a hard-wired value of h (at some non-uniformly fixed sequence), the distinguishing algorithm will use the auxiliary information $Z_n = \overline{X}_n 1^{|\overline{X}_n|}$ in order to compute $h(\overline{X}_n)$, which it will pass to A. Indeed, we rely on the hypothesis that h is efficiently computable.

The actual proof is quite simple in case the function f is also polynomial-time computable (which is not the case in general). In this special case, on input $(e, z, \overline{E}_e(\overline{\alpha}))$, where $z = (\overline{x}, 1^{|\overline{x}|})$ and $\overline{\alpha} \in \{\overline{x}, 1^{|\overline{x}|}\}$, the distinguishing algorithm computes $u = h(\overline{x})$ and $v = f(\overline{x})$, invokes A, and outputs 1 if and only if $A(e, \overline{E}_e(\overline{\alpha}), 1^{|\overline{x}|}, u) = v$.

> (We comment that in case $\overline{\alpha} = 1^{|\overline{x}|}$, we actually mean that $\overline{\alpha}$ is a sequence of $t(n)$ strings of the form $1^{\ell(n)}$, where t and ℓ are as in $\overline{x} = (x^{(1)},...,x^{(t(n))}) \in (\{0,1\}^{\ell(n)})^{t(n)}$.)

The proof becomes more involved in case f is not polynomial-time computable.[6] Again, the solution is in realizing that indistinguishability of encryption postulates a similar output profile in both cases, and in particular no value can occur non-negligibly more in one case than in the other. To clarify the point, we define $\Delta_v(\overline{x}_n)$ to be the difference between $\mathsf{Pr}[A(G_1(1^n),\overline{E}_{G_1(1^n)}(\overline{x}_n),h(\overline{x}_n))=v]$ and $\mathsf{Pr}[A(G_1(1^n),\overline{E}_{G_1(1^n)}(1^{|\overline{x}_n|}),h(\overline{x}_n))=v]$. We know that $\mathsf{E}[\Delta_{f(\overline{X}_n)}(\overline{X}_n)] > 1/p(n)$, but given $\overline{x}_n$ we cannot evaluate $\Delta_{f(\overline{x}_n)}(\overline{x}_n)$, since we do not have $f(\overline{x}_n)$. Instead, we let $\Delta(\overline{x}_n) \stackrel{\text{def}}{=} \max_v\{\Delta_v(\overline{x}_n)\}$, and observe that $\mathsf{E}[\Delta(\overline{X}_n)] \geq \mathsf{E}[\Delta_{f(\overline{X}_n)}(\overline{X}_n)] > 1/p(n)$. Furthermore, given $\overline{x}_n$ we can approximate $\Delta(\overline{x}_n)$ in polynomial-time, and can find (in polynomial-time) a value v such that $\Delta_v(\overline{x}_n) > \Delta(\overline{x}_n) - (1/2p(n))$, with probability at least $1-2^{-n}$.

> On approximating $\Delta(\overline{x}_n)$ etc.: By invoking algorithm A on $O(n\cdot p(n)^3)$ samples of the distributions $(G_1(1^n), \overline{E}_{G_1(1^n)}(\overline{x}_n), h(\overline{x}_n))$ and $G_1(1^n), \overline{E}_{G_1(1^n)}(1^{|\overline{x}_n|}), h(\overline{x}_n))$,

[6] Unlike in the non-uniform treatment, here we cannot hardwire values (such as the values of h and f on good sequences) into the algorithm D' (which is required to be uniform).

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we obtain (implicitly) an approximation of all $\Delta_v(\overline{x}_n)$'s up-to an additive deviation of $1/4p(n)$ (with error probability at most 2^{-n}). The approximation to $\Delta_v(\overline{x}_n)$, denoted $\widetilde{\Delta}_v(\overline{x}_n)$ is merely the difference between the fraction of samples (from both distributions) on which algorithm A returned 1. (Indeed, most $\Delta_v(\overline{x}_n)$'s are approximated by 0, but some $\Delta_v(\overline{x}_n)$'s may approximated by non-zero values.) We just output v for which the approximated value $\widetilde{\Delta}_v(\overline{x}_n)$ is largest. Thus, if for some v_0 it holds that $\Delta_{v_0}(\overline{x}_n) = \Delta(\overline{x}_n)$, then with probability at least $1-2^{-n}$ we output v such that

$$\begin{aligned}\Delta_v(\overline{x}_n) &\geq \widetilde{\Delta}_v(\overline{x}_n) - (1/4p(n)) \\ &\geq \widetilde{\Delta}_{v_0}(\overline{x}_n) - (1/4p(n)) \\ &\geq \Delta_{v_0}(\overline{x}_n) - (1/4p(n)) - (1/4p(n))\end{aligned}$$

Thus, $\Delta_v(\overline{x}_n) \geq \Delta(\overline{x}_n) - (1/2p(n))$.

Thus, on input $(e, z, \overline{E}_e(\overline{\alpha}))$, where $z = (\overline{x}, 1^{|\overline{x}|})$, the new algorithm, denoted D', operates in two stages.

1. In the first stage, D' ignores the ciphertext $\overline{E}_e(\overline{\alpha})$. Using z, algorithm D' recovers $\overline{x}$, and computes $u = h(\overline{x})$. Using $\overline{x}$ and u, algorithm D' estimates $\Delta(\overline{x})$, and finds a v as above.

2. In the second stage (using u and v found in the first stage), algorithm D' invokes A, and outputs 1 if and only if $A(e, \overline{E}_e(\overline{\alpha}), 1^{|x|}, u) = v$.

Let $V(\overline{x})$ be the value found in the first stage of algorithm A (i.e., obliviously of the ciphertext $\overline{E}_e(\overline{\alpha})$). The reader can easily verify that

$$\begin{aligned}&\left|\mathsf{Pr}\left[D'(G_1(1^n), Z_n, \overline{E}_{G_1(1^n)}(\overline{X}_n)) = 1\right] - \mathsf{Pr}\left[D'(G_1(1^n), Z_n, \overline{E}_{G_1(1^n)}(1^{\overline{X}_n})) = 1\right]\right| \\ &= \mathsf{E}\left[\Delta_{V(\overline{X}_n)}(\overline{X}_n)\right] \\ &\geq \left(1-2^{-n}\right) \cdot \mathsf{E}\left[\Delta(\overline{X}_n) - \frac{1}{2p(n)}\right] - 2^{-n} \\ &> \mathsf{E}\left[\Delta(\overline{X}_n)\right] - \frac{2}{3p(n)} > \frac{1}{3p(n)}\end{aligned}$$

Thus, we have derived a probabilistic polynomial-time algorithm (i.e., D') that distinguishes encryptions of $\overline{X}_n$ from encryptions of $\overline{Y}_n = 1^{|\overline{X}_n|}$, when getting auxiliary information $Z_n = \overline{X}_n 1^{|\overline{X}_n|}$. By hypothesis $\{\overline{X}_n\}$ is polynomial-time constructible, and it follows that so is $\{\overline{X}_n \overline{Y}_n Z_n\}$ Thus, we derive contradiction to Definition 5.2.14 (even under the special case postulated in the theorem), and the claim follows. □

Having established the important direction, we now turn to the opposite one. That is, we assume that (G, E, D) is (uniformly) semantically secure and prove that it has (uniformly) indistinguishable encryptions. Again, the proof is

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by contradiction. Suppose, without loss of generality, that there exists a probabilistic polynomial-time algorithm D', a polynomial-time constructible ensemble $\overline{T} \stackrel{\rm def}{=} \{\overline{T}_n = \overline{X}_n\overline{Y}_n Z_n\}_{n\in\mathbb{N}}$ (as in Definition 5.2.14), a positive polynomial p and infinitely many n's so that

$$\begin{aligned}\Pr\left[D'(Z_n, G_1(1^n), \overline{E}_{G_1(1^n)}(\overline{X}_n))=1\right] \\ > \Pr\left[D'(Z_n, G_1(1^n), \overline{E}_{G_1(1^n)}(\overline{Y}_n))=1\right] \mid + \frac{1}{p(n)}\end{aligned}$$

Let $t(n)$ and $\ell(n)$ be such that $\overline{X}_n$ (resp., $\overline{Y}_n$) consists of $t(n)$ strings, each of length $\ell(n)$. Suppose, without loss of generality, that $|Z_n| = m(n)\cdot\ell(n)$, and parse Z_n into $\overline{Z}_n = (Z_n^{(1)},...,Z_n^{(m(n))}) \in (\{0,1\}^{\ell(n)})^{m(n)}$ such that $Z_n = Z_n^{(1)}\cdots Z_n^{(m(n))}$. We define an auxiliary polynomial-time constructible ensemble $\overline{Q} \stackrel{\rm def}{=} \{\overline{Q}_n\}_{n\in\mathbb{N}}$ so that

$$\overline{Q}_n = \left\{\begin{array}{ll} 0^{\ell(n)}\overline{Z}_n\overline{X}_n\overline{Y}_n & \text{with probability } \frac{1}{2} \\ 1^{\ell(n)}\overline{Z}_n\overline{Y}_n\overline{X}_n & \text{with probability } \frac{1}{2}\end{array}\right.$$

That is, $\overline{Q}_n$ is a sequence of $1+m(n)+2t(n)$ strings, each of length $\ell(n)$, that contains $\overline{Z}_n\overline{X}_n\overline{Y}_n$ in addition to a bit (provided in the $\ell(n)$-bit long prefix) indicating whether the order of $\overline{X}_n$ and $\overline{Y}_n$ is switched or not. We define the function f so that to equal this "switch" indicator bit, and the function h to provide all information in $\overline{Q}_n$ except this switch bit. That is, we define f and h as follows:

- The function $f:\{0,1\}^*\rightarrow\{0,1\}$ is defined so that f returns the first bit of its input; that is, $f(\sigma^{\ell(n)}abc) = \sigma$, for $(a,b,c) \in (\{0,1\}^{l(n)})^{m(n)+2t(n)}$.

- The function $h : \{0,1\}^* \rightarrow \{0,1\}$ is defined so that h provides the information in the suffix without yielding information on the prefix; that is, $h(\sigma^{\ell(n)}abc) = abc$ if $\sigma = 0$ and $h(\sigma^{\ell(n)}abc) = acb$ otherwise. Thus, $h(\overline{Q}_n) = \overline{Z}_n\overline{X}_n\overline{Y}_n$; that is, it returns $\overline{T}_n$ to its original order (undoing the possible switch employed in $\overline{Q}_n$).

We stress that both h and f are polynomial-time computable.

We will show that the distinguishing algorithm D' (which distinguishes $\overline{E}(\overline{X}_n)$ from $\overline{E}(\overline{Y}_n)$, when also given $Z_n \equiv \overline{Z}_n$) can be transformed into a polynomial-size algorithm A that guesses the value of $f(\overline{Q}_n)$, from the encryption of $\overline{Q}_n$ (and the value of $h(\overline{Q}_n)$), and does so significantly better than with probability $\frac{1}{2}$. This violates semantic security, since no algorithm (regardless of its running-time) can guess $f(\overline{Q}_n)$ better than with probability $1/2$ when only given $h(\overline{Q}_n)$ and $1^{|\overline{Q}_n|}$ (since given $h(\overline{Q}_n)$ and $1^{|\overline{Q}_n|}$, the value of $f(\overline{Q}_n)$ is uniformly distributed over $\{0,1\}$).

On input $(e, \overline{E}_e(\overline{\alpha}), 1^{|\overline{\alpha}|}, h(\overline{\alpha}))$, where $\overline{\alpha} = \sigma^{\ell(n)}abc \in (\{0,1\}^{l(n)})^{1+m(n)+2t(n)}$ equals either $(0^{\ell(n)}, \overline{z}, \overline{x}, \overline{y})$ or $(1^{\ell(n)}, \overline{z}, \overline{y}, \overline{x})$, algorithm A proceeds in two stages:

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1. In the first stage, algorithm A ignores the ciphertext $\overline{E}_e(\overline{\alpha})$. It first extracts $\overline{x},\overline{y}$ and $z \equiv ovz$ out of $h(\overline{\alpha}) = \overline{z}\,\overline{x}\,\overline{y}$, and approximates $\Delta(z,\overline{x},\overline{y})$, which is defined to equal

$$\mathsf{Pr}\left[D'(z,G_1(1^n),\overline{E}_{G_1(1^n)}(\overline{x}))=1\right] - \mathsf{Pr}\left[D'(z,G_1(1^n),\overline{E}_{G_1(1^n)}(\overline{y}))=1\right] \tag{5.7}$$

Specifically, using $O(n \cdot p(n)^2)$ samples, algorithm A obtains an approximation, denoted $\widetilde{\Delta}(z,\overline{x},\overline{y})$, such that $|\widetilde{\Delta}(z,\overline{x},\overline{y}) - \Delta(z,\overline{x},\overline{y})| < 1/3p(n)$ with probability at least $1-2^{-n}$.

Algorithm A sets $\xi = 1$ if $\widetilde{\Delta}(z,\overline{x},\overline{y}) > 1/3p(n)$, sets $\xi = -1$ if $\widetilde{\Delta}(z,\overline{x},\overline{y}) < -1/3p(n)$, and sets $\xi = 0$ otherwise (i.e., $|\widetilde{\Delta}(z,\overline{x},\overline{y})| \leq 1/3p(n)$).

In case $\xi = 0$, algorithm A halts with an arbitrary reasonable guess (say a randomly selected bit). (We stress that all this is done obliviously of the ciphertext $\overline{E}_e(\overline{\alpha})$, which is only used next.)

2. In the second stage, algorithm A extracts the last block of ciphertexts (i.e., $\overline{E}_e(c)$) out of $\overline{E}_e(\overline{\alpha}) = \overline{E}_e(\sigma^{\ell(n)}abc)$, and invokes D' on input $(z,e,\overline{E}_e(c))$, where z is as extracted in the first stage. Using the value of ξ as determined in the first stage, algorithm A decides as follows:

   - In case $\xi = 1$, algorithm A outputs 1 if and only if the output of D' is 1.
   - In case $\xi = -1$, algorithm A outputs 0 if and only if the output of D' is 1.

Claim 5.2.15.2: Let p, $\overline{Q}_n$, h, f and A be as above.

$$\mathsf{Pr}\left[A(G_1(1^n),\overline{E}_{G_1(1^n)}(\overline{Q}_n),h(\overline{Q}_n))=f(\overline{Q}_n)\right] \;>\; \frac{1}{2}+\frac{1}{10\cdot p(n)^2}$$

Proof sketch: We focus on the case in which the approximation of $\Delta(z,\overline{x},\overline{y})$ computed by (the first stage of) A is within $1/3p(n)$ of the correct value. Thus, in case $\xi \neq 0$, the sign of ξ concurs with the sign of $\Delta(z,\overline{x},\overline{y})$. It follows that, for every possible $(z,\overline{x},\overline{y})$ such that $\xi = 1$ (it holds that $\Delta(z,\overline{x},\overline{y}) > 0$ and) the following holds

$$\begin{aligned}
&\mathsf{Pr}\left[A(G_1(1^n),\overline{E}_{G_1(1^n)}(\overline{Q}_n),h(\overline{Q}_n))=f(\overline{Q}_n)\,|\,(Z_n,\overline{X}_n,\overline{X}_n)=(z,\overline{x},\overline{y})\right] \\
&\quad= \frac{1}{2}\cdot\mathsf{Pr}\left[A(G_1(1^n),\overline{E}_{G_1(1^n)}(0^{\ell(n)},z,\overline{x},\overline{y}),h(0^{\ell(n)},z,\overline{x},\overline{y}))=0\right] \\
&\qquad+\frac{1}{2}\cdot\mathsf{Pr}\left[A(G_1(1^n),\overline{E}_{G_1(1^n)}(1^{\ell(n)},z,\overline{y},\overline{x}),h(1^{\ell(n)},z,\overline{y},\overline{x}))=1\right] \\
&\quad= \frac{1}{2}\cdot\mathsf{Pr}\left[D'(z,G_1(1^n),\overline{E}_{G_1(1^n)}(\overline{y}))=0\right] \\
&\qquad+\frac{1}{2}\cdot\mathsf{Pr}\left[D'(z,G_1(1^n),\overline{E}_{G_1(1^n)}(\overline{x}))=1\right] \\
&\quad= \frac{1}{2}\cdot(1+\Delta(z,\overline{x},\overline{y}))
\end{aligned}$$

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Similarly, for every possible $(z,\overline{x},\overline{y})$ such that $\xi=-1$ (it holds that $\Delta(z,\overline{x},\overline{y})<0$ and) the following holds

$$\begin{aligned}
&\mathsf{Pr}\left[A(G_1(1^n),\overline{E}_{G_1(1^n)}(\overline{Q}_n),h(\overline{Q}_n))=f(\overline{Q}_n)\,|\,(Z_n,\overline{X}_n,\overline{X}_n)=(z,\overline{x},\overline{y})\right]\\
&= \frac{1}{2}\cdot\mathsf{Pr}\left[A(G_1(1^n),\overline{E}_{G_1(1^n)}(0^{\ell(n)},z,\overline{x},\overline{y}),h(0^{\ell(n)},z,\overline{x},\overline{y}))=0\right]\\
&\quad+\frac{1}{2}\cdot\mathsf{Pr}\left[A(G_1(1^n),\overline{E}_{G_1(1^n)}(1^{\ell(n)},z,\overline{y},\overline{x}),h(1^{\ell(n)},z,\overline{y},\overline{x}))=1\right]\\
&= \frac{1}{2}\cdot\mathsf{Pr}\left[D'(z,G_1(1^n),\overline{E}_{G_1(1^n)}(\overline{y}))=1\right]\\
&\quad+\frac{1}{2}\cdot\mathsf{Pr}\left[D'(z,G_1(1^n),\overline{E}_{G_1(1^n)}(\overline{x}))=0\right]\\
&= \frac{1}{2}\cdot(1-\Delta(z,\overline{x},\overline{y}))
\end{aligned}$$

Thus, in both cases where $\xi\neq 0$, algorithm A succeeds with probability

$$\frac{1+\xi\cdot\Delta(z,\overline{x},\overline{y})}{2}=\frac{1+|\Delta(z,\overline{x},\overline{y})|}{2}$$

and in case $\xi=0$ it succeeds with probability $1/2$. Recall that if $\Delta(z,\overline{x},\overline{y})>\frac{2}{3p(n)}$ then $\xi=1$. Using the contradiction hypothesis that asserts that $\mathsf{E}[\Delta(Z_n,\overline{X}_n,\overline{Y}_n)]>\frac{1}{p(n)}$, we lower bound $\mathsf{Pr}[\Delta(Z_n,\overline{X}_n,\overline{X}_n)>\frac{2}{3p(n)}]$ by $\frac{1}{3p(n)}$. Thus, the overall success probability of algorithm A is at least

$$\frac{1}{3p(n)}\cdot\frac{1+(2/3p(n))}{2}+\left(1-\frac{1}{3p(n)}\right)\cdot\frac{1}{2}=\frac{1}{2}+\frac{1}{(3p(n))^2}$$

and the claim follows. □

This completes the proof of the opposite direction. ■

Discussion: The proof of the first (i.e., important) direction holds also in the single-message setting. In general, for any function t, in order to prove that semantic security holds with respect to t-long sequences of ciphertexts, we just use the hypothesis that t-long message-sequences have indistinguishable encryptions. In contrast, the proof of the second (i.e., opposite) direction makes an essential use of the multiple-message setting. In particular, in order to prove that t-long message-sequences have indistinguishable encryptions, we use the hypothesis that semantic security holds with respect to $(1+m+2t)$-long sequences of ciphertexts, where m depends on the length of the auxiliary input in the claim of ciphertext-indistinguishability. Thus, even if we only want to establish ciphertext-indistinguishability in the single-message setting, we do so by using semantic security in the multiple-message setting. Furthermore, we use the fact that given a sequence of ciphertexts, we can extract a certain subsequence of ciphertexts.

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5.2.5.3 Single-message versus multiple-message definitions

As in the non-uniform case, for the public-key model, single-message security implies multiple-message security. Again, this implication does NOT hold in the private-key model. The proofs of both statements are analogous to the proofs provided in the non-uniform case. Specifically:

1. For the public-key model, single-message uniform-indistinguishability of encryptions imply multiple-message uniform-indistinguishability of encryptions, which in turn implies multiple-message uniform-semantic security.

   In the proof of this result, we use the fact that all hybrids are polynomial-time constructible, and that we may select a random pair of neighboring hybrids (cf. the proof of Theorem 3.2.6). We also use the fact that an ensemble of triplets, $\{\overline{T}_n = \overline{X}_n \overline{Y}_n Z'_n\}_{n\in\mathbb{N}}$, with $\overline{X}_n = (X_n^{(1)}, ..., X_n^{(t(n))})$, $\overline{Y}_n = (Y_n^{(1)}, ..., Y_n^{(t(n))})$, as in Definition 5.2.14, induces an ensemble of triplets, $\{T_n = X_n Y_n Z_n\}_{n\in\mathbb{N}}$, for the case $t \equiv 1$. Specifically, we shall use $X_n = X_n^{(i)}$, $Y_n = Y_n^{(i)}$, and $Z_n = (\overline{X}_n, \overline{Y}_n, Z'_n, i)$, where i is uniformly distributed in $\{1, ..., t(n)\}$.

2. For the private-key model, single-message uniform-indistinguishability of encryptions does NOT imply multiple-message uniform-indistinguishability of encryptions. The proof is exactly as in the non-uniform case.

5.2.5.4 The gain of a uniform treatment

Suppose that one is content with the uniform-complexity level of security, which is what we advocate below. Then the gain in using the uniform-complexity treatment is that a uniform-complexity level of security can be obtained using only uniform complexity assumptions (rather than non-uniform complexity assumptions). Specifically, the results presented in the next section are based on non-uniform assumptions such as the existence of functions that cannot be inverted by polynomial-size circuits (rather than by probabilistic polynomial-time algorithms). These non-uniform assumption are used in order to satisfy the non-uniform definitions presented in the main text (above). Using any of these constructions, while making the analogous uniform assumptions, yields encryption schemes with the analogous uniform-complexity security. (We stress that this is no coincidence, but is rather an artifact of these results being proven by a uniform reducibility argument.)

However, something is lost when relying on these (seemingly weaker) uniform complexity assumptions. Namely, the security we obtain is only against the (seemingly weaker) uniform adversaries. We believe that this loss in security is immaterial. Our belief is based on the thesis that uniform complexity is the right model of "real world" cryptography. We believe that it is reasonable to consider only objects (i.e., inputs) generated by uniform and efficient procedures and the effect that these objects have on uniformly and efficient observers (i.e.,

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adversaries). In particular, schemes secure against probabilistic polynomial-time adversaries can be used in any setting consisting of probabilistic polynomial-time machines with inputs generated by probabilistic polynomial-time procedures. We believe that the cryptographic setting is such a case.

5.3 Constructions of Secure Encryption Schemes

In this subsection we present constructions of secure private-key and public-key encryption schemes. Here and throughout this section security means *semantic security in the multiple-message setting.* Recall that this is equivalent to ciphertext-indistinguishability (in the multiple-message setting). Also recall that for public-key schemes it suffices to prove ciphertext-indistinguishability in the single-message setting. The main results of this section are

- Using any (non-uniformly robust) pseudorandom function, one can construct secure private-key encryption schemes. Recall, that the former can be constructed using any (non-uniformly strong) one-way function.
- Using any (non-uniform strong) trapdoor one-way permutation, one can construct secure public-key encryption schemes.

In addition, we review some popular suggestions for private-key and public-key encryption schemes.

Probabilistic Encryption: Before starting, we recall that a secure *public-key* encryption scheme must employ a probabilistic (i.e., randomized) encryption algorithm. Otherwise, given the encryption-key as (additional) input, it is easy to distinguish the encryption of the all-zero message from the encryption of the all-ones message. The same holds for *private-key* encryption schemes when considering the multi-message setting.[7] For example, using a deterministic (private-key) encryption algorithm allows the adversary to distinguish two encryptions of the same message from the encryptions of a pair of different messages. Thus, the common practice of using pseudorandom permutations as "block-ciphers" (see definition below) is NOT secure (again, one can distinguish two encryptions of the same message from encryptions of two different messages). This explains the linkage between the above robust security definitions and *randomized* (a.k.a *probabilistic*) encryption schemes. Indeed, all our encryption schemes will employ randomized encryption algorithms.[8]

[7] We note that the above does not hold with respect to private-key schemes in the single-message setting (or for the augmented model of state-based ciphers discussed in Section 5.3.1). In such a case, the private-key can be augmented to include a seed for a pseudorandom generator, the output of which can be used to eliminate randomness from the encryption algorithm. (Question: why does the argument fail in the public-key setting and in the multi-message private-key setting?)

[8] The (private-key) stream-ciphers discussed in Section 5.3.1 are an exception, but– as we point out– they do not adhere to our (basic) formulation of encryption schemes (as in Definition 5.1.1).

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5.3.1 * Stream–Ciphers

It is common practice to use "pseudorandom generators" as a basis for private-key stream ciphers (see definition below). Specifically, the pseudorandom generator is used to produce a stream of bits that are XORed with the corresponding plaintext bits to yield corresponding ciphertext bits. That is, the generated pseudorandom sequence (which is determined by the a-priori shared key) is used as a "one-time pad" instead of a truly random sequence, with the advantage that the generated sequence may be much longer than the key (whereas this is not possible for a truly random sequence). This common practice is indeed sound provided one actually uses pseudorandom generators (as defined in Section 3.3), rather than using programs that are called "pseudorandom generators" but actually produce sequences that are easy to predict (such as the linear congruential generator or some modifications of it that output a constant fraction of the bits of each resulting number).

As we shall see, using any pseudorandom generator one may obtain a secure *private-key stream cipher* that allows to encrypt a stream of plaintext bits. We note that such a stream cipher does not conform with our formulation of an encryption scheme (i.e., as in Definition 5.1.1), because in order to encrypt several messages one is required to maintain a counter (so to prevent reusing parts of the pseudorandom "one-time pad"). In other words, we obtain a secure encryption scheme with a *variable state* that is modified after the encryption of each message. We stress that constructions of secure and *stateless* encryption schemes (i.e., conforming with Definition 5.1.1) are known and are presented in Sections 5.3.3 and 5.3.4. The traditional interest in stream ciphers is due to efficiency considerations. We discuss this issue at the end of Section 5.3.3. But before doing so, let us formalize the above discussion.

5.3.1.1 Definitions

We start by extending the simple mechanism of encryption schemes (as presented in Definition 5.1.1). The key-generation algorithm remains unchanged, but both the encryption and decryption algorithm take an additional input and emit an additional output, corresponding to their state before and after the operation. The length of the state is not allowed to grow by too much during each application of the encryption algorithm (see Item 3 in Definition 5.3.1 below), or else the efficiency of the entire "repeated encryption" process can not be guaranteed. For sake of simplicity, we incorporate the key in the state of the corresponding algorithm. Thus, the initial state of each of the algorithms is set to equal its corresponding key. Furthermore, one may think of the intermediate states as of updated values of the corresponding key. For clarity, the reader may consider of the special case in which the state contains the initial key, the number of times the scheme was invoked (or the total number of bits in such invocations), and auxiliary information that allows to speed-up the computation of the next ciphertext (or plaintext).

For simplicity, we assume below that the decryption algorithm (i.e., D) is

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deterministic (otherwise formulating the reconstruction condition would be more complex). Intuitively, the main part of the reconstruction condition (i.e., Item 2 in Definition 5.3.1) is that the (proper) iterative encryption-decryption process recovers the original plaintexts. The additional requirement in Item 2 is that the state of the decryption algorithm is updated correctly as long as it is fed with strings of length equal to the length of the valid ciphertexts. This extra requirement implies that given the initial decryption-key and the current ciphertext as well as the lengths of all previous ciphertexts (which may be actually incorporated in the current ciphertext), one may recover the current plaintext. This fact is interesting for two reasons:

A theoretical reason: It implies that, without loss of generality (alas with possible loss in efficiency), the decryption algorithm may be stateless. Furthermore, without loss of generality (alas with possible loss in efficiency), the state of the encryption algorithm may consist of the initial encryption-key and the lengths of the plaintexts encrypted so far.

A practical reason: It allows to recover from the loss of some of the ciphertexts. That is, assuming that all ciphertexts have the same (known) length (which is typically the case in the relevant applications), if the receiver knows (or is given) the total number of ciphertexts sent so far then it can recover the current plaintext from the current ciphertext, even if some of the previous ciphertexts were lost.

We comment that in traditional stream ciphers, the plaintexts (and ciphertexts) are individual bits or blocks of a fixed number of bits (i.e., $|\alpha^{(i)}| = |\beta^{(i)}| = \ell$ for all i's).

Definition 5.3.1 (state-based cipher – the mechanism): *A* state-based encryption scheme *is a triple, (G, E, D), of probabilistic polynomial-time algorithms satisfying the following three conditions*

1. *On input 1^n, algorithm G outputs a pair of bit strings.*
2. *For every pair $(e^{(0)}, d^{(0)})$ in the range of $G(1^n)$, and every sequence of plaintexts $\alpha^{(i)}$'s, the following holds: if $(e^{(i)}, \beta^{(i)}) \leftarrow E(e^{(i-1)}, \alpha^{(i)})$ and $(d^{(i)}, \gamma^{(i)}) \leftarrow D(d^{(i-1)}, \beta^{(i)})$, for $i = 1, 2, \ldots$, then $\gamma^{(i)} = \alpha^{(i)}$ for every i. Furthermore, for every i and every $\beta \in \{0,1\}^{|\beta^{(i)}|}$, it holds that $D(d^{(i-1)}, \beta) = (d^{(i)}, \cdot)$.*
3. *There exists a polynomial p such that for every pair $(e^{(0)}, d^{(0)})$ in the range of $G(1^n)$, and every sequence of $\alpha^{(i)}$'s and $e^{(i)}$'s as above, it holds that $|e^{(i)}| \leq |e^{(i-1)}| + |\alpha^{(i)}| \cdot p(n)$. Similarly for the $d^{(i)}$'s.*

That is, as in Definition 5.1.1, the encryption-decryption process operates properly (i.e., the decrypted message equals the plaintext), provided that the corresponding algorithms get the corresponding keys (or states). Note that in Definition 5.3.1 the keys are modified by the encryption-decryption process, and

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so correct decryption requires holding the correctly-updated decryption-key. We stress that the furthermore clause in Item 2 guarantees that the decryption-key is correctly updated as long as the decryption process is fed with strings of the correct lengths (but not necessarily with the correct ciphertexts). As discuss above, this extra condition has interesting theoretical and practical consequences to be further emphasized in Construction 5.3.3 (below). We comment that in Construction 5.3.3, it holds that $|e^{(i)}| \leq |e^{(0)}| + \log_2 \sum_{j=1}^{i} |\alpha^{(j)}|$, which is much stronger than the requirement in Item 3.

We stress that Definition 5.3.1 refers to the encryption of multiple messages (and is meaningless when considering the encryption of a single message). However, Definition 5.3.1 by itself does not explain why one should encrypt the ith message using the updated encryption-key $e^{(i-1)}$, rather than reusing the initial encryption-key $e^{(0)}$ in all encryptions (where decryption is done by reusing the initial decryption-key $d^{(0)}$). Indeed, the reason for updating these keys is provided by the following security definition that refers to the encryption of multiple messages, and holds only in case the encryption-keys in use are properly updated (in the multiple-message encryption process). Below we present only the semantic security definition for private-key schemes.

Definition 5.3.2 (semantic security – state-based cipher): *For a state-based encryption scheme, (G,E,D), and any $\overline{x} = (x^{(1)},...,x^{(t)})$, we let $\overline{E}_e(\overline{x}) = (y^{(1)},...,y^{(t)})$ be the result of the following t-step (possibly random) process, where $e^{(0)} \stackrel{\rm def}{=} e$. For $i = 1,...,t$, we let $(e^{(i)}, y^{(i)}) \leftarrow E(e^{(i-1)}, x^{(i)})$, where each of the t invocations E utilizes independently chosen random coins. The scheme (G,E,D) is* semantically secure *in the state-based private-key model if for every polynomial $t(\cdot)$ and every probabilistic polynomial-time algorithm A there exists a probabilistic polynomial-time algorithm A' such that for every ensemble $\{\overline{X}_n = (X_n^{(1)},...,X_n^{(t(n))})\}_{n\in\mathbb{N}}$, with $|X_n^{(i)}| = \mathrm{poly}(n)$, every pair of functions $f,h : \{0,1\}^* \to \{0,1\}^*$, every polynomial $p(\cdot)$ and all sufficiently large n*

$$\begin{aligned} &\mathsf{Pr}\left[A(1^n, \overline{E}_{G_1(1^n)}(\overline{X}_n), 1^{|\overline{X}_n|}, h(\overline{X}_n)) = f(\overline{X}_n)\right] \\ &\quad < \ \mathsf{Pr}\left[A'(1^n, t(n), 1^{|\overline{X}_n|}, h(\overline{X}_n)) = f(\overline{X}_n)\right] + \frac{1}{p(n)} \end{aligned}$$

Note that Definition 5.3.2 (only) differs from Definition 5.2.8 in the preamble defining the random variable $\overline{E}_e(\overline{x})$, which mandates that the encryption-key $e^{(i-1)}$ is used in the ith encryption. Furthermore, Definition 5.3.2 guarantees nothing regarding an encryption process in which the plaintext sequence $x^{(1)},...,x^{(t)}$ is encrypted by $E(e,x^{(1)}), E(e,x^{(2)}),...,E(e,x^{(t)})$ (i.e., the initial encryption-key e itself is used in all encryptions, as in Definition 5.2.8).

5.3.1.2 A sound version of a common practice

Using any (on-line) pseudorandom generator, one can easily construct a secure state-based private-key encryption scheme. Recall that on-line pseudorandom generators are a special case of variable-output pseudorandom generators (see

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Section 3.3.3), in which a hidden state is maintained and updated so to allow generation of the next output bit in time polynomial in the length of the initial seed, regardless of the number of bits generated so far. Specifically, the next (hidden) state and output bit are produced by applying a (polynomial-time computable) function $g:\{0,1\}^n \to \{0,1\}^{n+1}$ to the current state (i.e., $s'\sigma \leftarrow g(s)$, where s is the current state, s' is the next state and σ is the next output bit). The suggested state-based private-key encryption scheme will be initialized with a key equal to the seed of such a generator, and will maintain and update a state allowing it to quickly produce the next output bit of the generator. The stream of plaintext bits will be encrypted by XORing these bits with the corresponding output bits of the generator.

Construction 5.3.3 (how to construct stream ciphers (i.e., state-based private-key encryption schemes)): *Let g be a polynomial-time computable function such that $|g(s)| = |s| + 1$ for all $s \in \{0,1\}^*$.*

key-generation and initial state: *On input 1^n, uniformly select $s \in \{0,1\}^n$, and output the key-pair (s, s). The initial state of each algorithm is set to $(s, 0, s)$.*

(We maintain the initial key s and a step-counter in order to allow recovery from loss of ciphertexts.)

encrypting the next plaintext bit x with state (s, t, s'): *Let $s''\sigma = g(s')$, where $|s''| = |s'|$ and $\sigma \in \{0,1\}$. Output the ciphertext bit $x \oplus \sigma$, and set the new state to $(s, t + 1, s'')$.*

decrypting the ciphertext bit y with state (s, t, s'): *Let $s''\sigma = g(s')$, where $|s''| = |s'|$ and $\sigma \in \{0,1\}$. Output the plaintext bit $y \oplus \sigma$, and set the new state to $(s, t + 1, s'')$.*

When notified that some ciphertext bits may have been lost and that the current ciphertext bit has index t', the decryption procedure first recovers the correct current state, denoted $s_{t'}$. This is done by computing $s_i\sigma_i = g(s_{i-1})$, for $i = 1, ..., t'$, where $s_0 \stackrel{\text{def}}{=} s$.

Note that both the encryption and decryption algorithms are deterministic, and that the state after encryption of t bits has length $2n + \log_2 t < 3n$ (for $t < 2^n$).

Recall that g (as in Construction 5.3.3) is called a next step function of an on-line pseudorandom generator if for every polynomial p the ensemble $\{G_n^p\}_{n\in\mathbb{N}}$ is pseudorandom (with respect to polynomial-size circuits), where G_n^p is defined by the following random process:

Uniformly select $s_0 \in \{0,1\}^n$;
For $i = 1$ to $p(n)$, let $s_i\sigma_i \leftarrow g(s_{i-1})$, where $\sigma_i \in \{0,1\}$ (and $s_i \in \{0,1\}^n$);
Output $\sigma_1\sigma_2 \cdots \sigma_{p(n)}$.

Also recall that if g is (itself) a pseudorandom generator then it constitutes a next step function of an on-line pseudorandom generator (see Exercise 21 of Chapter 3). Thus:

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Proposition 5.3.4 *If g is a pseudorandom generator (with respect to polynomial-size circuits) then Construction 5.3.3 constitutes a secure state-based private-key encryption scheme.*

Proof Idea: Consider an ideal version of Construction 5.3.3 in which a truly random sequence is used instead of the output produced by the on-line pseudorandom generator defined by g. The ideal version coincides with the traditional one-time pad, and thus is perfectly secure. The security of the actual Construction 5.3.3 follows by the pseudorandomness of the on-line generator. ■

5.3.2 Preliminaries: Block–Ciphers

Many encryption schemes are more conveniently presented by first presenting a restricted type of encryption scheme that we call a *block-cipher*.[9] In contrast to encryption schemes (as defined in Definition 5.1.1), block-ciphers (defined below) are only required to operate on plaintext of a specific length (which is a function of the security parameter). As we shall see, given a secure block-cipher we can easily construct a (general) secure encryption scheme.

5.3.2.1 Definitions

We start by considering the syntax (i.e., Definition 5.1.1).

Definition 5.3.5 (block-cipher): *A* block-cipher *is a triple, (G, E, D), of probabilistic polynomial-time algorithms satisfying the following two conditions*

1. *On input 1^n, algorithm G outputs a pair of bit strings.*
2. *There exists a polynomially-bounded function $\ell : \mathbb{N} \to \mathbb{N}$, called the* block length*, so that for every pair (e, d) in the range of $G(1^n)$, and for each $\alpha \in \{0,1\}^{\ell(n)}$, algorithms E and D satisfy*

$$\mathsf{Pr}[D_d(E_e(\alpha))=\alpha] = 1$$

Typically, we use either $\ell(n) = \Theta(n)$ or $\ell(n) = 1$. Analogously to Definition 5.1.1, the above definition does not distinguish private-key encryption schemes from public-key ones. The difference between the two types is captured in the security definitions, which are essentially as before with the modification that we only consider plaintexts of length $\ell(n)$. For example, the analogue of Definition 5.2.1 reads

Definition 5.3.6 (semantic security – private-key block-ciphers): *A block-cipher, (G, E, D), with block length ℓ is* semantically secure (in the private-key model)

[9] In using the term *block-cipher*, we abuse standard terminology by which a block-cipher must, in addition to operating on plaintext of specific length, produce ciphertexts equal in length to the length of the corresponding plaintexts. We comment that the latter cannot be semantically secure; see Exercise 22.

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if for every probabilistic polynomial-time algorithm A there exists a probabilistic polynomial-time algorithm A' such that for every ensemble $\{X_n\}_{n\in\mathbb{N}}$, with $|X_n| = \ell(n)$, and f, h, $p(\cdot)$ and n as in Definition 5.2.1

$$\Pr\left[A(1^n, E_{G_1(1^n)}(X_n), 1^{|X_n|}, h(X_n)) = f(X_n)\right] < \Pr\left[A'(1^n, 1^{|X_n|}, h(X_n)) = f(X_n)\right] + \frac{1}{p(n)}$$

5.3.2.2 Transforming block-ciphers into general encryption schemes

There are obvious ways of transforming a block-cipher into a general encryption scheme. The basic idea is to break the plaintexts (for the resulting scheme) into blocks and encode each block separately by using the block-cipher. Thus, the security of the block-cipher (in the multiple-message settings) implies the security of the resulting encryption scheme. The only technicality we need to deal with is how to encrypt plaintexts of length that is not an integer multiple of the block-length (i.e., $\ell(n)$). This is easily resolved by padding the last block (while indicating the end of the actual plaintext).[10]

Construction 5.3.7 (from block-ciphers to general encryption schemes): *Let (G, E, D) be a block-cipher with block length function ℓ. We construct an encryption scheme, (G', E', D') as follows. The key-generation algorithm, G', is identical to G. To encrypt a message α* (with encryption-key e generated under security parameter n), *we break it into consecutive blocks of length $\ell(n)$, while possibly augmenting the last block. Let $\alpha_1, ..., \alpha_t$ be the resulting blocks. Then*

$$E'_e(\alpha) \stackrel{\rm def}{=} (|\alpha|, E_e(\alpha_1), ..., E_e(\alpha_t))$$

To decrypt the ciphertext $(m, \beta_1, ..., \beta_t)$ (with decryption-key d), *we let $\alpha_i = D_d(\beta_i)$ for $i = 1, ..., t$, and let the plaintext be the m-bit long prefix of the concatenated string $\alpha_1 \cdots \alpha_t$.*

The above construction yields ciphertexts which reveal the exact length of the plaintext. Recall that this is not prohibited by the definitions of security, and that we cannot hope to entirely hide the length. However, we can easily construct encryption schemes that hide some information about the length of the plaintext; see examples in Exercise 4. Also, note that the above construction applies even to the special case where ℓ is identically 1.

Proposition 5.3.8 *Let (G, E, D) and (G', E', D') be as in Construction 5.3.7. Suppose that the former a secure private-key*[11] (resp., public-key) *block-cipher. Then the latter is a secure private-key* (resp., public-key) *encryption scheme.*

---

[10] We choose to use a very simple indication of the end of the actual plaintext (i.e., include its length in the ciphertext). In fact, it suffices to include the length of the plaintext modulo $\ell(n)$. Another natural alternative is to use a padding of the form $10^{(\ell(n)-|\alpha|-1) \bmod \ell(n)}$, while observing that no padding is ever required in case $\ell(n) = 1$.

[11] Recall that throughout this section *security* means security in the multiple-message setting.

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Proof Sketch: The proof is by a reducibility argument. Assuming towards the contradiction that the encryption scheme (G', E', D') is not secure, we conclude that neither is (G, E, D), contradicting our hypothesis. Specifically, we rely on the fact that in both schemes security means security in the multiple-message setting. Note that in case the security of (G', E', D') is violated via $t(n)$ messages of length $L(n) = \text{poly}(n)$, the security of (G, E, D) is violated via $t(n) \cdot \lceil L(n)/\ell(n) \rceil$ messages of length $\ell(n)$. Also, the argument may utilize any of the two notions of security (i.e., semantic security or ciphertext-indistinguishability). ■

5.3.3 Private-key encryption schemes

Secure private-key encryption schemes can be easily constructed using any efficiently computable pseudorandom function ensemble (see Section 3.6). Specifically, we present a block cipher with block length $\ell(n) = n$. The key-generation algorithm consists of selecting a seed, denoted s, for such a function, denoted f_s. To encrypt a message $x \in \{0,1\}^n$ (using key s), the encryption algorithm uniformly selects a string $r \in \{0,1\}^n$ and produces the ciphertext $(r, x \oplus f_s(r))$. To decrypt the ciphertext (r, y) (using key s), the decryption algorithm just computes $y \oplus f_s(r)$. Formally, we have

Construction 5.3.9 (a private-key block-cipher based on pseudorandom functions): *Let $F = \{F_n\}$ be an efficiently computable function ensemble and let I and V be the algorithms associated with it. That is, $I(1^n)$ selects a function with distribution F_n and $V(s, x)$ returns $f_s(x)$, where f_s is the function associated with the string s. We define a private-key block cipher, (G, E, D), with block length $\ell(n) = n$ as follows*

key-generation: $G(1^n) = (k, k)$, *where* $k \leftarrow I(1^n)$.

encrypting plaintext $x \in \{0,1\}^n$ (using the key k): $E_k(x) = (r, V(k, r) \oplus x)$, *where r is uniformly chosen in $\{0,1\}^n$.*

decrypting ciphertext (r, y) (using the key k): $D_k(r, y) = V(k, r) \oplus y$.

Clearly, for every k (in the range of $I(1^n)$) and $x \in \{0,1\}^n$,

$$D_k(E_k(x)) \;=\; D_k(U_n, f_k(U_n) \oplus x) \;=\; f_k(U_n) \oplus (f_k(U_n) \oplus x) \;=\; x$$

Below we assume that F is *pseudorandom with respect to polynomial-size circuits*, meaning that no polynomial-size circuit having "oracle gates" can distinguish the case the answers are provided by a random function from the case in which the answers are provided by a function in F. Alternatively, one may consider probabilistic polynomial-time oracle machines that obtain a non-uniform polynomially-long auxiliary input. That is,

> *for every probabilistic polynomial-time oracle machine M for every pair of positive polynomial p and q, for all sufficiently large n's and*

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all $z \in \{0,1\}^{p(n)}$,

$$\left|\Pr\left[M^f(z)=1\right] - \Pr\left[M^{f_{I(1^n)}}(z)=1\right]\right| < \frac{1}{q(n)}$$

where f *is a uniformly selected function mapping* $\{0,1\}^n$ *to* $\{0,1\}^n$.

Recall, that such (non-uniformly strong) pseudorandom functions can be constructed using any non-uniformly strong one-way function.

Proposition 5.3.10 *Let* F *and* (G, E, D) *be as in Construction 5.3.9, and suppose that* F *is pseudorandom with respect to polynomial-size circuits. Then* (G, E, D) *is secure.*

The proof of Proposition 5.3.10 is given below. Combining Propositions 5.3.8 and 5.3.10 (with the above), we obtain

Theorem 5.3.11 *If there exist (non-uniformly strong) one-way functions then there exist secure private-key encryption schemes.*

The converse holds too; see Exercise 1.

Proof of Proposition 5.3.10: The proof consists of two steps (suggested as a general methodology in Section 3.6):

1. Prove that an idealized version of the scheme, in which one uses a uniformly selected function $f:\{0,1\}^n \to \{0,1\}^n$, rather than the pseudorandom function f_s, is secure (in the sense of ciphertext-indistinguishability).

2. Conclude that the real scheme (as presented above) is secure (since otherwise one could distinguish a pseudorandom function from a truly random one).

Specifically, in the ideal version the messages $x^{(1)}, ..., x^{(t)}$ are encrypted by $(r^{(1)}, f(r^{(1)}) \oplus x^{(1)}), ..., (r^{(t)}, f(r^{(t)}) \oplus x^{(t)})$, where the $r^{(j)}$'s are independently and uniformly selected, and f is a random function. Thus, with probability greater than $1 - t^2 \cdot 2^{-n}$, the $r^{(j)}$'s are all distinct and so the values $f(r^{(j)}) \oplus x^{(j)}$ are independently and uniformly distributed, regardless of the $x^{(j)}$'s. It follows that the ideal version is ciphertext-indistinguishable. Now, if the actual scheme is not ciphertext-indistinguishable, then for some sequence of $r^{(j)}$'s a polynomial-size circuit can distinguish the $f(r^{(j)}) \oplus x^{(j)}$'s from the $f_s(r^{(j)}) \oplus x^{(j)}$'s, where f is random and f_s is pseudorandom. But this contradicts the hypothesis that polynomial-size circuits cannot distinguish between the two cases. ■

Discussion: Note that we could have gotten rid of the randomization if we had allowed the encryption algorithm to be history dependent (as discussed in Section 5.3.1 above). Specifically, in such a case, we could have used a counter in

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the role of r. Furthermore, if the encryption scheme is used for FIFO communication between the parties and both can maintain the counter value then there is no need for the sender to send the counter value. However, in the later case Construction 5.3.3 is preferable (because the adequate pseudorandom generator may be more efficient than a pseudorandom function as used in Construction 5.3.9). We note that in case the encryption scheme is not used for FIFO communication and one may need to decrypt messages with arbitrary varying counter values, it is typically better to use Construction 5.3.9. Furthermore, in many cases it may be preferable to select a value (i.e., r) at random rather than rely on a counter that must stored in a reliable manner between applications (of the encryption algorithm).

The ciphertexts produced by Construction 5.3.9 are longer than the corresponding plaintexts. This is unavoidable in case of secure (history-independent) encryption schemes (see Exercise 22). In particular, the common practice of using pseudorandom permutations as block-ciphers[12] is NOT secure (e.g., one can distinguish two encryptions of the same message from encryptions of two different messages).

Recall that by combining Constructions 5.3.7 and 5.3.9 (and referring to Propositions 5.3.8 and 5.3.10), we obtain a (full-fledged) private-key encryption scheme. A more efficient scheme is obtained by a direct combination of the ideas underlying both constructions:

Construction 5.3.12 (a private-key encryption scheme based on pseudorandom functions): *Let $F = \{F_n\}$ (and I and V) be as in Construction 5.3.9; that is, $F = \{F_n\}$ is an efficiently computable function ensemble and I and V be the selection and evaluation algorithms associated with it. We define a private-key encryption scheme, (G, E, D), as follows:*

key-generation: $G(1^n) = (i, i)$, *where* $i \leftarrow I(1^n)$.

encrypting plaintext $\alpha \in \{0,1\}^*$ (using the key i): *Break α into consecutive blocks of length n, while possibly augmenting the last block. Let $\alpha_1, ..., \alpha_t$ be the resulting blocks. Associate $\{0,1\}^n$ with the set of integer residues modulo 2^n, select uniformly $r \in \{0,1\}^n$, and compute $r_j = r + j \bmod 2^n$, for $j = 1, ..., t$. Finally, form the ciphertext $(r, |\alpha|, V(i, r_1) \oplus \alpha_1, ..., V(i, r_t) \oplus \alpha_t)$. That is,*

$$E_i(x) = (r, |\alpha|, V(i, (r + 1 \bmod 2^n)) \oplus \alpha_1, ..., V(i, (r + t \bmod 2^n)) \oplus \alpha_t)$$

decrypting ciphertext $(r, m, y_1, ..., y_t)$ (using the key i): *For $j = 1, ..., t$, compute $\alpha_j = V(i, (r + j \bmod 2^n)) \oplus y_j$, and output the m-bit long prefix of $\alpha_1 \cdots \alpha_t$. That is, $D_i(r, m, y_1, ..., y_t)$ is the m-bit long prefix of*

$$(V(i, (r + 1 \bmod 2^n)) \oplus y_1) \cdots (V(i, (r + t \bmod 2^n)) \oplus y_t)$$

Clearly, Construction 5.3.12 constitutes a private-key encryption scheme (provided that F is pseudorandom with respect to polynomial-size circuits). See Exercise 23.

[12] That is, letting $E_i(x) = p_i(x)$, where p_i is the permutation associated with the string i.

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5.3.4 Public-key encryption schemes

As mentioned above, randomization during the encryption process can be avoided in private-key encryption schemes that employ a varying state (not allowed in our basic Definition 5.1.1). In case of public-key encryption schemes, randomization during the encryption process is essential (even if the encryption scheme employs a varying state). Thus, the *randomized encryption paradigm* plays an even more pivotal role in the construction of public-key encryption scheme. To demonstrate this paradigm we start with a very simple (and quite wasteful) construction. But before doing so, we recall the notion of trapdoor permutations.

Trapdoor permutations: All our constructions employ a collection of trapdoor permutations, as in Definition 2.4.5. Recall that such a collection, $\{p_\alpha\}_\alpha$, comes with four probabilistic polynomial-time algorithms, denoted here by I, S, F and B (for *index*, *sample*, *forward* and *backward*), such that the following (syntactic) conditions hold

1. On input 1^n, algorithm I selects a random n-bit long *index* α of a permutation p_α, along with a corresponding trapdoor τ;
2. On input α, algorithm S *samples* the domain of p_α, returning a random element in it;
3. For x in the domain of p_α, given α and x, algorithm F returns $p_\alpha(x)$ (i.e., $F(\alpha, x) = p_\alpha(x)$);
4. For y in the range of p_α if (α, τ) is a possible output of $I(1^n)$ then, given τ and y, algorithm B returns $p_\alpha^{-1}(y)$ (i.e., $B(\tau, y) = p_\alpha^{-1}(y)$).

The hardness condition refers to the difficulty of inverting p_α on a random element of its range, when given only the range-element and α. That is, let $I_1(1^n)$ denote the first element in the output of $I(1^n)$ (i.e., the index), then for every polynomial-size circuit family $\{C_n\}$, every polynomial p and all sufficiently large n's

$$\mathsf{Pr}[C_n(I_1(1^n), p_{I_1(1^n)}(S(I_1(1^n))) = S(I_1(1^n))] \; < \; \frac{1}{p(n)}$$

Namely, C_n fails to invert p_α on $p_\alpha(x)$, where α and x are selected by I and S as above. Recall the above collection can be easily modified to have a hard-core predicate (cf. Theorem 2.5.2). For simplicity, we continue to refer to the collection as $\{p_\alpha\}$, and *let b denote the corresponding hard-core predicate.*

5.3.4.1 Simple schemes

We are now ready to present a very simple (alas quite wasteful) construction of a secure public-key encryption scheme. It is a block-cipher with $\ell \equiv 1$.

Construction 5.3.13 (a simple public-key block-cipher scheme): *Let $\{p_\alpha\}$, I, S, F, B and b be as above.*

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key-generation: *The key-generation algorithm consists of selecting at random a permutation p_α together with a trapdoor τ for it: The permutation (or rather its description) serves as the public-key, whereas the trapdoor serves as the private-key. That is, $G(1^n) = I(1^n)$, which means that the index-trapdoor pair generated by I is associated with the key-pair of G.*

encryption: *To encrypt a bit σ, using the encryption-key α, the encryption algorithm randomly selects an element, r, in the domain of p_α and produces the ciphertext $(p_\alpha(r), \sigma \oplus b(r))$. That is, $E_\alpha(\sigma) = (F(\alpha, r), \sigma \oplus b(r))$, where $r \leftarrow S(\alpha)$.*

decryption: *To decrypt the ciphertext (y, ς), using the decryption-key τ, the decryption algorithm just computes $\varsigma \oplus b(p_\alpha^{-1}(y))$, where the inverse is computed using the trapdoor τ of p_α. That is, $D_\tau(y, \varsigma) = \varsigma \oplus b(B(\tau, y))$.*

Clearly, for every possible (α, τ) output of G and for every $\sigma \in \{0,1\}$, it holds that

$$\begin{aligned}
D_\tau(E_\alpha(\sigma)) &= D_\tau(F(\alpha, S(\alpha)), \sigma \oplus b(S(\alpha))) \\
&= (\sigma \oplus b(S(\alpha))) \oplus b(B(\tau, F(\alpha, S(\alpha)))) \\
&= \sigma \oplus b(S(\alpha)) \oplus b(p_\alpha^{-1}(p_\alpha(S(\alpha)))) \\
&= \sigma \oplus b(S(\alpha)) \oplus b(S(\alpha)) \;=\; \sigma
\end{aligned}$$

The security of the above public-key encryption scheme follows from the (non-uniform) one-way feature of the collection $\{p_\alpha\}$ (or rather from the hypothesis that b is a corresponding hard-core predicate).

Proposition 5.3.14 *Suppose that b is a (non-uniformly strong) hard-core of the collection $\{p_\alpha\}$. Then Construction 5.3.13 constitute a secure public-key block-cipher (with block-length $\ell \equiv 1$).*

Proof: Recall that by the equivalence theorems (i.e., Theorems 5.2.5 and 5.2.11), it suffices to show single-message ciphertext-indistinguishability. Furthermore, by Proposition 5.2.7 and the fact that here there are only two plaintexts (i.e., 0 and 1), it suffices to show that one cannot predict which of the two plaintexts (selected at random) is being encrypted (significantly better than with success probability 1/2). We conclude by noting that a good guess for the plaintext σ, given the encryption-key α and the ciphertext $E_\alpha(\sigma) = (f_\alpha(r), \sigma \oplus b(r))$, where $r \leftarrow S(\alpha)$, yields a good guess for $b(r)$ given $(\alpha, f_\alpha(r))$. That is, the latter guess is correct with probability equal to the probability that former guess is correct. Thus, violation of the security of the encryption scheme yields a contradiction to the the hypothesis that b is a hard-core of $\{p_\alpha\}$. Details follow.

Recall that by saying that b is a hard-core of $\{p_\alpha\}$ we mean that for every polynomial-size circuit family $\{C_n\}$, every polynomial p and all sufficiently large n's

$$\mathsf{Pr}[C_n(I_1(1^n), p_{I_1(1^n)}(S(I_1(1^n))) = b(S(I_1(1^n)))] \;<\; \frac{1}{2} + \frac{1}{p(n)} \qquad (5.8)$$

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By Proposition 5.2.7, it suffices to show that for randomly chosen α (i.e., $\alpha \leftarrow I_1(1^n)$) and uniformly distributed $\sigma \in \{0,1\}$, no polynomial-size circuit given the encryption-key α and the ciphertext $E_\alpha(\sigma)$, can predict σ non-negligibly better than with success probability 1/2. The actual proof uses a reducibility argument: Suppose towards the contradiction that there exists a polynomial-size circuit family $\{C'_n\}$, a polynomial p' and infinitely many n's such that

$$\mathsf{Pr}[C'_n(I_1(1^n), E_{I_1(1^n)}(\sigma)) = \sigma] \; > \; \frac{1}{2} + \frac{1}{p'(n)} \tag{5.9}$$

where σ is uniformly distributed in $\{0,1\}$. Recall that $E_\alpha(\sigma) = (p_\alpha(r), \sigma \oplus b(r))$, where $r \leftarrow S(\alpha)$ is a random sample in p_α's domain, and consider the following probabilistic circuit C''_n: On input α and y (in the range of p_α), the circuit C''_n uniformly selects $\varsigma \in \{0,1\}$, invokes C'_n on input $(\alpha,(y,\varsigma))$, and outputs $C'_n(\alpha,(y,\varsigma)) \oplus \varsigma$. In the following analysis of the behavior of C''_n, we let $\alpha \leftarrow I_1(1^n)$, $r \leftarrow S(\alpha)$, and consider uniformly distributed $\varsigma, \sigma \in \{0,1\}$:

$$\begin{aligned}
\mathsf{Pr}[C''_n(\alpha, p_\alpha(r)) = b(r)] &= \mathsf{Pr}[C'_n(\alpha,(p_\alpha(r),\varsigma)) \oplus \varsigma = b(r)] \\
&= \mathsf{Pr}[C'_n(\alpha,(p_\alpha(r),\varsigma)) = \varsigma \oplus b(r)] \\
&= \mathsf{Pr}[C'_n(\alpha,(p_\alpha(r),\sigma \oplus b(r)) = (\sigma \oplus b(r)) \oplus b(r)] \\
&= \mathsf{Pr}[C'_n(\alpha, E_\alpha(\sigma)) = \sigma] \\
&> \frac{1}{2} + \frac{1}{p'(n)}
\end{aligned}$$

where the inequality is due to Eq. (5.9). Removing the randomization from C''_n (i.e., by fixing the best possible choice), we derive a contradiction to Eq. (5.8). The proposition follows. ∎

Using Propositions 5.3.8 and 5.3.14, and recalling that Theorem 2.5.2 applies also to collections of one-way functions and to the non-uniform setting, we obtain

Theorem 5.3.15 *If there exist collections of (non-uniformly hard) trapdoor permutations then there exist secure public-key encryption schemes.*

A generalization: As admitted above, Construction 5.3.13 is quite wasteful. Specifically, it is wasteful in *bandwidth*; that is, the relation between the length of the plaintext and the length of the ciphertext. In Construction 5.3.13 the relation between these lengths equals the security parameter (i.e., the length of description of individual elements in the domain of the permutation). However, the idea underlying Construction 5.3.13 can yield efficient public-key schemes, provided we use trapdoor permutations having hard-core functions with large range (see Section 2.5.3). To demonstrate the point, we use the following assumption relating to the RSA collection of trapdoor permutations (cf. Subsections 2.4.3 and 2.4.4).

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Large hard-core conjecture for RSA: *The first $n/2$ least significant bits of the argument constitute a* (non-uniformly strong) *hard-core function of the RSA function when applied with n-bit long moduli.*

We stress that the conjecture is NOT know to follow from the assumption that the RSA collection is (non-uniformly) hard to invert. What can be proved under the latter assumption is only that the first $O(\log n)$ least significant bits of the argument constitute a (non-uniformly strong) hard-core function of RSA (with n-bit long moduli). Still, if the above conjecture holds then one obtains a secure public-key encryption scheme with efficiency comparable to that of "plain RSA" (see discussion below). Furthermore, this scheme coincides with the common practice of randomly padding messages (using padding equal in length to the message) before encrypting them (by applying the RSA function). That is, we consider the following scheme:

Construction 5.3.16 (Randomized RSA – a public-key block-cipher scheme): *This scheme employs the RSA collection of trapdoor permutations (cf. Subsections 2.4.3 and 2.4.4). The following description is, however, self-contained.*

key-generation: *The key-generation algorithm consists of selecting at random two n-bit primes, P and Q, setting $N = P \cdot Q$, selecting at random a pair (e, d) so that $e \cdot d \equiv 1 \pmod{(P-1)\cdot(Q-1)}$, and outputting the tuple $((N, e), (N, d))$, where (N, e) is the encryption-key and (N, d) is the decryption-key. That is, $((N, e), (N, d)) \leftarrow G(1^n)$, where N, e and d are as specified above.*

(Note that N is $2n$-bit long.)

encryption: *To encrypt an n-bit string σ (using the encryption-key (N, e)), the encryption algorithm randomly selects an element, $r \in \{0, ..., N-1\}$, and produces the ciphertext $(r^e \bmod N, \sigma \oplus \text{LSB}(r))$, where $\text{LSB}(r)$ denotes the n least significant bits of r. That is, $E_{(N,e)}(\sigma) = (r^e \bmod N, \sigma \oplus \text{LSB}(r))$.*

decryption: *To decrypt the ciphertext $(y, \varsigma) \in \{0, ..., N-1\} \times \{0,1\}^n$ (using the decryption-key (N, d)), the decryption algorithm just computes $\varsigma \oplus \text{LSB}(y^d \bmod N)$, where $\text{LSB}(\cdot)$ is as above. That is, $D_{(N,d)}(y, \varsigma) = \varsigma \oplus \text{LSB}(y^d \bmod N)$.*

The bandwidth of the above scheme is much better than in Construction 5.3.13: a plaintext of length n is encrypted via a ciphertext of length $3n$. Furthermore, Randomized RSA is almost as efficient as "plain RSA" (or the RSA function itself).

To see that Randomized RSA satisfies the syntactic requirements of an encryption scheme, consider any possible output of $G(1^n)$, denoted $((N, e), (N, d))$, and any $\sigma \in \{0,1\}^n$. Then, for r uniformly selected in $\{0, ..., N-1\}$, it holds that

$$\begin{aligned} D_{(N,d)}(E_{(N,e)}(\sigma)) &= D_{(N,d)}((r^e \bmod N), \sigma \oplus \text{LSB}(r)) \\ &= (\sigma \oplus \text{LSB}(r)) \oplus \text{LSB}((r^e \bmod N)^d \bmod N) \\ &= \sigma \oplus \text{LSB}(r) \oplus \text{LSB}(r^{ed} \bmod N) \;=\; \sigma \end{aligned}$$

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where the last equality is due to $r^{ed} \equiv r \pmod N$. The security of Randomized RSA (as a public-key encryption scheme) follows from the large hard-core conjecture for RSA, analogously to the proof of Proposition 5.3.14.

Proposition 5.3.17 *Suppose that the large hard-core conjecture for RSA does hold. Then Construction 5.3.16 constitute a secure public-key block-cipher* (*with block-length* $\ell(n) = n$).

Proof Sketch: Recall that by the equivalence theorems (i.e., Theorems 5.2.5 and 5.2.11), it suffices to show single-message ciphertext-indistinguishability. Considering any two strings x and y, we need to show that $((N,e), r^e \bmod N, x \oplus \text{LSB}(r))$ and $((N,e), r^e \bmod N, y \oplus \text{LSB}(r))$ are indistinguishable, where N, e and r are selected at random as in the construction. It suffices to show that, for every x, the distributions $((N,e), r^e \bmod N, x \oplus \text{LSB}(r))$ and $((N,e), r^e \bmod N, x \oplus s)$ are indistinguishable, where $s \in \{0,1\}^n$ is uniformly distributed, independently of anything else. The latter claim follows from the hypothesis that the n least significant bits are a hard-core function for RSA with moduli of length $2n$. ■

Discussion: We wish to stress that *encrypting messages by merely applying the RSA function to them* (*without randomization*), *yields an insecure encryption scheme.* Unfortunately, this procedure (referred to about as 'plain RSA'), is quite common in practice. The fact that plain RSA is definitely insecure is a special case of the fact that any public-key encryption scheme that employs a deterministic encryption algorithm is insecure. We warn that the fact that in such deterministic encryption schemes one can distinguish encryptions of two specific messages (e.g., the all-zero message and the all-one message) is not "merely of theoretical concern" – it may seriously endanger some applications! In contrast, Randomized RSA (as defined in Construction 5.3.16) *may be* secure, provided a quite reasonable conjecture (i.e., the large hard-core conjecture for RSA) holds. Thus, the common practice of applying the RSA function to a randomly-padded version of the plaintext is way superior to using the RSA function directly (i.e., without randomization): *the randomized version is likely to be secure, whereas the non-randomized* (*or plain*) *version is definitely insecure.*

We note that Construction 5.3.16 (or alternatively Construction 5.3.13) generalizes to any collection of trapdoor permutations having a corresponding large hard-core function. Suppose that $\{p_\alpha\}$ is such a collection, and h (or rather $\{h_\alpha\}$) is a corresponding hard-core function (resp., a corresponding collection of hard-core functions) such that any element in the domain of p_α is mapped to an $\ell(|\alpha|)$-bit long string. Then we can encrypt an $\ell(|\alpha|)$-bit long plaintext, x, by $(p_\alpha(r), h(r) \oplus x)$ (resp., $(p_\alpha(r), h_\alpha(r) \oplus x)$), where $r \leftarrow S(\alpha)$ (as in Construction 5.3.13). This yields a secure public-key encryption scheme with bandwidth that relates to the relation between $\ell(|\alpha|)$ and the length of a description of individual elements in the domain of p_α.

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5.3.4.2 An alternative scheme

An alternative construction of a public-key encryption scheme is presented below. Rather than encrypting each plaintext bit (or block of bits) by an independently selected element in the domain of the trapdoor permutation (as done in Construction 5.3.13), we select only one such element (for the entire plaintext), and generate from it additional bits, one per each bit of the plaintext. These additional bits are determine by successive applications of the trapdoor permutation, and only the last result is included in the ciphertext. In a sense, the construction of the encryption scheme (below) augments the construction of a pseudorandom generator based on one-way permutations (i.e., Construction 3.4.4).

Construction 5.3.18 (a public-key encryption scheme): *Let $\{p_\alpha\}$, I, S, F, B and b be as in Construction 5.3.13. We use the notation $p_\alpha^{i+1}(x) = p_\alpha(p_\alpha^i(x))$ and $p_\alpha^{-(i+1)}(x) = p_\alpha^{-1}(p_\alpha^{-i}(x))$.*

key-generation: *The key-generation algorithm consists of selecting at random a permutation p_α together with a trapdoor, exactly as in Construction 5.3.13. That is, $G(1^n) = I(1^n)$, which means that the index-trapdoor pair generated by I is associated with the key-pair of G.*

encryption: *To encrypt a string σ, using the encryption-key α, the encryption algorithm randomly selects an element, r, in the domain of p_α and produces the ciphertext $(p_\alpha^{|\sigma|}(r), \sigma \oplus G_\alpha^{(|\sigma|)}(r))$, where*

$$G_\alpha^{(\ell)}(r) \stackrel{\rm def}{=} b(r) \cdot b(p_\alpha(r)) \cdots b(p_\alpha^{\ell-1}(r)) \tag{5.10}$$

That is, $E_\alpha(\sigma) = (p_\alpha^{|\sigma|}(S(\alpha)), \sigma \oplus G_\alpha^{(|\sigma|)}(S(\alpha)))$.

decryption: *To decrypt the ciphertext (y, ς), using the decryption-key τ, the decryption algorithm just computes $\varsigma \oplus G_\alpha^{(|\varsigma|)}(p_\alpha^{-|\varsigma|}(y))$, where the inverse is computed using the trapdoor τ of p_α. That is, $D_\tau(y, \varsigma) = \varsigma \oplus G_\alpha^{(|\varsigma|)}(p_\alpha^{-|\varsigma|}(y))$.*

We stress that the above encryption scheme is a full-fledged one (rather than a block-cipher). Its bandwidth tends to 1 with the length of the plaintext; that is, a plaintext of length $\ell = \text{poly}(n)$ is encrypted via a ciphertext of length $m + \ell$, where m denotes the length of the description of individual elements in the domain of p_α. Clearly, for every possible (α, τ) output of G (and $r \leftarrow S(\alpha)$), it holds that

$$\begin{aligned} D_\tau(E_\alpha(\sigma)) &= D_\tau(p_\alpha^{|\sigma|}(r), \sigma \oplus G_\alpha^{(|\sigma|)}(r)) \\ &= (\sigma \oplus G_\alpha^{(|\sigma|)}(r)) \oplus G_\alpha^{(|\sigma|)}(p_\alpha^{-|\sigma \oplus G_\alpha^{(|\sigma|)}(r)|}(p_\alpha^{|\sigma|}(r))) \\ &= \sigma \oplus G_\alpha^{(|\sigma|)}(r) \oplus G_\alpha^{(|\sigma|)}(r) \;=\; \sigma \end{aligned}$$

The security of the above public-key encryption scheme follows from the (non-uniform) one-way feature of the collection $\{p_\alpha\}$, but here we restrict the sampling algorithm S to produce almost uniform distribution over the domain (so that this distribution is preserved under successive applications of p_α).

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Proposition 5.3.19 *Suppose that b is a (non-uniformly strong) hard-core of the trapdoor collection $\{p_\alpha\}$. Furthermore, suppose that this trapdoor collection utilizes a domain sampling algorithm S so that the statistical difference between $S(\alpha)$ and the uniform distribution over the domain of p_α is negligible in terms of $|\alpha|$. Then Construction 5.3.18 constitute a secure public-key encryption scheme.*

Proof: Again, we prove single-message ciphertext-indistinguishability. As in the proof of Proposition 5.3.17, it suffices to show that, for every σ, the distributions $(\alpha, p_\alpha^{|\sigma|}(S(\alpha)), \sigma \oplus G_\alpha^{(|\sigma|)}(S(\alpha)))$ and $(\alpha, p_\alpha^{|\sigma|}(S(\alpha)), \sigma \oplus s)$ are indistinguishable, where $s \in \{0,1\}^{|\sigma|}$ is uniformly distributed, independently of anything else. The latter claim holds by a minor extension to Proposition 3.4.6: the latter refers to the case $S(\alpha)$ is uniform over the domain of p_α, but can be extended to the case in which there is a negligible statistical difference between the distributions.

> Details: We need to prove that for every polynomial ℓ and every sequence of pairs $(\sigma'_n, \sigma''_n) \in \{0,1\}^{\ell(n)} \times \{0,1\}^{\ell(n)}$, the distributions $D'_n \stackrel{\text{def}}{=} (\alpha, p_\alpha^{\ell(n)}(S(\alpha)), \sigma'_n \oplus G_\alpha^{(\ell(n))}(S(\alpha)))$ and $D''_n \stackrel{\text{def}}{=} (\alpha, p_\alpha^{\ell(n)}(S(\alpha)), \sigma''_n \oplus G_\alpha^{(\ell(n))}(S(\alpha)))$ are indistinguishable, where $\alpha \leftarrow I_1(1^n)$. We prove the above in two steps:
>
> 1. We first prove that for every sequence of σ_n's, the distributions $D_n \stackrel{\text{def}}{=} (\alpha, p_\alpha^{\ell(n)}(S(\alpha)), \sigma_n \oplus G_\alpha^{(\ell(n))}(S(\alpha)))$ and $R_n \stackrel{\text{def}}{=} (\alpha, p_\alpha^{\ell(n)}(S(\alpha)), \sigma_n \oplus U_{\ell(n)})$ are indistinguishable, where $U_{\ell(n)}$ denotes a random variable uniformly distributed over $\{0,1\}^{\ell(n)}$ and $\alpha \leftarrow I_1(1^n)$.
>
>   Suppose first that $S(\alpha)$ is uniform over the domain of p_α. Then the indistinguishability of $\{D_n\}_{n\in\mathbb{N}}$ and $\{R_n\}_{n\in\mathbb{N}}$ follows directly from Proposition 3.4.6 (as adapted to circuits): the adapted form refers to the indistinguishability of $(\alpha, p_\alpha^{\ell(n)}(S(\alpha)), G_\alpha^{(\ell(n))}(S(\alpha)))$ and $(\alpha, p_\alpha^{\ell(n)}(S(\alpha)), U_{\ell(n)})$, and yields the desired claim by noting that σ_n can be incorporated in the prospective distinguisher. The extension (to the case that $S(\alpha)$ has negligible statistical difference to the uniform distribution over the domain of p_α) is straightforward.
>
> 2. Applying the previous item to D'_n and $R'_n \stackrel{\text{def}}{=} (\alpha, p_\alpha^{\ell(n)}(S(\alpha)), \sigma'_n \oplus U_{\ell(n)})$, we conclude that $\{D'_n\}_{n\in\mathbb{N}}$ and $\{R'_n\}_{n\in\mathbb{N}}$ are indistinguishable. Similarly, $\{D''_n\}_{n\in\mathbb{N}}$ and $\{R''_n\}_{n\in\mathbb{N}}$, where $R''_n \stackrel{\text{def}}{=} (\alpha, p_\alpha^{\ell(n)}(S(\alpha)), \sigma''_n \oplus U_{\ell(n)})$, are indistinguishable. Furthermore, $\{R'_n\}_{n\in\mathbb{N}}$ and $\{R''_n\}_{n\in\mathbb{N}}$ are identically distributed. Thus, $\{D'_n\}_{n\in\mathbb{N}}$ and $\{D''_n\}_{n\in\mathbb{N}}$ are indistinguishable.

The proposition follows. ■

An instantiation: Assuming that factoring Blum Integers (i.e., products of two primes each congruent to $3 \pmod 4$) is hard, one may use the modular squaring function (which induces a permutation over the quadratic residues modulo the product of these integers) in role of the trapdoor permutation used in Construction 5.3.18. This yields a secure public-key encryption scheme with efficiency comparable to that of plain RSA (see further discussion below).

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Construction 5.3.20 (The Blum-Goldwasser Public-Key Encryption Scheme): *Consult Appendix A for the relevant number theoretic background, and note that for* $P \equiv 3 \pmod 4$ *the number* $(P+1)/4$ *is an integer. For simplicity, we present a block-cipher with arbitrary block-length* $\ell(n) = \mathrm{poly}(n)$*; a full-fledged encryption scheme can be derived by an easy modification (see Exercise 24).*

key-generation: *The key-generation algorithm consists of selecting at random two* n*-bit primes,* P *and* Q*, each congruent to 3 mod 4, and outputting the pair* $(N, (P, Q))$*, where* $N = P \cdot Q$*.*

Actually, for sake of efficiency, the key-generator also computes

$$\begin{aligned} d_P &= ((P+1)/4)^{\ell(n)} \bmod P-1 & (\text{in } \{0, ..., P-2\}) \\ d_Q &= ((Q+1)/4)^{\ell(n)} \bmod Q-1 & (\text{in } \{0, ..., Q-2\}) \\ c_P &= Q \cdot (Q^{-1} \bmod P) & (\text{in } \{0, ..., N-Q\}) \\ c_Q &= P \cdot (P^{-1} \bmod Q) & (\text{in } \{0, ..., N-P\}) \end{aligned}$$

It outputs the pair (N, T)*, where* N *serves as the encryption-key and* $T = (P, Q, N, c_P, d_P, c_Q, d_Q)$ *serves as decryption-key.*

encryption: *To encrypt the message* $\sigma \in \{0,1\}^{\ell(n)}$*, using the encryption-key* N*:*

1. *Uniformly select* $s_0 \in \{1, ..., N\}$*.*
   (Note that if $\mathrm{GCD}(s_0, N) = 1$ then $s_0^2 \bmod N$ is a uniformly distributed quadratic residue modulo N.)
2. *For* $i = 1, .., \ell(n)+1$*, compute* $s_i \leftarrow s_{i-1}^2 \bmod N$ *and* $b_i = \mathrm{lsb}(s_i)$*, where* $\mathrm{lsb}(s)$ *is the least significant bit of* s*.*

The ciphertext is $(s_{\ell(n)+1}, \varsigma)$*, where* $\varsigma = \sigma \oplus b_1 b_2 \cdots b_{\ell(n)}$*.*

decryption: *To decrypt of the ciphertext* (r, ς) *using the decryption-key* $T = (P, Q, N, c_P, d_P, c_Q, d_Q)$*, one first retrieves* s_1 *and then computes the* b_i*'s as above. Instead of extracting modular square roots successively* $\ell(n)$ *times, we extract the* $2^{\ell(n)}$*-th root, which can be done as efficiently as extracting a single square root:*

1. *Let* $s' \leftarrow r^{d_P} \bmod P$*, and* $s'' \leftarrow r^{d_Q} \bmod Q$*.*
2. *Let* $s_1 \leftarrow c_P \cdot s' + c_Q \cdot s'' \bmod N$*.*
3. *For* $i = 1, .., \ell(n)$*, compute* $b_i = \mathrm{lsb}(s_i)$ *and* $s_{i+1} \leftarrow s_i^2 \bmod N$*.*

The plaintext is $\varsigma \oplus b_1 b_2 \cdots b_{\ell(n)}$*.*

Again, one can easily verify that the above construction constitutes an encryption scheme: the main fact to verify is that the value of s_1 as reconstructed in the decryption stage equals the value used in the encryption stage. This follows by combining the Chinese Reminder Theorem with the fact that for every quadratic residue $s \bmod N$ it holds that $s \equiv (s^{2^\ell} \bmod N)^{d_P} \pmod P$ and $s \equiv (s^{2^\ell} \bmod N)^{d_Q} \pmod Q$.

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Details: Recall that for a prime $P \equiv 3 \pmod 4$, and every integer i, we have $i^{(P+1)/2)} \equiv i \pmod P$. Thus, for every integer j, we have

$$\begin{aligned}(j^{2^\ell} \bmod N)^{d_P} &\equiv \left(j^{2^\ell} \bmod N\right)^{((P+1)/4)^\ell} \pmod P \\ &\equiv j^{((P+1)/2)^\ell} \pmod P \\ &\equiv j \pmod P\end{aligned}$$

Similarly, $j \equiv (j^{2^\ell} \bmod N)^{d_Q} \pmod Q$. Observing that c_P and c_Q are as in the Chinese Reminder Theorem (i.e., $i \equiv c_P \cdot (i \bmod P) + c_Q \cdot (i \bmod Q) \pmod N$, for every integer i), we conclude that s_1 as recovered in Step 2 of the decryption process equals s_1 as first computed in Step 2 of the encryption process.

Encryption amounts to $\ell(n) + 1$ modular multiplications, whereas decryption amounts to $\ell(n)+2$ such multiplications and 2 modular exponentiations (relative to half-sized moduli). Counting modular exponentiations with respect to n-bit moduli as $O(n)$ (i.e., at least n and at most $2n$) modular multiplications (with respect to n-bit moduli), we conclude that the entire encryption-decryption process requires work comparable to $2\ell(n) + 4n$ modular multiplications. For comparison to (Randomized) RSA, note that encrypting/decrypting $\ell(n)$-bit messages (in Randomized RSA) amounts to $\lceil \ell(n)/n \rceil$ modular exponentiations, and so the total work is comparable to $2 \cdot (\ell(n)/n) \cdot 1.5n = 3\ell(n)$ (for general exponent e, or half that much in case $e = 3$).

The security of the Blum-Goldwasser scheme (i.e., Construction 5.3.20) follows immediately from Proposition 5.3.19 and the fact that the least significant bit (i.e., lsb) is a hard-core for the modular squaring function. Recalling that inverting the latter is computationally equivalent to factoring, we get:

Corollary 5.3.21 *Suppose that factoring is infeasible in the sense that for every polynomial-size circuit $\{C_n\}$, every positive polynomial p and all sufficiently large n's*

$$\mathsf{Pr}[C_n(P_n \cdot Q_n) = P_n] < \frac{1}{p(n)}$$

where P_n and Q_n are uniformly distributed n-bit long primes. Then Construction 5.3.20 constitutes a secure public-key encryption scheme.

Thus, the conjectured infeasibility of factoring (which is a necessary condition for security of RSA), yields a secure public-key encryption scheme with efficiency comparable to that of (plain or Randomized) RSA. In contrast, recall that plain RSA itself is not secure (as it employs a deterministic encryption algorithm), whereas Randomized RSA (i.e., Construction 5.3.16) is not known to be secure under standard assumption such as intractability of factoring (or even of inverting the RSA function).[13]

[13] Recall that Randomized RSA is secure provided that the $n/2$ least significant bits constitute a hard-core function for n-bit RSA moduli. This is a reasonable conjecture, but it seems stronger than the conjecture that RSA is hard to invert: assuming that RSA is hard to invert, we only know that the $O(\log n)$ least significant bits constitute a hard-core function for n-bit moduli.

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5.4 * Beyond eavesdropping security

Our treatment so far has referred only to a "passive" attack in which the adversary merely eavesdrops on the line over which ciphertexts are being sent. Stronger types of attacks, culminating in the so-called Chosen Ciphertext Attack, may be possible in various applications. Specifically, in some settings it is feasible for the adversary to make the sender encrypt a message of the adversary's choice, and in some settings the adversary may even make the receiver decrypt a ciphertext of the adversary's choice. This gives rise to *chosen plaintext attacks* and to *chosen ciphertext attacks*, respectively, which are not covered by the security definitions considered in previous sections. Thus, our main goal in this section is to provide a treatment to such types of "active" attacks. In addition, we also discuss the related notion of non-malleable encryption schemes (see Section 5.4.5).

5.4.1 Overview

We start with an overview of the type of attacks and results considered in the current (rather long) section.

5.4.1.1 Types of attacks

The following mini-taxonomy of attacks is certainly not exhaustive.

Passive attacks. We first re-consider passive attacks as referred to in the definitions given in previous sections. In case of public-key schemes we distinguish two sub-cases:

1. A *key-oblivious*, passive attack, as captured in the abovementioned definitions. By 'key-obliviousness' we refer to the postulation that the choice of plaintext does not depend on the public-key.

2. A *key-dependent*, passive attack, in which the choice of plaintext may depend on the public-key.

(In Definition 5.2.2 the choice of plaintext means the random variable X_n, whereas in Definition 5.2.4 it means the pair (x_n, y_n). In both these definitions, the choice of the plaintext is key-oblivious.)

Chosen Plaintext Attacks. Here the attacker may obtain the encryption of any plaintext of its choice (under the key being attacked). Indeed, such an attack does not add power in case of public-key schemes.

Chosen Ciphertext Attacks. Here the attacker may obtain the decryption of any ciphertext of its choice (under the key being attacked). That is, the attacker is given oracle access to the decryption function corresponding to the decryption-key in use. We distinguish two types of such attacks.

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1. In an *a-priori chosen* ciphertext attack, the attacker is given this oracle access prior to being presented the ciphertext that it should attack (i.e., the ciphertext for which it has to learn partial information). That is, the attack consists of two stages: in the first stage the attacker is given the above oracle access, and in the second stage the oracle is removed and the attacker is given a 'test ciphertext' (i.e., a target to be learned).

2. In an *a-posteriori chosen* ciphertext attack, after being given the target ciphertext, the oracle is not removed but the adversary's access to it is restricted in that it is not allowed to make a query equal to the target ciphertext.

In both cases, the adversary may make queries that do not correspond to a legitimate ciphertext, and the answer will be accordingly (i.e., a special 'failure' symbol). Furthermore, in both cases the adversary may effect the selection of the target ciphertext (by specifying a distribution from which the corresponding plaintext is to be drawn).

Formal definitions of all types of attacks listed above are given in the following corresponding subsections (i.e., in Sections 5.4.2, 5.4.3 and 5.4.4, respectively). In addition, in Section 5.4.5, we consider the related notion of *malleability*; that is, attacks aimed at generating ciphertexts related to the secret plaintext rather than gaining information about it.

5.4.1.2 Constructions

As in the basic case, actively-secure private-key encryption schemes can be constructed based on the existence of one-way functions, whereas actively-secure public-key encryption schemes are based on the existence of trapdoor permutations. In both cases, withstanding a-posteriori chosen ciphertext attacks is harder than withstanding a-priori chosen ciphertext attacks. We will present the following results.

For private-key schemes: We will show that the private-key encryption scheme based on pseudorandom functions (i.e., Construction 5.3.9), is secure also under *a-priori chosen ciphertext attacks*, but is not secure under an *a-posteriori chosen ciphertext attack*. We will also show how to transform any passively-secure private-key encryption scheme into a scheme *secure under (a-posteriori) chosen ciphertext attacks*, by using a message authentication scheme on top of the basic encryption. Thus, the latter construction relies on message authentication schemes as defined in Section 6.1. We mention that message authentication schemes can be constructed using pseudorandom functions; see Section 6.3.

For public-key schemes: Assuming the existence of trapdoor permutations, we will present constructions of public-key encryption schemes that are secure against (a-priori and a-posteriori) chosen ciphertext attacks. The constructions

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utilize various forms of non-interactive zero-knowledge proofs (see Section 4.10), which can be constructed under the former assumption. We warn that these constructions are rather complex. We will start with the construction of a public-key encryption scheme that is secure against a-priori chosen ciphertext attacks, and then turn to the more complex scheme that is secure also under a-posteriori chosen ciphertext attacks.

As a corollary to the relation between these strong notions of security and non-malleable encryption scheme, we will conclude that the abovementioned schemes are non-malleable.

5.4.1.3 Methodological comments

As hinted above, we do not cover all possible intermediate types of attacks, but rather focus on some natural ones. For example, we only consider key-dependent attacks on public-key encryption schemes (but not on private-key schemes).

The attacks are presented in increasing order of strength; hence, security under such attacks yields increasingly stronger notions. This fact may be best verified when considering the indistinguishability variants of these security definitions.

A uniform-complexity treatment seems more appealing in the current section (i.e., more than in the previous sections). However, for sake of consistency with the basic definitions (i.e., the previous sections of this chapter), we use non-uniform formulations of the various definitions. To obtain the corresponding uniform-complexity formulations, one should merely restrict the (polynomial-size) circuit families to be constructible by a uniform polynomial-time machine. We stress that all the results extend to the uniform-complexity setting (because all our reductions are either uniform or can be adapted to be uniform using the techniques of Section 5.2.5).

As mentioned above, non-interactive zero-knowledge proofs play a central role in the construction of public-key encryption schemes that are secure under chosen ciphertext attacks. At that point, we will assume that the reader is fairly comfortable with the notion of zero-knowledge proofs. Furthermore, although we recall the relevant definition of non-interactive zero-knowledge, which will serve as our starting point towards stronger notions, we recommend to study first the more basic definitions (and results) regarding non-interactive zero-knowledge proofs (as presented in Section 4.10). In our constructions of encryption schemes that are secure under a-posteriori chosen ciphertext attacks, we shall use some results from Chapter 6. In case of private-key encryption schemes (cf. Section 5.4.4.3), we will use a message authentication scheme, but do so in a self-contained way. In case of public-key encryption schemes (cf. Section 5.4.4.4), we will use signature schemes having an extra property in order to construct a certain non-interactive zero-knowledge proof, which we use for the construction of the encryption scheme. At that point we shall refer to a specific result proved in Chapter 6.

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5.4.2 Key-dependent passive attacks

The following discussion as well as the entire subsection *refers only to public-key encryption schemes.* For sake of simplicity, we present the single-message definitions of security; and note that, as in the basic case (for public-key encryption schemes), the single-message definitions of security are equivalent to the multiple-message ones.

In Definitions 5.2.2 and 5.2.4 the plaintext distribution (or pair) is fixed obliviously of the encryption-key. This suffices for the natural case in which the (high level) application (using the encryption scheme) is oblivious of the encryption-key.[14] However, in some settings, the adversary may have partial control on the application. Furthermore, in the public-key case, the adversary knows the encryption-key in use, and so (if it may partially control the application then) it may be able to cause the application to invoke the encryption scheme on plaintexts that are related to the encryption-key in use. Thus, for such settings, we need stronger definitions of security that postulate that partial information about the plaintext remains secret even if the plaintext does depend on the encryption-key in use. Note that here we merely consider the dependence of the "target" plaintext (i.e., the one for which the adversary wishes to obtain partial information) on the encryption-key, and ignore the fact that the above motivation also suggests that the adversary can obtain the encryptions of additional plaintexts chosen by it (as discussed in Section 5.4.3). However, it is easy to see that (in the public-key setting discussed here) these additional encryptions are of no use because the adversary can generate them by itself (see Section 5.4.3).

5.4.2.1 Definitions

Recall that we seek a definition that guarantees that partial information about the plaintext remains secret even if the plaintext does depend on the encryption-key in use. That is, we seek a strengthening of semantic security (as defined in Definition 5.2.2) in which one allows the plaintext distribution ensemble (denoted $\{X_n\}_{n\in\mathbb{N}}$ in Definition 5.2.2) to depend on the encryption-key in use (i.e., for encryption-key e we consider the distribution X_e over $\{0,1\}^{\mathrm{poly}(|e|)}$). Furthermore, we also allow the partial information functions (denoted f and h in Definition 5.2.2) to depend on the encryption-key in use (i.e., for encryption-key e, we consider the functions f_e and h_e). In the actual definition it is important to restrict the scope of the functions h_e's and the distributions X_e's so that their dependency on e is polynomial-time computable (see Exercise 25). This yields the definition presented in Exercise 26, which is equivalent to the following formulation.

[14] Indeed, it is natural (and even methodologically imperative) that a high-level application that uses encryption as a tool, is oblivious of the keys used by that tool. However, this refers only to proposer operation of the application, and deviation may be caused (in some settings) by an improper behavior (i.e., an adversary).

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Definition 5.4.1 (semantic security under key-dependent passive attacks): *The sequence $\{(f_e, h_e, X_e)\}_{e\in\{0,1\}^*}$ is* admissible *for the current definition if*

1. *The functions $f_e : \{0,1\}^* \to \{0,1\}^*$ are polynomially-bounded; that is, there exists a polynomial ℓ such that $|f_e(x)| \leq \ell(|x| + |e|)$.*
2. *There exists a non-uniform family of polynomial-size* (h-evaluation) *circuits $\{H_n\}_{n\in\mathbb{N}}$ such that for every e in the range of $G_1(1^n)$ and every $x \in \{0,1\}^{\text{poly}(|e|)}$ it holds that $H_n(e, x) = h_e(x)$.*
3. *There exists a non-uniform family of* (probabilistic) *polynomial-size* (sampling) *circuits $\{S_n\}_{n\in\mathbb{N}}$ such that for every e in the range of $G_1(1^n)$ and for some $m = \text{poly}(|e|)$, the random variables $S_n(e, U_m)$ and X_e are identically distributed. We stress that for every e, the length of X_e is fixed.*

An encryption scheme, (G, E, D), is semantically secure under key-dependent passive attacks *if for every probabilistic polynomial-time algorithm A, there exists a probabilistic polynomial-time algorithm A' such that for every admissible sequence $\{(f_e, h_e, X_e)\}_{e\in\{0,1\}^*}$, every positive polynomial $p(\cdot)$ and all sufficiently large n:*

$$\begin{aligned} &\mathsf{Pr}\left[A(e, E_e(X_e), 1^{|X_e|}, h_e(X_e)) = f_e(X_e)\right] \\ &\qquad < \mathsf{Pr}\left[A'(e, 1^{|X_e|}, h_e(X_e)) = f_e(X_e)\right] + \frac{1}{p(n)} \end{aligned}$$

where $(e, d) \leftarrow G(1^n)$, and the probability is taken over the internal coin tosses of algorithms G, E, A and A', as well as over X_e.

We stress that the performance of A' is measured against the same distribution of triplets (f_e, h_e, X_e) (i.e., $e \leftarrow G_1(1^n)$) as the one considered for algorithm A. Unlike in other versions of the definition of semantic security, here it is important to let A' have the encryption-key e because the task (i.e., the evaluation of $f_e(X_e)$) as well as its main input (i.e., the value $h_e(X_e)$) are related to e. (Indeed, if e were not given to A' then no encryption scheme (G, E, D) could have satisfied the revised Definition 5.4.1: Considering $h_e(x) = x \oplus e$ (for $|x| = |e|$) and $f_e(x) = x$, note that it is easy for A to compute x from e and $h_e(x)$ (which are explicit in $(e, E_e(x), 1^{|x|}, h_e(x))$), whereas no A' can compute x from $(1^n, 1^{|x|}, h_e(x))$.)

Using Exercise 13.2, one may verify that Definition 5.2.2 is a special case of Definition 5.4.1. An analogous modification (or generalization) of Definition 5.2.4 yields the following:

Definition 5.4.2 (indistinguishability of encryptions under key-dependent passive attacks): *The sequence $\{(x_e, y_e)\}_{e\in\{0,1\}^*}$ is* admissible *for the current definition if there exists a non-uniform family of polynomial-size circuits $\{P_n\}_{n\in\mathbb{N}}$ that maps each encryption-key $e \in \{0,1\}^*$ to the corresponding pair of* (equal length) *strings (x_e, y_e). That is, for every e in the range of $G_1(1^n)$, it holds that $P_n(e) = (x_e, y_e)$. An encryption scheme, (G, E, D), has* indistinguishable

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encryptions under key-dependent passive attacks *if for every non-uniform family of polynomial-size circuits* $\{C_n\}$*, every admissible sequence* $\{(x_e, y_e)\}_{e\in\{0,1\}^*}$*, every positive polynomial* $p(\cdot)$ *and all sufficiently large* n*:*

$$|\mathsf{Pr}\left[C_n(e, E_e(x_e)) = 1\right] - \mathsf{Pr}\left[C_n(e, E_e(y_e)) = 1\right]| < \frac{1}{p(n)}$$

where $(e, d) \leftarrow G(1^n)$*, and the probability is taken over the internal coin tosses of algorithms* G *and* E*.*

As in the basic case, the two definitions are equivalent.

Theorem 5.4.3 (equivalence of definitions for key-dependent passive attacks): *The public-key encryption scheme* (G, E, D) *is semantically secure under key-dependent passive attacks if and only if it has indistinguishable encryptions under key-dependent passive attacks.*

Proof Sketch: In order to show that indistinguishable encryptions implies semantic security, we follow the proof of Proposition 5.2.6. Specifically, A' is constructed and analyzed almost as before, with the exception that A' gets and uses the encryption-key e (rather than generating a random encryption-key by itself).[15] That is, we let $A'(e, 1^{|x|}, h_e(x)) = A(e, E_e(1^{|x|}), 1^{|x|}, h_e(x))$, and show that for every (deterministic) polynomial-size circuit families $\{C_n\}_{n\in\mathbb{N}}$ and $\{H_n\}_{n\in\mathbb{N}}$ (and all sufficiently large n):

$$\begin{aligned} &\mathsf{Pr}\left[A(e, E_e(C_n(e)), 1^{|C_n(e)|}, H_n(e, C_n(e))) = f_e(C_n(e))\right] \qquad (5.11)\\ &< \ \mathsf{Pr}\left[A(e, E_e(1^{|C_n(e)|}), 1^{|C_n(e)|}, H_n(e, C_n(e))) = f_e(C_n(e))\right] + \frac{1}{\mathrm{poly}(n)} \end{aligned}$$

where $e \leftarrow G_1(1^n)$. Once proven, Eq. (5.11) implies that (G, E, D) satisfies Definition 5.4.1.

> On how Eq. (5.11) implies Definition 5.4.1: The issue is that Eq. (5.11) refers to deterministic circuits (i.e. C_n's), whereas Definition 5.4.1 refers to probabilistic circuits (i.e. S_n's). This small gap can be bridged by fixing a sequence of coins for the probabilistic (sampling) circuits. Specifically, starting with any admissible (for Definition 5.4.1) sequence $\{(f_e, h_e, X_e)\}_{e\in\{0,1\}^*}$, where $H_n(e, x) = h_e(x)$ and $X_e \equiv S_n(e, U_{\mathrm{poly}(n)})$, we consider some sequence of coins r_n (for S_n) that maximizes the gap between $\mathsf{Pr}[A(e, E_e(x_e), 1^{|x_e|}, H_n(e, x_e)) = f_e(x_e)]$ and $\mathsf{Pr}[A'(e, 1^{|x_e|}, H_n(e, x_e)) = f_e(x_e)]$, where e is random and $x_e = S_n(e, r_n)$. Recalling that $A'(e, 1^m, \gamma) = A(e, E_e(1^m), 1^m, \gamma)$ and incorporating the sequence of r_n's in A, we obtain a contradiction to Eq. (5.11) (i.e., by letting $C_n(e) = S_n(e, r_n) = x_e$).

Assuming (to the contrary of the above claim) that Eq. (5.11) does not hold, we obtain a sequence of admissible pairs $\{(x_e, y_e)\}_{e\in\{0,1\}^*}$ for Definition 5.4.2

[15] Here we use the convention by which A' gets e along with $h_e(x)$ (and $1^{|x|}$). This is important because A' must feed a matching pair $(e, h_e(x))$ to A.

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such that their encryptions can be distinguished (in contradiction to our hypothesis). Specifically, we set $x_e \stackrel{\rm def}{=} C_n(e)$ and $y_e \stackrel{\rm def}{=} 1^{|x_e|}$, and let $C'_n(e,\alpha) \stackrel{\rm def}{=} A(e,\alpha,1^{|x_e|},H_n(e,x_e))$, where $x_e = C_n(e)$. Thus, we obtain a (poly(n)-size) circuit C'_n such that

$$|\mathsf{Pr}[C'_n(e,E_e(x_e))=f_e(x_e)] - \mathsf{Pr}[C'_n(e,E_e(y_e))=f_e(x_e)]| \;>\; \frac{1}{\mathrm{poly}(n)}$$

where e is distributed according to $G_1(1^n)$. Using an idea as in the proof of Theorem 5.2.15, we derive a (poly(n)-size) circuit C''_n that distinguishes $(e,E_e(x_e))$ from $(e,E_e(y_e))$, where $e \leftarrow G_1(1^n)$, in contradiction to our hypothesis.

> Details: Recall that the idea was to proceed in two stages. First, using only e (which also yields x_e, y_e), we find an arbitrary value v such that $|\mathsf{Pr}[C'_n(e,E_e(x_e))=v] - \mathsf{Pr}[C'_n(e,E_e(y_e))=v]|$ is large. In the second stage, we use this value v in order to distinguish the case in which we are given an encryption of x_e from the case in which we are given an encryption of y_e. (We comment if $(e,x) \mapsto f_e(x)$ were computable by a poly(n)-size circuit then converting C'_n into a distinguisher C''_n would have been much easier; we further comment that as a corollary to the current proof, one can conclude that the restricted form is equivalent to the general one.)

This concludes the proof that indistinguishable encryptions (as per Definition 5.4.2) implies semantic security (as per Definition 5.4.1), and we now turn to the opposite direction.

Suppose that (G,E,D) does not have indistinguishable encryptions, and consider an admissible sequence $\{(x_e,y_e)\}_{e\in\{0,1\}^*}$ that witnesses this failure. Following the proof of Proposition 5.2.7, we define a probability ensemble $\{X_e\}_{e\in\{0,1\}^*}$ and function ensembles $\{h_e\}_{e\in\{0,1\}^*}$ and $\{f_e\}_{e\in\{0,1\}^*}$, in an analogous manner:

- The distribution X_e is uniformly distributed over $\{x_e,y_e\}$.
- The function f_e satisfies $f_e(x_e)=1$ and $f_e(y_e)=0$.
- The function h_e is defined such that $h_e(X_e)$ equals the description of the circuit C_n that distinguishes $(e,E_e(x_e))$ from $(e,E_e(y_e))$, where $e \leftarrow G_1(1^n)$ (and $(x_e,y_e)=P_n(e)$).

Using the admissibility of the sequence $\{(x_e,y_e)\}_e$ (for Definition 5.4.2) it follows that $\{(f_e,h_e,X_e)\}_e$ is admissible for Definition 5.4.1. Using the same algorithm A as in the proof of Proposition 5.2.7 (i.e., $A(e,\beta,C_n)=C_n(e,\beta)$, where β is a ciphertext and $C_n = h_e(X_e)$), and using the same analysis, we derive a contradiction to the hypothesis that (G,E,D) satisfies Definition 5.4.1.

> Details: Without loss of generality, suppose that
>
> $$\mathsf{Pr}\,[C_n(e,E_e(x_e))=1] > \mathsf{Pr}\,[C_n(e,E_e(y_e))=1] + \frac{1}{p(n)}$$
>
> for $e \leftarrow G_1(1^n)$. Then,
>
> $$\mathsf{Pr}\,[A(e,E_e(X_e),h_e(X_e))=f_e(X_e)] > \frac{1}{2} + \frac{1}{2p(n)}$$

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On the other hand, for every algorithm A'

$$\Pr\left[A'(e, 1^{|X_e|}, h_e(X_e)) = f_e(X_e)\right] \leq \frac{1}{2}$$

because $(e, 1^{|X_e|}, h_e(X_e))$ contains no information about the value of $f_e(X_e)$ (which is uniformly distributed in $\{0,1\}$). This violates Definition 5.4.1, and so our initial contradiction hypothesis (i.e., that one can distinguish encryptions under (G, E, D)) must be false.

The theorem follows. ■

Multiple-message security: Definitions 5.4.1 and 5.4.2 can be easily generalized to handle the encryption of many messages (as in Section 5.2.4), yielding again two equivalent definitions. Since we are in the public-key setting, one can show (analogously to Theorem 5.2.11) that the single-message definitions of security are equivalent to the multiple-message ones (i.e., by showing that Definition 5.4.2 implies its multiple-message generalization). One important observation is that admissibility for the multiple-message definition enables one to carry out a hybrid argument (as in the proof of Theorem 5.2.11). For details see Exercise 27. The bottom-line is that *we can freely use any of the four definitions, and security for that definition implies security for any of the other definitions.*

5.4.2.2 Constructions

All the results presented in Section 5.3.4 extend to security under key-dependent passive attacks. That is, for each of the constructions presented in Section 5.3.4, the same assumption used to prove security under key-oblivious passive attacks actually suffices for proving security under key-dependent passive attacks. Before demonstrating this fact, we comment that (in general) security under key-oblivious passive attacks *does not* necessarily imply security under key-dependent passive attacks; see Exercise 28.

Initial observations: We start by observing that Construction 5.3.7 (from block-ciphers to general encryption schemes) maintains its security in our context. That is:

Proposition 5.4.4 (extension of Proposition 5.3.8): *Let (G, E, D) and (G', E', D') be as in Construction 5.3.7; i.e., (G', E', D') be the full-fledged encryption constructed based on the block-cipher (G, E, D). Then if (G, E, D) is secure under key-dependent passive attacks, then so is (G', E', D').*

Proof Idea: As in the proof of Proposition 5.3.8, we merely observe that multiple-message security of (G', E', D') is equivalent to multiple-message security of (G, E, D). ■

We next observe that Construction 5.3.13 (a block-cipher with block length $\ell \equiv 1$) maintains its security also under a key-dependent passive attack. This is a special case of the following observation:

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Proposition 5.4.5 *Let (G,E,D) be a block-cipher with logarithmically bounded block-length (i.e., $\ell(n)=O(\log n)$). If (G,E,D) is secure under key-oblivious passive attacks then it is also secure under key-dependent passive attacks.*

Proof Sketch: Here we use the definition of ciphertext-indistinguishability in the single-message setting. The key observation is that the set of possible messages is relatively small, and so selecting a message in a key-dependent manner does not give much advantage over selecting a message at random (i.e., obliviously of the key).

Consider an arbitrary admissible set of pairs, $\{(x_e,y_e)\}_{e\in\{0,1\}^*}$, where $|x_e|=|y_e|=O(\log|e|)$, and a circuit family $\{C_n\}$ that tries to distinguish $(e,E_e(x_e))$ from $(e,E_e(y_e))$. We shall show that $\{C_n\}$ necessarily fails.

Let $\{P_n\}_{n\in\mathbb{N}}$ be the circuit family producing the abovementioned admissible set (i.e., $P_n(e)=(x_e,y_e)$). Fixing some $n\in\mathbb{N}$ and an arbitrary $(x,y)\in\{0,1\}^*\times\{0,1\}^*$, we consider a circuit $C_n^{x,y}$ (depending on the circuits C_n and P_n and the pair (x,y)) that, on input (e,α), operates as follows:

1. Using the hard-wired circuit P_n and the input (key) e, the circuit $C_n^{x,y}$ checks whether (x_e,y_e) equals the hard-wired pair (x,y) (i.e., $C_n^{x,y}$ checks whether $P_n(e)=(x,y)$). In case the check fails, $C_n^{x,y}$ outputs an arbitrary value (e.g., 1) obliviously of the ciphertext α.

2. Otherwise (i.e., $P_n(e)=(x,y)$), the circuit $C_n^{x,y}$ invokes C_n on its own input and answers accordingly (i.e., outputs $C_n(e,\alpha)$).

Since (G,E,D) is secure under key-oblivious passive attacks it follows that (for every $(x,y)\in\{0,1\}^m\times\{0,1\}^m$, where $m\leq\text{poly}(n)$) the circuit $C_n^{x,y}$ cannot distinguish the case $\alpha=E_e(x)$ from the case $\alpha=E_e(y)$. Thus, for some negligible function $\mu:\mathbb{N}\to[0,1]$ and every pair $(x,y)\in\{0,1\}^m\times\{0,1\}^m$, the following holds

$$\begin{aligned}\mu(n) &> \left|\mathsf{Pr}_e[C_n^{x,y}((e,E_e(x))=1]-\mathsf{Pr}_e[C_n^{x,y}((e,E_e(y))=1]\right|\\ &= \left|\mathsf{Pr}_e\left[\begin{array}{c}C_n((e,E_e(x_e))=1\\ \wedge\ (x_e,y_e)=(x,y)\end{array}\right]-\mathsf{Pr}_e\left[\begin{array}{c}C_n((e,E_e(y_e))=1\\ \wedge\ (x_e,y_e)=(x,y)\end{array}\right]\right|\end{aligned}$$

where $e\leftarrow G_1(1^n)$. Since the above holds for any pair $(x,y)\in\{0,1\}^m\times\{0,1\}^m$, and since $|x_e|=|y_e|=\ell(n)$ it follows that

$$\begin{aligned}&\left|\mathsf{Pr}_e[C_n((e,E_e(x_e))=1]-\mathsf{Pr}_e[C_n((e,E_e(y_e))=1]\right|\\ &\leq \sum_{|x|=|y|=\ell(n)}\left|\mathsf{Pr}_e\left[\begin{array}{c}C_n((e,E_e(x_e))=1\\ \wedge\ (x_e,y_e)=(x,y)\end{array}\right]-\mathsf{Pr}_e\left[\begin{array}{c}C_n((e,E_e(y_e))=1\\ \wedge\ (x_e,y_e)=(x,y)\end{array}\right]\right|\\ &< 2^{2\ell(n)}\cdot\mu(n)\end{aligned}$$

and the proposition follows. ■

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A feasibility result: Combining Theorem 5.3.15 with Propositions 5.4.4 and 5.4.5, we obtain a feasibility result:

Theorem 5.4.6 *If there exist collections of (non-uniformly hard) trapdoor permutations then there exist public-key encryption schemes that are secure under key-dependent passive attacks.*

More efficient schemes: In order to obtain more efficient schemes, we directly analyze the efficient constructions presented in Section 5.3.4. For example, extending the proof of Proposition 5.3.19, we obtain:

Proposition 5.4.7 *Suppose that b is a (non-uniformly strong) hard-core of the trapdoor collection $\{p_\alpha\}$. Furthermore, suppose that this trapdoor collection utilizes a domain sampling algorithm S so that the statistical difference between $S(\alpha)$ and the uniform distribution over the domain of p_α is negligible in terms of $|\alpha|$. Then Construction 5.3.18 constitute a public-key encryption scheme that is secure under key-dependent passive attacks.*

Proof Sketch: Again, we prove single-message ciphertext-indistinguishability. We rely heavily on the admissibility condition. In analogy to the proof of Proposition 5.3.19, it suffices to show that, for every polynomial-size circuit family $\{C_n\}$, the distributions $(\alpha, p_\alpha^\ell(S(\alpha)), C_n(\alpha) \oplus G_\alpha^{(\ell)}(S(\alpha)))$ and $(\alpha, p_\alpha^\ell(S(\alpha)), C_n(\alpha) \oplus s)$ are indistinguishable, for a randomly generated α and $\ell = |C_n(\alpha)|$, where $s \in \{0,1\}^\ell$ is uniformly distributed (independently of anything else).[16] Incorporating $\{C_n\}$ in the potential distinguisher, it suffices to show that the distributions $(\alpha, p_\alpha^\ell(S(\alpha)), G_\alpha^{(\ell)}(S(\alpha)))$ and $(\alpha, p_\alpha^\ell(S(\alpha)), s)$ are indistinguishable. The latter claim follows as in the proof of Proposition 5.3.19 (i.e., by a minor extension to Proposition 3.4.6). The proposition follows. ■

5.4.3 Chosen plaintext attack

So far, we have discussed only passive attacks (in two variants: key-oblivious versus key-dependent, discussed in Section 5.2 and 5.4.2, respectively). Turning to active attacks, we start with mild active attacks in which the adversary may obtain (from some legitimate user) ciphertexts corresponding to plaintexts of its choice. Such attacks will be called *chosen plaintext attack*, and are possible (as well as are all that is possible) in some applications. For example, in some settings the adversary may (directly or indirectly) control the encrypting module (but not the decrypting module).

Intuitively, a chosen plaintext attack poses additional threat in case of private-key encryption schemes (see Exercise 29), but not in the case of public-key encryption schemes. In fact, we will show that, in the case of public-key encryption schemes, a chosen plaintext attack can be emulated by a passive key-dependent attack.

[16] Recall that here α serves as an encryption-key and $C_n(\alpha)$ is a key-dependent plaintext. Typically, $C_n(\alpha)$ would be the first or second element in the plaintext pair $(x_\alpha, y_\alpha) = P_n(\alpha)$.

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5.4.3.1 Definitions

We start by rigorously formulating the framework of chosen plaintext attacks. Intuitively, such attacks proceeds in four stages corresponding to the generation of a key (by a legitimate party), the adversary's requests (answered by the legitimate party) to encrypt plaintexts under this key, the generation of a challenge ciphertext (under this key and according to a templet specified by the adversary), and additional requests to encrypt plaintexts (under the same key). That is, a chosen plaintext attack proceeds as follows:

1. *Key generation:* A key-pair $(e, d) \leftarrow G(1^n)$ is generated (by a legitimate party). In the *public-key setting* the adversary is given $(1^n, e)$, whereas in the *private-key setting* the adversary is only given 1^n.

2. *Encryption requests:* Based on the information obtained so far, the adversary may request (the legitimate party) to encrypt plaintexts of its (i.e., the adversary's) choice. A request to encrypt the plaintext x is answered with a value taken from the distribution $E_e(x)$, where e is as determined in Step 1. After making several such requests, the adversary moves to the next stage.

3. *Challenge generation:* Based on the information obtained so far, the adversary specifies a challenge templet and is given an actual challenge.

   When defining semantic security the challenge templet is a triplet of circuits (S_m, h_m, f_m), where S_m specifies a distribution of m-bit long plaintexts (and $h_m, f_m : \{0,1\}^m \to \{0,1\}^*$), and the actual challenge is a pair $(E_e(x), h_m(x))$ where x is distributed according to $S_m(U_{\text{poly}(n)})$. When defining indistinguishability of encryptions the challenge templet is merely a pair of equal-length strings, and the actual challenge is an encryption of one of these two strings.

4. *Additional encryption requests:* Based on the information obtained so far, the adversary may request to encrypt additional plaintexts of its choice. These requests are handled as in Step 2. After making several such requests, the adversary produces an output and halts.

In the actual definition, the adversary's strategy will be decoupled into two parts corresponding to its actions before and after the generation of the actual challenge. Each part will be represented by a (probabilistic polynomial-time) oracle machine, where the oracle is an "encryption oracle" (with respect to the key generated in Step 1). The first part, denoted A_1, captures the adversary's behavior during Step 2. It is given a security parameter (and possibly an encryption-key), and its output is a pair (τ, σ), where τ is the templet generated in the beginning of Step 3 and σ is a state information passed to the second part of the adversary. The second part of the adversary, denoted A_2, captures the adversary's behavior during Step 4. It is given the state σ (of the first part) as well as the actual challenge (generated Step 3), and produces the actual output of the adversary.

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In accordance to using non-uniform formulations, we let each of the two oracle machines have a (non-uniform) auxiliary input. In fact, it suffices to provide only the first machine with such a (non-uniform) auxiliary input, because it can pass auxiliary input to the second machine in the state information σ. (Similarly, in the case of public-key schemes, it suffices to provide only the first machine with the encryption-key.) We comment that we provide these machines with probabilistic oracles; that is, in response to a plaintext query x, the oracle E_e returns a random ciphertext $E_e(x)$ (i.e., the result of a probabilistic process applied to e and x). Thus, in the case of public-key schemes, the four-step attack process can be written as follows:

$$
\begin{array}{rcl}
(e,d) & \leftarrow & G(1^n) \\
(\tau,\sigma) & \leftarrow & A_1^{E_e}(e,z) \\
c & \stackrel{\rm def}{=} & \text{an actual challenge generated according to the templet } \tau \\
\text{output} & \leftarrow & A_2^{E_e}(\sigma,c)
\end{array}
$$

where z denotes (non-uniform) auxiliary input given to the adversary. In case of private-key schemes, the adversary (i.e., A_1) is given 1^n instead of e.

Semantic security: Instantiating the above framework to derive a definition of semantic security amounts to specifying the challenge generation (as hinted above) and to postulating that the success probability in such an attack should be met by a corresponding benign process. Specifically, the challenge generation consists of the adversary specifying a triplet of circuits, denoted (S_m, h_m, f_m), and being presented with an encryption of $x \leftarrow S_m(U_{\text{poly}(n)})$ along with the partial information $h_m(x)$. The adversary's goal is to guess $f_m(x)$, and semantic security amount to saying that the adversary's success probability can be matched by a *corresponding algorithm* that is only given $h_m(x)$ and $1^{|x|} = 1^m$. Like the adversary, the corresponding algorithm is decoupled into two parts, the first is in charge of outputting a challenge templet, and the second is in charge of solving the challenge, where state information is passed from the first part to the second part. It is important to require that the challenge templet produced by the corresponding algorithm is distributed exactly as the challenge templet produced by the adversary. (See further discussion below.)

Definition 5.4.8 (semantic security under chosen plaintext attacks):

For public-key schemes: *A public-key encryption scheme, (G, E, D), is said to be* semantically secure under chosen plaintext attacks *if for every pair of probabilistic polynomial-time oracle machines, A_1 and A_2, there exists a pair of probabilistic polynomial-time algorithms, A'_1 and A'_2, such that the following two conditions hold:*

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1. *For every positive polynomial $p(\cdot)$, and all sufficiently large n and $z \in \{0,1\}^{\mathrm{poly}(n)}$:*

$$\Pr\left[\begin{array}{ll} v = f_m(x) & \text{where} \\ & (e,d) \leftarrow G(1^n) \\ & ((S_m,h_m,f_m),\sigma) \leftarrow A_1^{E_e}(e,z) \\ & c \leftarrow (E_e(x),h_m(x))\,,\ \text{where } x \leftarrow S_m(U_{\mathrm{poly}(n)}) \\ & v \leftarrow A_2^{E_e}(\sigma,c) \end{array}\right]$$
$$< \ \Pr\left[\begin{array}{ll} v = f_m(x) & \text{where} \\ & ((S_m,h_m,f_m),\sigma) \leftarrow A'_1(1^n,z) \\ & x \leftarrow S_m(U_{\mathrm{poly}(n)}) \\ & v \leftarrow A'_2(\sigma,1^{|x|},h_m(x)) \end{array}\right] + \frac{1}{p(n)}$$

Recall that (S_m,h_m,f_m) is a triplet of circuits produced as in Step 3 of the foregoing description, and that x is a sample from the distribution induced by S_m.

2. *For every n and z, the first element* (i.e., the (S_m,h_m,f_m) part) *in the random variables $A'_1(1^n,z)$ and $A_1^{E_{G_1(1^n)}}(G_1(1^n),z)$ are identically distributed.*

For private-key schemes: *The definition is identical except that algorithm A_1 gets the security parameter 1^n instead of the encryption-key e.*

Note that as in almost all other definitions of semantic security (with the exception of Definition 5.4.1), algorithm A'_1 does not get a (random) encryption-key as input (but may rather generate one by itself).[17] Since the challenge templet is not fixed (or determined by e) but rather chosen by A and A' themselves, it is very important to require that in both cases the challenge templet is distributed identically (or approximately so): there is no point in relating the success probability of A and A', unless these probabilities refer to same distribution of problems (i.e., challenge templets). (The issue arises also in Definition 5.4.1 where it was resolved by forcing A' to refer to the challenge templet determined by the public-key e.)[18]

Definition 5.4.8 implies Definition 5.4.1, but this may not be evident from the definitions themselves (most importantly, because here f_m is computationally bounded whereas in Definition 5.4.1 the function is computationally unbounded). Still the validity of the claim follows from the equivalence of the two definitions to the corresponding notions of indistinguishability of encryptions (and the fact that the implication is evident for the latter formulations).

[17] In fact, A'_1 is likely to start by generating $e \leftarrow G_1(1^n)$, because it has to generate a challenge templet that is distributed as the one produced by A_1 on input $e \leftarrow G_1(1^n)$.

[18] Indeed, an alternative solution could have been the one adopted here and in the sequel; that is, allow A' to select the challenge templet by itself provided that the selection yields a distribution similar to the one faced by A, as induced by the public-key e.

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Indistinguishability of encryptions: Deriving the corresponding definition of indistinguishability of encryptions (from the above framework) is considerably simpler. Here the challenge generation consists of the adversary specifying two equal-length strings and the adversary is presented with the encryption of one of them. The adversary's goal is to distinguish the two possible cases.

Definition 5.4.9 (indistinguishability of encryptions under chosen plaintext attacks):

For public-key schemes: *A public-key encryption scheme, (G, E, D), is said to have* indistinguishable encryptions under chosen plaintext attacks *if for every pair of probabilistic polynomial-time oracle machines, A_1 and A_2, for every positive polynomial $p(\cdot)$, and all sufficiently large n and $z \in \{0,1\}^{\mathrm{poly}(n)}$:*

$$|p_{n,z}^{(1)} - p_{n,z}^{(2)}| < \frac{1}{p(n)}$$

where

$$p_{n,z}^{(i)} \stackrel{\text{def}}{=} \mathsf{Pr}\left[\begin{array}{ll} v=1 & \text{where} \\ & (e,d) \leftarrow G(1^n) \\ & ((x^{(1)},x^{(2)}),\sigma) \leftarrow A_1^{E_e}(e,z) \\ & c \leftarrow E_e(x^{(i)}) \\ & v \leftarrow A_2^{E_e}(\sigma,c) \end{array}\right]$$
$$\text{where } |x^{(1)}| = |x^{(2)}|.$$

For private-key schemes: *The definition is identical except that A_1 gets the security parameter 1^n instead of the encryption-key e.*

Clearly, Definition 5.4.9 implies Definition 5.4.2 as a special case. Furthermore, for public-key schemes, the two definitions are equivalent (see Proposition 5.4.10), whereas for private-key schemes Definition 5.4.9 is strictly stronger (see Exercise 29).

Proposition 5.4.10 *Let (G, E, D) be a public-key encryption scheme that has indistinguishable encryptions under key-dependent passive attacks. Then, (G, E, D) has indistinguishable encryptions under chosen plaintext attack.*

Proof Sketch: They key observation is that, in the public-key model, a chosen plaintext attack can be emulated by a passive key-dependent attack. Specifically, the (passive) attacker can emulate access to an encryption oracle by itself (by using the encryption-key given to it). Thus, we obtain an attacker as in Definition 5.4.9, with the important exception that it never makes oracle calls (but rather emulates E_e by itself). Put in other words, we have an attacker as in Definition 5.4.2, with the minor exception that it is a probabilistic polynomial-time machine with auxiliary input (rather than being a polynomial-size circuit) and that it distinguishes a pair of plaintext distributions rather than a pair of

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(fixed) plaintexts. However, fixing the best possible coins for this attacker (and incorporating them as well as z in an adequate circuit), we obtain an attacker exactly as in Definition 5.4.2 such that its distinguishing gap is at least as large as the one of the (initial) chosen plaintext attacker. ■

Equivalence of semantic security and ciphertext-indistinguishability. As in previous cases, we show that the two formulations of (chosen plaintext attack) security (i.e., semantic security and indistinguishable encryptions) are in fact equivalent.

Theorem 5.4.11 (equivalence of definitions for chosen plaintext attacks): *A public-key* (resp., private-key) *encryption scheme* (G, E, D) *is semantically secure under chosen plaintext attacks if and only if it has indistinguishable encryptions under chosen plaintext attacks.*

Proof Sketch: In order to show that indistinguishable encryptions implies semantic security, we follow again the ideas underlying the proof of Proposition 5.2.6. Specifically, for both the private-key and public-key cases, A'_1 and A'_2 are constructed as follows:

1. $A'_1(1^n, z) \stackrel{\text{def}}{=} (\tau, \sigma')$, where (τ, σ') is generated as follows:

   First, A'_1 generates an instance of the encryption scheme; that is, A'_1 lets $(e, d) \leftarrow G(1^n)$. Next, A'_1 invokes A_1, while emulating the oracle E_e, and sets $(\tau, \sigma) \leftarrow A_1^{E_e}(1^n, z)$. Finally, A'_1 sets $\sigma' \stackrel{\text{def}}{=} (e, \sigma)$.

   We warn that the generation of the key-pair by A'_1 should not be confused with the generation of the key-pair in the probabilistic evaluation of the combined algorithm $A = (A_1, A_2)$. In particular, the generated encryption-key e, allows A'_1 to emulate the encryption oracle E_e (also in the private-key case). Furthermore, A'_1 outputs the encryption-key e as part of the state passed by it to A'_2, whereas A_1 does not necessarily do so (and, in fact, cannot do so in case of private-key model). This will allow A'_2 too to emulate the encryption oracle E_e.

2. $A'_2((e, \sigma), 1^m, \gamma) \stackrel{\text{def}}{=} A_2^{E_e}(\sigma, (E_e(1^m), \gamma))$, where typically $\gamma = h_m(x)$ and $m = |x|$.

Since A'_1 merely emulates the generation of a key-pair and the actions of A_1 with respect to such a pair, the equal distribution condition (i.e., Item 2 in Definition 5.4.8) holds. Using the (corresponding) indistinguishability of encryption hypothesis, we show that (even in the presence of an encryption oracle E_e) the distributions $(\sigma, (E_e(x), h(x)))$ and $(\sigma, (E_e(1^{|x|}), h(x)))$ are indistinguishable, where $(e, d) \leftarrow G(1^n)$, $((S, h, f), \sigma) \leftarrow A_1^{E_e}(y, z)$ (with $y = e$ or $y = 1^n$ depending on the model), and $x \leftarrow S(U_{\text{poly}(n)})$.

> Details: Suppose that given $((S, h, f), \sigma)$ generated by $A_1^{E_e}(y, z)$ and oracle access to E_e, where $e \leftarrow G_1(1^n)$ (and y is as above), one can distinguish

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> $(\sigma, (E_e(x), h(x)))$ and $(\sigma, (E_e(1^{|x|}), h(x)))$, where $x \leftarrow S(U_{\text{poly}(n)})$. Then we obtain a distinguisher as in Definition 5.4.9 as follows. The first part of the distinguisher invokes A_1 (while answering its oracle queries by forwarding these queries to its own E_e oracle), and obtains $((S, h, f), \sigma) \leftarrow A_1^{E_e}(y, z)$. It sets $x^{(1)} \leftarrow S(U_{\text{poly}(n)})$ and $x^{(2)} = 1^{|x^{(1)}|}$. and outputs $((x^{(1)}, x^{(2)}), (\sigma, h(x^{(1)})))$. That is, $(x^{(1)}, x^{(2)})$ is the challenge templet, and it is answered with $E_e(x^{(i)})$, where i is either 1 or 2. The second part of the new distinguisher, gets as input a challenge ciphertext $\alpha \leftarrow E_e(x^{(i)})$ and the state generated by the first part $(\sigma, L(x^{(1)}))$, and invokes the distinguisher of the contradiction hypothesis with input $(\sigma, (\alpha, h(x^{(1)})))$, while answering its oracle queries by forwarding these queries to its own E_e oracle. Thus, the new distinguisher violates the condition in Definition 5.4.9, in contradiction to the hypothesis that (G, E, D) has indistinguishable encryptions.

It follows that indistinguishable encryptions (as per Definition 5.4.9) implies semantic security (as per Definition 5.4.8). (Here, this implication is easier to prove than in previous cases, because the function f is computable via a circuit that is generated as part of the challenge templet (and, w.l.o.g., is part of σ.)

We now turn to the opposite direction. Suppose that (G, E, D) does not have indistinguishable encryptions, and consider the pairs $(x^{(1)}, x^{(2)})$ produced as a challenge templet by the distinguishing adversary. Following the ideas of the proof of Proposition 5.2.7, we let the semantic-security adversary generate a corresponding challenge templet (S, h, f) such that

- The circuit S samples uniformly in $\{x^{(1)}, x^{(2)}\}$.
- The function f satisfies $f(x^{(1)}) = 1$ and $f(x^{(2)}) = 0$.
- The function h is defined arbitrarily subject to $h(x^{(1)}) = h(x^{(2)})$.

We stress that when the semantic-security adversary invokes the distinguishing adversary, the former uses its own oracle to answer the queries made by the latter. The reader may easily verify that the semantic-security adversary has a noticeable advantage in guessing $f(S(U_{\text{poly}(n)}))$ (by using the distinguishing gap between $E_e(x^{(1)})$ and $E_e(x^{(2)})$), whereas no algorithm that only gets $h(S(U_{\text{poly}(n)}))$ can have any advantage in such a guess. We derive a contradiction to the hypothesis that (G, E, D) satisfies Definition 5.4.8, and the theorem follows. ■

Multiple-message security: Definitions 5.4.8 and 5.4.9 can be easily generalized to handle challenges in which multiple plaintexts are encrypted. As in previous cases, the corresponding (multiple-plaintext) definitions are equivalent. Furthermore, the multiple-plaintext definitions are equivalent to the single-plaintext definition, both for public-key and private-key schemes. We stress the equivalence for private-key schemes (which does not hold for the basic definitions presented in Section 5.1; see Proposition 5.2.12). To see the equivalence it is best to consider the notion of indistinguishable encryptions. In this case,

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the argument used in the proof of Theorem 5.2.11 can be applied here by using an encryption oracle (rather than by generating encryptions using knowledge of the encryption-key, which is only possible in the public-key setting).

5.4.3.2 Constructions

In view of Proposition 5.4.10 (and Theorem 5.4.11), we focus on private-key encryption schemes (because a public-key encryption scheme is secure under chosen plaintext attacks if and only if it is secure under passive key-dependent attacks). All the results presented in Section 5.3.3 extend to security under chosen plaintext attacks. Specifically, we prove that Constructions 5.3.9 and 5.3.12 remain secure also under a chosen plaintext attack.

Proposition 5.4.12 *Let F and (G,E,D) be as in Construction 5.3.9, and suppose that F is pseudorandom with respect to polynomial-size circuits. Then the private-key encryption scheme (G,E,D) is secure under chosen plaintext attacks. The same holds with respect to Construction 5.3.12.*

Proof Sketch: We focus on Construction 5.3.9, and follow the technique underlying the proof of Proposition 5.3.10. That is, we consider an idealized version of the scheme, in which one uses a uniformly selected function $f:\{0,1\}^n \to \{0,1\}^n$, rather than the pseudorandom function f_s. Essentially, all that the adversary obtains by encryption queries in the ideal version is pairs $(r, f(r))$, where the r's are uniformly and independently distributed in $\{0,1\}^n$. As to the challenge itself, the plaintext is "masked" by the value of f at another uniformly and independently distributed element in $\{0,1\}^n$. Thus, unless the latter element happens to equal one of the r's used by the encryption oracle (which happens with negligible probability), the challenge plaintext is perfectly masked. Thus, the ideal version is secure under a chosen plaintext attack, and the same holds for the real scheme (since otherwise one derives a contradiction to the hypothesis that F is pseudorandom). ■

Summary: Private-key and public-key encryption schemes that are secure under chosen plaintext attacks exist if and only if corresponding schemes that are secure under passive (key-dependent) attacks exist.

5.4.4 Chosen ciphertext attack

We now turn to stronger forms of active attacks in which the adversary may obtain (from some legitimate user) plaintexts corresponding to ciphertexts of its choice. We consider two types of such attacks, called *chosen ciphertext attacks*: In the milder type, called *a-priori chosen ciphertext attacks*, such decryption requests can be made only before the challenge ciphertext (for which the adversary should gain knowledge) is presented. In the stronger type, called *a-posteriori chosen ciphertext attacks*, such decryption requests can be made also after the

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challenge ciphertext is presented, as long as one does not request to decrypt this very (challenge) ciphertext.

Both types of attacks address security threats in realistic applications: In some settings the adversary may experiment with the decryption module, before the actual ciphertext in which it is interested is sent. Such a setting corresponds to an a-priori chosen ciphertext attack. In other settings, one may invoke the decryption module on inputs of one's choice at any time but all these invocations are recorded, and real damage is caused only by knowledge gained with respect to a ciphertext for which a decryption request was not recorded. In such a setting protection against a-posteriori chosen ciphertext attacks is adequate. Furthermore, in both cases, decryption requests can be made also with respect to strings that are not valid ciphertexts, in which case the decryption module returns a special error symbol.

Typically, in settings in which a mild or strong form of a chosen *ciphertext* attack is possible, a chosen *plaintext* attack is possible too. Thus, we actually consider combined attacks in which the adversary may ask for encryption and decryption of strings of its choice. Indeed (analogously to Proposition 5.4.10), in case of public-key schemes (but not in case of private-key schemes) the combined attack is equivalent to a "pure" chosen *ciphertext* attack.

Organization: We start by providing security definitions for the two types of attacks discussed above. In Section 5.4.4.2, we further extend the definitional treatment of security (and derive a seemingly stronger notion that is in fact equivalent to the notions in Section 5.4.4.1). In Section 5.4.4.3 (resp., Section 5.4.4.4) we discuss the construction of private-key (resp., public-key) encryption schemes that are secure under chosen ciphertext attacks.

5.4.4.1 Definitions for two types of attacks

Following Section 5.4.3.1 and bearing in mind that we wish to define two types of (i.e., a-priori and a-posteriori chosen ciphertext) attacks, we first formulate the framework of chosen ciphertext attacks. As in the case of chosen plaintext attacks, we consider attacks that proceeds in four stages corresponding to the generation of a pair of keys (by a legitimate party), the adversary's requests (answered by the legitimate party) to encrypt and/or decrypt strings under the corresponding key, the generation of a challenge ciphertext (under this key and according to a templet specified by the adversary), and additional requests to encrypt and/or decrypt strings. That is, a chosen ciphertext attack proceeds as follows:

1. *Key generation:* A key-pair $(e, d) \leftarrow G(1^n)$ is generated (by a legitimate party). In the *public-key setting* the adversary is given $(1^n, e)$, whereas in the *private-key setting* the adversary is only given 1^n.

2. *Encryption and decryption requests:* Based on the information obtained so far, the adversary may request (the legitimate party) to encrypt and/or

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decrypt strings of its (i.e., the adversary's) choice. A request to encrypt the plaintext x is answered with a value taken from the distribution $E_e(x)$, where e is as determined in Step 1. A request to decrypt a valid (w.r.t. E_e) ciphertext y is answered with the value $D_d(y)$, where d is as determined in Step 1. A request to decrypt a string y that is not a valid ciphertext (w.r.t. E_e) is answered with a special error symbol. After making several such requests, the adversary moves to the next stage.

3. *Challenge generation:* Based on the information obtained so far, the adversary specifies a challenge templet and is given an actual challenge. This is done as in the corresponding step in the framework of chosen plaintext attacks.

4. *Additional encryption and decryption requests:* Based on the information obtained so far, the adversary may request to encrypt additional plaintexts of its choice. In addition, in case of an a-posteriori chosen ciphertext attack (but not in the case of a-priori chosen ciphertext attack), the adversary may make additional decryption requests with the only (natural) restriction that it is not allowed to ask to decrypt the challenge ciphertext. All requests are handled as in Step 2. After making several such requests, the adversary produces an output and halts.

In the actual definition, as in the case of chosen plaintext attacks, the adversary's strategy will be decoupled into two parts corresponding to its actions before and after the generation of the actual challenge. Each part will be represented by a (probabilistic polynomial-time) two-oracle machine, where the first oracle is an "encryption oracle" and the second is a "decryption oracle" (both with respect to the corresponding key generated in Step 1). As in the case of chosen plaintext attacks, the two parts are denoted A_1 and A_2, and A_1 passes a state information (denoted σ) to A_2. Again, in accordance to using non-uniform formulations, we provide A_1 with a (non-uniform) auxiliary input. Thus, in the case of (a-posteriori chosen ciphertext attacks on) public-key schemes, the four-step attack process can be written as follows:

$$\begin{array}{rcl} (e,d) & \leftarrow & G(1^n) \\ (\tau,\sigma) & \leftarrow & A_1^{E_e,D_d}(e,z) \\ c & \stackrel{\rm def}{=} & \text{an actual challenge generated according to the templet } \tau \\ \text{output} & \leftarrow & A_2^{E_e,D_d}(\sigma,c) \end{array}$$

where A_2 is not allowed to make a query regarding the ciphertext in c, and z denotes the (non-uniform) auxiliary input given to the adversary. In case of private-key schemes, the adversary (i.e., A_1) is given 1^n instead of e. In case of a-priori chosen ciphertext attacks, A_2 is not allowed to query D_d (or, equivalently, A_2 is only given oracle access to the oracle E_e).

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Semantic security: As in the case of chosen plaintext attacks, a definition of semantic security is derived by an adequate specification of the challenge generation and the meaning of success. As before, the challenge generation consists of the adversary specifying a triplet of circuits, denoted (S, h, f), and being presented with an encryption of $x \leftarrow S(U_{\text{poly}(n)})$ along with the partial information $h(x)$. The adversary's goal is to guess $f(x)$, and semantic security amount to saying that the adversary's success probability can be matched by a *corresponding algorithm* that is only given $h(x)$ and $1^{|x|}$. Again, the corresponding algorithm is decoupled into two parts, the first is in charge of outputting a challenge templet, and the second is in charge of solving the challenge, where state information is passed from the first part to the second part. Furthermore, again, it is important to require that the challenge templet produced by the corresponding algorithm is distributed exactly as the challenge templet produced by the adversary.

Definition 5.4.13 (semantic security under chosen ciphertext attacks):

For public-key schemes: *A public-key encryption scheme, (G, E, D), is said to be* semantically secure under a-priori chosen ciphertext attacks *if for every pair of probabilistic polynomial-time oracle machines, A_1 and A_2, there exists a pair of probabilistic polynomial-time algorithms, A'_1 and A'_2, such that the following two conditions hold:*

1. *For every positive polynomial $p(\cdot)$, and all sufficiently large n and $z \in \{0,1\}^{\text{poly}(n)}$:*

$$\Pr\left[\begin{array}{ll} v = f(x) & \text{where} \\ & (e,d) \leftarrow G(1^n) \\ & ((S,h,f),\sigma) \leftarrow A_1^{E_e,D_d}(e,z) \\ & c \leftarrow (E_e(x), h(x))\,,\ \text{where } x \leftarrow S(U_{\text{poly}(n)}) \\ & v \leftarrow A_2^{E_e}(\sigma, c) \end{array}\right]$$
$$< \Pr\left[\begin{array}{ll} v = f(x) & \text{where} \\ & ((S,h,f),\sigma) \leftarrow A'_1(1^n, z) \\ & x \leftarrow S(U_{\text{poly}(n)}) \\ & v \leftarrow A'_2(\sigma, 1^{|x|}, h(x)) \end{array}\right] + \frac{1}{p(n)}$$

2. *For every n and z, the first element* (i.e., the (S,h,f) part) *in the random variables $A'_1(1^n, z)$ and $A_1^{E_{G_1(1^n)},D_{G_2(1^n)}}(G_1(1^n), z)$ are identically distributed.*

Semantic security under a-posteriori chosen ciphertext attacks *is defined analogously, except that A_2 is given oracle access to both E_e and D_d with the restriction that when given the challenge $c = (c', c'')$, machine A_2 is not allowed to make the query c' to the oracle D_d.*

For private-key schemes: *The definition is identical except that algorithm A_1 gets the security parameter 1^n instead of the encryption-key e.*

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Clearly, the a-posteriori version of Definition 5.4.13 implies its a-priori version, which in turn implies Definition 5.4.8. It is easy to see that these implications are strict (see Exercises 31 and 30, respectively).

Indistinguishability of encryptions: As in the case of chosen plaintext attacks, deriving the corresponding definition of indistinguishability of encryptions (from the above framework) is considerably simpler: the challenge generation consists of the adversary specifying two equal-length strings and the adversary is presented with the encryption of one of them.

Definition 5.4.14 (indistinguishability of encryptions under chosen ciphertext attacks):

For public-key schemes: *A public-key encryption scheme, (G, E, D), is said to have* indistinguishable encryptions under a-priori chosen ciphertext attacks *if for every pair of probabilistic polynomial-time oracle machines, A_1 and A_2, for every positive polynomial $p(\cdot)$, and all sufficiently large n and $z \in \{0,1\}^{\mathrm{poly}(n)}$:*

$$|p_{n,z}^{(1)} - p_{n,z}^{(2)}| \;<\; \frac{1}{p(n)}$$

where

$$p_{n,z}^{(i)} \stackrel{\mathrm{def}}{=} \mathsf{Pr}\left[\begin{array}{ll} v=1 & \text{where} \\ & (e,d) \leftarrow G(1^n) \\ & ((x^{(1)},x^{(2)}),\sigma) \leftarrow A_1^{E_e,D_d}(e,z) \\ & c \leftarrow E_e(x^{(i)}) \\ & v \leftarrow A_2^{E_e}(\sigma,c) \end{array}\right]$$

$$\text{where } |x^{(1)}| = |x^{(2)}|.$$

Indistinguishability of encryptions under a-posteriori chosen ciphertext attacks *is defined analogously, except that A_2 is given oracle access to both E_e and D_d with the restriction that when given the challenge c, machine A_2 is not allowed to make the query c to the oracle D_d.*

For private-key schemes: *The definition is identical except that A_1 gets the security parameter 1^n instead of the encryption-key e.*

Clearly, the a-posteriori version of Definition 5.4.14 implies its a-priori version, which in turn implies Definition 5.4.9 as a special case. Again, it is easy to see that these implications are strict (see Exercises 31 and 30, respectively).

Terminology: We use CCA as a shorthand for chosen ciphertext attack.

Equivalence of semantic security and ciphertext-indistinguishability. Again, we show that the two formulations of security (i.e., semantic security and indistinguishable encryptions) are in fact equivalent.

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Theorem 5.4.15 (equivalence of definitions for CCA): *A public-key* (resp., private-key) *encryption scheme* (G, E, D) *is semantically secure under a-priori CCA if and only if it has indistinguishable encryptions under a-priori CCA. An analogous claim holds for a-posteriori CCA.*

Proof Sketch: We adapt the proof of Theorem 5.4.11 to the current setting. The adaptation is straightforward, and we focus on the case of a-posteriori CCA security (while commenting on the case of a-priori CCA security).

In order to show that indistinguishable encryptions implies semantic security, given and adversary (A_1, A_2) we construct the following matching algorithm A'_1, A'_2:

1. $A'_1(1^n, z) \stackrel{\rm def}{=} (\tau, \sigma')$, where (τ, σ') is generated as follows:

   First, A'_1 generates an instance of the encryption scheme; that is, A'_1 lets $(e, d) \leftarrow G(1^n)$. Next, A'_1 invokes A_1, while emulating the oracles E_e and D_d, and sets $(\tau, \sigma) \leftarrow A_1^{E_e, D_d}(1^n, z)$. Finally, A'_1 sets $\sigma' \stackrel{\rm def}{=} ((e, d), \sigma)$. (In case of a-priori CCA security, we may set $\sigma' \stackrel{\rm def}{=} (e, \sigma)$, as in the proof of Theorem 5.4.11.)

   We comment that the generated key-pair (e, d), allows A'_1 to emulate the encryption and decryption oracles E_e and D_d.

2. $A'_2(((e, d), \sigma), 1^m, \gamma) \stackrel{\rm def}{=} A_2^{E_e, D_d}(\sigma, (E_e(1^m), \gamma))$, where typically $\gamma = h(x)$ and $m = |x|$. (In case of a-priori CCA security, we may set $A'_2((e, \sigma), 1^m, \gamma) \stackrel{\rm def}{=} A_2^{E_e}(\sigma, (E_e(1^m), \gamma))$, as in the proof of Theorem 5.4.11.)

Again, since A'_1 merely emulates the generation of a key-pair and the actions of A_1 with respect to such a pair, the equal distribution condition (i.e., Item 2 in Definition 5.4.13) holds. Using the (corresponding) indistinguishability of encryption hypothesis, we show that (even in the presence of the encryption oracle E_e and a restricted decryption oracle D_d) the distributions $(\sigma, (E_e(x), h(x)))$ and $(\sigma, (E_e(1^{|x|}), h(x)))$ are indistinguishable, where $(e, d) \leftarrow G(1^n)$, $((S, h, f), \sigma) \leftarrow A_1^{E_e}(y, z)$ (with $y = e$ or $y = 1^n$ depending on the model), and $x \leftarrow S(U_{{\rm poly}(n)})$. The main thing to notice is that the oracle queries made by a possible distinguisher of the above distributions can be handled by a distinguisher of encryptions (as in Definition 5.4.14), by passing these queries to its own oracles. It follows that indistinguishable encryptions (as per Definition 5.4.14) implies semantic security (as per Definition 5.4.13).

We now turn to the opposite direction. Here the construction of a challenge templet (as per Definition 5.4.13) is exactly as the corresponding construction in the proof of Theorem 5.4.11. Again, the thing to notice is that the oracle queries made by a possible distinguisher of encryptions (as in Definition 5.4.14) can be handled by the semantic-security adversary, by passing these queries to its own oracles. We derive a contradiction to the hypothesis that (G, E, D) satisfies Definition 5.4.13, and the theorem follows. ■

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Multiple-message security: Definitions 5.4.13 and 5.4.14 can be easily generalized to handle challenges in which multiple plaintexts are encrypted. We stress that in case of a-posteriori CCA the adversary is not allowed to make a decryption query that equals *any* of the challenge ciphertexts. As in previous cases, the corresponding (multiple-plaintext) definitions are equivalent. Furthermore, as in case of chosen plaintext attacks, the multiple-plaintext definitions are equivalent to the single-plaintext definition (both for public-key and private-key schemes). We stress that the above notion of multiple-message CCA security refers to a *single* challenge-generation step in which a sequence of messages (rather than a single message) can be specified. A more general notion of multiple-message CCA security allows multiple challenge-generation steps that may be interleaved with the query steps. This notion generalizes the notion of chosen ciphertext attacks, and is discussed in the next subsection. Actually, we will focus on this generalization when applied to a-posteriori chosen ciphertext attacks, although a similar generalization can be applied to a-priori chosen ciphertext attacks (and in fact also to chosen plaintext attacks).

5.4.4.2 A third equivalent definition of a-posteriori CCA-security

In continuation to the last paragraph, we consider general attacks during which several challenge templets may be produced (at arbitrary times and possibly interleaved with encryption and decryption queries).[19] Each of these challenge templets will be answered similarly to the way such templets were answered above (i.e., by selecting a plaintext from the specified distribution and providing its encryption together with the specified partial information). Unlike in Section 5.4.4.1, we will even allow attacks that make decryption queries regarding ciphertexts obtained as (part of the) answer to previous challenge templets. After such an attack, the adversary will try to obtain information about the unrevealed plaintexts, and security holds if its success probability can be met by a corresponding benign adversary that does not see the ciphertexts. Indeed, the above discussion requires clarification and careful formulation, provided next.

We start with a description of *the actual attacks*. It will be convenient to change the formalism and consider the generation of challenge templets as challenge queries that are answered by a special oracle called the tester, and denoted $T_{e,r}$, where e is an encryption-key and r is a random string of adequate length. On query a *challenge templet* of the form (S, h), where S is a sampling circuit and h is a function (evaluation circuit), the (randomized) oracle $T_{e,r}$ returns the pair $(E_e(x), h(x))$, where $x = S(r)$. (Indeed, we may assume without loss of generality that all queries (S, h) satisfy that S is a sampling circuit mapping $|r|$-bit long strings into string of the length that fits h's input.) We stress that r is not known to the adversary, and that this formalism supports the generation of dependent challenges as well as independent ones.[20] A multiple-challenge

[19] Note that in this section we generalize the notion of an a-posteriori chosen ciphertext attack. When generalizing the notion of an a-priori chosen ciphertext attack, we disallow decryption queries after the first challenge templet is produced.

[20] Independently distributed plaintexts can be obtained by sampling circuits that refer to

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CCA is allowed queries to $T_{e,r}$ as well as *unrestricted* queries to both E_e and the corresponding D_d, including decryption queries referring to previously obtained challenge ciphertexts. It terminates by outputting a function f and a value v, hoping that $f(x^1,...,x^t) = v$, where $x^i = S^i(r)$ and (S^i,h^i) is the i challenge query made by the adversary. Note that the description of f may encode various information gathered by the adversary during its attack (e.g., it may even encode its entire computation transcript).

We now turn to describe *the benign adversary* (which does not see the ciphertexts). Such an adversary is given oracle access to a corresponding oracle, T_r, that behave as follows. On query a challenge templet of the form (S,h), the oracle returns $h(x)$, where $x = S(r)$. (Again, r is not known to the adversary.) Like the real adversary, the benign adversary also terminates by outputting a function f and a value v, hoping that $f(x^1,...,x^t) = v$, where $x^i = S^i(r)$ and (S^i,h^i) is the i challenge query made by the adversary.

Security amounts to asserting the the effect of any efficient multiple-challenge CCA can be simulated by a efficient benign adversary that does not see the ciphertexts. As in Definition 5.4.13, the simulation has to satisfy two conditions: First, the probability that $f(x^1,...,x^t) = v$ in the CCA must be met by the probability that a corresponding event holds in the benign model (where the adversary does not see ciphertexts). Second, the challenge queries as well as the function f should be distributed similarly in the two models. Actually, each decryption query (of the real attacker) that refer to a ciphertext c that is contained in the answer given to a challenge query (S,h) is considered (or counted) as a (fictitious) challenge query $(S,\texttt{id})$, where id is the identity function. Note that this convention is justified by the fact that the challenge query $(S,\texttt{id})$ is equivalent to the decryption query c (followed by the encryption query $x = D_d(c)$). Put in other words, if the real adversary made a decryption query that refers to a ciphertext c contained in the answer given to the challenge (S,h) (and thus obtained $D_d(c) = D_d(E_e(S(r))) = S(r)$), then it is only fair that we allow the benign adversary (which sees no ciphertexts) to make the challenge query $(S,\texttt{id})$ and so obtain $\texttt{id}(S(r)) = S(r)$.

In order to obtain the actual definition, we need to define the trace of the execution of the above two types of adversaries. For a multiple-challenge CCA adversary, denoted A, the trace is defined as the sequence of challenge queries made during the attack, augmented by (fictitious) challenge queries such that the (fictitious challenge) query $(S,\texttt{id})$ is included if and only if the adversary made a decryption query c such that $(c,\cdot)$ is the answer given to a previous challenge query of the form $(S,\cdot)$. For the benign adversary, denoted B, the trace is defined as the sequence of challenge queries made during the attack.

Definition 5.4.16 (multiple-challenge CCA security):

For public-key schemes: *A public-key encryption scheme, (G,E,D), is said*

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disjoint parts of the random string r. On the other hand, making a pair of queries of the form $(S,\cdot)$ and $(C\circ S,\cdot)$, where C is a deterministic circuit, will yield a pair of plaintexts of the form $x \stackrel{\rm def}{=} S(r)$ and $C(x)$.

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to be secure under multiple-challenge chosen ciphertext attacks *if for every probabilistic polynomial-time oracle machine A there exists a probabilistic polynomial-time oracle machine B such that the following two conditions hold:*

1. *For every positive polynomial $p(\cdot)$, and all sufficiently large n and $z \in \{0,1\}^{\text{poly}(n)}$:*

$$\Pr\left[\begin{array}{ll} v = f(x^1,...,x^t) & \text{where} \\ & (e,d) \leftarrow G(1^n) \text{ and } r \leftarrow U_{\text{poly}(n)} \\ & (f,v) \leftarrow A^{E_e,D_d,T_{e,r}}(e,z) \\ & x^i \leftarrow S^i(r), \text{ for } i=1,...,t. \end{array}\right]$$
$$< \Pr\left[\begin{array}{ll} v = f(x^1,...,x^t) & \text{where} \\ & r \leftarrow U_{\text{poly}(n)} \\ & (f,v) \leftarrow B^{T_r}(1^n,z) \\ & x^i \leftarrow S^i(r), \text{ for } i=1,...,t. \end{array}\right] + \frac{1}{p(n)}$$

   where S^i is the first part of the i^{th} challenge query made by A (resp., *B*) *to $T_{e,r}$* (resp., to T_r).

2. *The following two probability ensembles, indexed by $n \in \mathbb{N}$ and $z \in \{0,1\}^{\text{poly}(n)}$, are computationally indistinguishable:*
   (a) *The trace of $A^{E_{G_1(1^n)},D_{G_2(1^n)},T_{G_1(1^n),U_{\text{poly}(n)}}}(G_1(1^n),z)$, augmented by its output.*
   (b) *The trace of $B^{T_{U_{\text{poly}(n)}}}(1^n,z)$ augmented by its output.*

For private-key schemes: *The definition is identical except that machine A gets the security parameter 1^n instead of the encryption-key e.*

To get more comfortable with Definition 5.4.16, consider the special case in which the real CCA adversary does not make decryption queries to ciphertexts obtained as part of answers to challenge queries. (In the proof of Theorem 5.4.17, such adversaries will be called canonical and will be showed to be as powerful as the general ones.) The trace of such adversaries equals the sequence of challenge queries made during the attack, which simplifies Condition 2. Furthermore, consider the special case in which such an adversary makes a single challenge query, and further restrict it to make only a query of the form $(S, 0)$, where 0 is the all-zero function (i.e., which yields no information). Still, even this very restricted case (of Definition 5.4.16) easily implies security under a-posteriori CCA (cf. Exercise 32). More importantly, the following holds:

Theorem 5.4.17 (a-posteriori-CCA implies Definition 5.4.16): *Let (G, E, D) be a public-key* (resp., private-key) *encryption scheme that is secure under a-posteriori CCA. Then (G, E, D) is secure under multiple-challenge chosen ciphertext attacks.*

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Proof Sketch: As a bridge between the multiple-challenge CCA and the corresponding benign adversary that does not see the ciphertext, we consider canonical adversaries that can *perfectly simulate* any multiple-challenge CCA without making decryption queries to ciphertexts obtained as part of answers to challenge queries. Instead, these canonical adversaries make corresponding queries of the form $(S, \mathtt{id})$, where id is the identity function and $(S, \cdot)$ is the challenge-query that was answered with the said ciphertext. Specifically, suppose that a multiple-challenge CCA has made the challenge query (S, h), which was answered by (c, v) where $c = E_e(x)$, $v = h(x)$ and $x = S(r)$, and at a later stage makes the decryption query c, which is to be answered by $D_d(c) = x$. Then, the corresponding canonical adversary makes the challenge query (S, h) as the original adversary, receiving the same pair (c, v), but later instead of making the decryption query c the canonical adversary makes the challenge query $(S, \mathtt{id})$, which is answered by $\mathtt{id}(S(r)) = x = D_d(c)$. Note that the trace of the corresponding canonical adversary is identical to the trace of the original CCA adversary (and the same holds with respect to their outputs).

Thus, given an a-posteriori-CCA secure encryption scheme, it suffices to establish Definition 5.4.16 when the quantification is restricted to *canonical* adversaries A. Indeed, as in previous cases, we construct a benign adversary B in the natural manner: On input $(1^n, z)$, machine B generates $(e, d) \leftarrow G(1^n)$, and invokes A on input (y, z), where $y = e$ if we are in the public-key case and $y = 1^n$ otherwise. Next, B emulates all oracles expected by A, while using its own oracle T_r. Specifically, the oracles E_e and D_d are perfectly emulated by using the corresponding keys (known to B), and the oracle $T_{e,r}$ is (non-perfectly) emulated using the oracle T_r (i.e., the query (S, h) is forwarded to T_r, and the answer $h(S(r))$ is augmented with $E_e(1^m)$, where m is the number of output bits in S). Note that the latter emulation (i.e., the answer $(E_e(1^{|S(r)|}), h(S(r)))$) is non-perfect since the answer of $T_{e,r}$ would have been $(E_e(S(r)), h(S(r)))$, yet (as we shall show) A cannot tell the difference.

In order to show that B satisfies both conditions of Definition 5.4.16 (w.r.t the above A), we will show that the following two ensembles are computationally indistinguishable:

1. The global view in real attack of A on (G, E, D). That is, we consider the output of the following experiment:
   (a) $(e, d) \leftarrow G(1^n)$ and $r \leftarrow U_{\text{poly}(n)}$.
   (b) $(f, v) \leftarrow A^{E_e, D_d, T_{e,r}}(y, z)$, where $y = e$ if we are in the public-key case and $y = 1^n$ otherwise. Furthermore, we let $((S^1, h^1), ..., (S^t, h^t))$ denote the trace of the execution $A^{E_e, D_d, T_{e,r}}(y, z)$.
   (c) The output is $((S^1, h^1), ..., (S^t, h^t)), (f, v), r$.

2. The global view in an attack emulated by B. That is, we consider the output of an experiment as above, except that $A^{E_e, D_d, T_{e,r}}(y, z)$ is replaced by $A^{E_e, D_d, T'_{e,r}}(y, z)$, where on query (S, h) the oracle $T'_{e,r}$ replies with $(E_e(1^{|S(r)|}), h(S(r)))$ rather than with $(E_e(S(r)), h(S(r)))$.

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Note that computational indistinguishability of the above ensembles immediately implies Condition 2 of Definition 5.4.16, whereas Condition 1 also follows because using r we can determine whether or not $f(S^1(r),...,S^t(r)) = v$ holds (for (f,v) and $S^1,...,S^t$). Also note that the above ensembles may be computationally indistinguishable only in case A is *canonical* (which we have assumed to be the case).[21]

The computational indistinguishability of the above ensembles is proven using a hybrid argument, which in turn relies on the hypothesis that (G,E,D) has indistinguishable encryptions under a-posteriori-CCAs. Specifically, we introduce $t+1$ *mental experiments* that are hybrids of the above two ensembles (which we wish to relate). Each of these mental experiments is given oracle access to E_e and D_d, where $(e,d) \leftarrow G(1^n)$ is selected from the outside. The ith hybrid experiment uses these two oracles (in addition to y which equals e in the public-key case and 1^n otherwise), in order to emulate an execution of $A^{E_e,D_d,\Pi^i_{e,r}}(y,z)$, where r is selected by the experiment itself and $\Pi^i_{e,r}$ is a hybrid of $T_{e,r}$ and $T'_{e,r}$. Specifically, $\Pi^i_{e,r}$ is a history-dependent process that answers like $T_{e,r}$ on the first i queries and like $T'_{e,r}$ on the rest. Thus, for $i = 0,...,t$, we define the ith hybrid experiment as a process that given y (which equals e or 1^n) and oracle access to E_e and D_d, where $(e,d) \leftarrow G(1^n)$, behaves as follows:

1. The process selects $r \leftarrow U_{\text{poly}(n)}$.
2. The process emulates an execution of $A^{E_e,D_d,\Pi^i_{e,r}}(y,z)$, where $y = e$ if we are in the public-key case and $y = 1^n$ otherwise, by using the oracles E_e and D_d. Specifically, the answers of $\Pi^i_{e,r}$ are emulated using the knowledge of r and oracle access to E_e: the jth query to $\Pi^i_{e,r}$, denoted (S^j,h^j), is answered by $(E_e(S^j(r)),h^j(S^j(r)))$ if $j \leq i$ and is answered by $(E_e(1^{|S^j(r)|}),h^j(S^j(r)))$ otherwise. (The process answers A's queries to E_e and D_d by forwarding them to its own corresponding oracles.)
3. As before, (f,v) denotes the output of $A^{E_e,D_d,\Pi^i_{e,r}}(y,z)$ and $((S^1,h^1),...,(S^t,h^t))$ denotes its trace. The process outputs $((S^1,h^1),...,(S^t,h^t)),(f,v),r$.

We stress that since A is *canonical*, none of the D_d-queries equals a ciphertext obtained as part of the answer of a $\Pi^i_{e,r}$-query.

Clearly, the distribution of the 0-hybrid is identical to the distribution of the global view in an attack emulated by B, whereas the distribution of the t-hybrid is identical to the distribution of the global view in a real attack by A.

[21] Non-canonical adversaries can easily distinguish the two types of views by distinguishing the oracle $T_{e,r}$ from oracle $T'_{e,r}$. For example, suppose we make a challenge query with a sampling-circuit S that generates some distribution over $\{0,1\}^m \setminus \{1^m\}$, next make a decryption query on the ciphertext obtained in the challenge query, and output the answer. Then, in case we query the oracle $T_{e,r}$, we output $D_d(E_e(S(r))) \neq 1^m$; whereas in case we query the oracle $T'_{e,r}$, we output $D_d(E_e(1^m)) = 1^m$. Recall that, at this point, we are guaranteed that A is canonical (and indeed it might have been derived for perfectly-emulating some non-canonical A'). An alternative way of handling non-canonical adversaries is to let B handled the disallowed (decryption) queries by making the corresponding challenge query, and returning its answer rather than the decryption value. (Note that B that emulates $T'_{r,e}$ can detect which queries are disallowed.)

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On the other hand, distinguishing the i-hybrid from the $(i+1)$-hybrid yields a successful *a-posteriori-CCA* (in the sense of distinguishing encryptions). That is, assuming that one can distinguish the i-hybrid from the $(i+1)$-hybrid, we construct a a-posteriori-CCA adversary (as per Definition 5.4.14) as follows. For $(e,d) \leftarrow G(1^n)$, given $y=e$ if we are in the public-key case and $y=1^n$ otherwise, the attacker (having oracle access to E_e and D_d) behaves as follows

1. The attacker selects $r \leftarrow U_{\text{poly}(n)}$.
2. The attacker emulates an execution of $A^{E_e,D_d,\Pi^j_{e,r}}(y,z)$, where $j \in \{i,i+1\}$ (is unknown to the attacker), as follows. The queries to E_e and D_d are answered by using the corresponding oracles available to the attacker, and the issue is answering the queries to $\Pi^j_{e,r}$. The first i queries to $\Pi^j_{e,r}$ are answered as in both $\Pi^i_{e,r}$ and $\Pi^{i+1}_{e,r}$ (i.e., query (S,h) is answered by $(E_e(S(r)),h(S(r))))$, and the last $t-(i+1)$ queries are also answered as in both $\Pi^i_{e,r}$ and $\Pi^{i+1}_{e,r}$ (i.e., by $(E_e(1^{|S(r)|}),h(S(r)))$, this time). The $i+1$ query, denoted (S^{i+1},h^{i+1}), is answered by producing the *challenge templet* $(S^{i+1}(r),1^{|S^{i+1}(r)|})$, which is answered by the challenge ciphertext c (where $c \in \{E_e(S^{i+1}(r)),E_e(1^{|S^{i+1}(r)|})\}$), and replying with $(c,h^{i+1}(S^{i+1}(r)))$.

   Note that if $c = E_e(S^{i+1}(r))$ then we emulate $\Pi^{i+1}_{e,r}$, whereas if $c = E_e(1^{|S^{i+1}(r)|})$ then we emulate $\Pi^i_{e,r}$.
3. Again, (f,v) denotes the output of $A^{E_e,D_d,\Pi^j_{e,r}}(y,z)$, and $((S^1,h^1),...,(S^t,h^t))$ denotes its trace. The attacker feeds $((S^1,h^1),...,(S^t,h^t)),(f,v),r$ to the hybrid distinguisher (which we have assumed to exist towards the contradiction), and outputs whatever the latter does.

The above is an a-posteriori-CCA as in Definition 5.4.14: it produces a *single* challenge (i.e., the pair of plaintexts $(S^{i+1}(r),1^{|S^{i+1}(r)|})$), and distinguishes the case it is given the ciphertext $c = E_e(S^{i+1}(r))$ from the case it is given the ciphertext $c = E_e(1^{|S^{i+1}(r)|})$, without querying D_d on the challenge ciphertext c. The last assertion follows by the hypothesis that A is *canonical*, and so none of the D_d-queries that A makes equals the ciphertext c obtained as (part of) the answer to the $i+1$st $\Pi^j_{e,r}$-query. Thus, distinguishing the $i+1$st and ith hybrids implies distinguishing encryptions under an a-posteriori-CCA, which contradicts our hypothesis regarding (G,E,D). The theorem follows. ■

Further generalization. Recall that we have allowed arbitrary challenge queries of the form (S,h) that were answered by $(E_e(S(r)),L(S(r)))$. Instead, we may allow queries of the form (S,h) that are answered by $(E_e(S(r)),h(r))$; that is, h is applied to r itself rather than to $S(r)$. Actually, given the independence of h from S, one could have replaced the challenge queries by two types of queries: partial-information (on r) *queries* that correspond to the h's (and are answered by $h(r)$), and encrypted partial-information queries that correspond to the S's (and are answered by $E_e(S(r))$). As shown in Exercise 33, all these forms are in fact equivalent.

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5.4.4.3 Constructing CCA-secure private-key schemes

In this section we show simple constructions of CCA-secure private-key encryption schemes. We start with a-priori CCA, and next turn to a-posteriori CCA.

Security under a-priori CCA. All the results presented in Section 5.3.3 extend to security under a-priori chosen ciphertext attacks. Specifically, we prove that Constructions 5.3.9 and 5.3.12 remain secure also under an a-priori CCA.

Proposition 5.4.18 *Let F and (G,E,D) be as in Construction 5.3.9, and suppose that F is pseudorandom with respect to polynomial-size circuits. Then the private-key encryption scheme (G,E,D) is secure under a-priori chosen ciphertext attacks. The same holds with respect to Construction 5.3.12.*

Proof Sketch: As in the proof of 5.4.12, we focus on Construction 5.3.9, and consider an idealized version of the scheme, in which one uses a uniformly selected function $f:\{0,1\}^n \rightarrow \{0,1\}^n$ (rather than the pseudorandom function f_s). Again, all that the adversary obtains by *encryption queries* in the ideal version is pairs $(r, f(r))$, where the r's are uniformly and independently distributed in $\{0,1\}^n$. Similarly, *decryption queries* provide the adversary with pairs $(r, f(r))$, but here the r's are selected by the adversary. Still in an a-priori CCA, all decryption queries are made before the challenge is presented, and so these r's are selected (by the adversary) independent of the challenge. Turning to the challenge itself, we observe that the plaintext is "masked" by the value of f at another uniformly and independently distributed element in $\{0,1\}^n$, denoted r_C. We stress that r_C is independent of all r's selected in decryption queries (because these occur before r_C is selected), as well as being independent of all r's selected by the encryption oracle (regardless of whether these queries are made prior or subsequently to the challenge). Now, unless r_C happens to equal one of the r's that appear in the pairs $(r, f(r))$ obtained by the adversary (which happens with negligible probability), the challenge plaintext is perfectly masked. Thus, the ideal version is secure under an a-priori CCA, and the same holds for the real scheme. ■

Security under a-posteriori CCA. Unfortunately, Constructions 5.3.9 and 5.3.12 are not secure under an a-posteriori chosen ciphertext attacks: Given a challenge ciphertext $(r, x \oplus f_s(r))$, the adversary may obtain $f_s(r)$ by making the query (r, y'), for any $y' \neq x \oplus f_s(r)$. This query is allowed and is answered with x' such that $y' = x' \oplus f_s(r)$. Thus, the adversary may recover the challenge plaintext x from the challenge ciphertext (r, y), where $y \stackrel{\rm def}{=} x \oplus f_s(r)$, by computing $y \oplus (y' \oplus x')$. Thus, we should look for new private-key encryption schemes if we want to obtain one that is secure under a-posteriori CCA. Actually, we show how to transform any private-key encryption scheme that is secure under chosen *plaintext* attack (CPA) into one that is secure under a-posteriori CCA.

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The idea underlying our transformation (of CPA-secure schemes into CCA-secure ones) is to eliminate the adversary's gain from chosen ciphertext attacks by making it infeasible to produce a legitimate ciphertext (other than the ones given explicitly to the adversary). Thus, an a-posteriori CCA adversary can be emulated by a chosen *plaintext* attack (CPA) adversary, while almost preserving the success probability.

The question is indeed how to make it infeasible for the (a-posteriori CCA) adversary to produce a legitimate ciphertext (other than the ones explicitly given to it). One answer is to use "Message Authentication Codes" (with unique valid tags) as defined in Section 6.1. That is, we augment each ciphertext with a corresponding authentication tag (which is "hard to forge"), and consider an augmented ciphertext to be valid only if it consists of a valid (string,tag)-pair. For sake of self-containment (and concreteness), we will use below a specific implementation of such MACs via pseudorandom functions. Incorporating this MAC in Construction 5.3.9, we obtain the following

Construction 5.4.19 (a private-key block-cipher secure against a-posteriori-CCA): *As in Construction 5.3.9, let $F = \{F_n\}$ be an efficiently computable function ensemble and let I be the function-selection algorithm associated with it; i.e., $I(1^n)$ selects a function f_s with distribution F_n. We define a private-key block cipher, (G, E, D), with block length $\ell(n) = n$ as follows*

key-generation: $G(1^n) = ((k, k'), (k, k'))$, *where both* $k \leftarrow I(1^n)$ *and* $k' \leftarrow I(1^n)$.

encrypting plaintext $x \in \{0,1\}^n$ (using the key (k, k')):

$$E_{k,k'}(x) = ((r, f_k(r) \oplus x), f_{k'}(r, f_k(r) \oplus x)),$$

where r is uniformly chosen in $\{0,1\}^n$.

decrypting ciphertext (r, y) (using the key (k, k')): $D_{k,k'}((r, y), t) = f_k(r) \oplus y$ *if* $f_{k'}(r, y) = t$ *and* $D_{k,k'}((r, y), t) = \perp$ *otherwise.*

Proposition 5.4.20 *Let F and (G, E, D) be as in Construction 5.4.19, and suppose that F is pseudorandom with respect to polynomial-size circuits. Then the private-key encryption scheme (G, E, D) is secure under a-posteriori chosen ciphertext attacks.*

Proof Sketch: Following the motivation preceding the construction, we emulate any a-posteriori-CCA adversary by a CPA adversary. Specifically, we need to show how to answer *description queries* made by the CCA adversary. Let us denote such a generic query by $((r, y), t)$, and consider the following three cases:

1. If $((r, y), t)$ equals the answer given to some (previous) *encryption query* x, then we answer the current query with x.

   Clearly, the answer we give is always correct.

2. If $((r, y), t)$ equals the challenge ciphertext then this query is not allowed.

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3. Otherwise, we answer that $((r,y),t)$ is not a valid ciphertext.

   We need to show that our answer is indeed correct. Recall that in this case $((r,y),t)$ did not appear before as an answer to an encryption query. Since for every (r,y) there is a *unique* t such that $((r,y),t)$ is a valid ciphertext, the case hypothesis implies that one of the following sub-cases must occur:

   Case 1: Some $((r,y),t')$, with $t' \neq t$, has appeared before as an answer to an encryption query. In this case $((r,y),t)$ is definitely not a valid ciphertext (because $((r,y),t')$ is the unique valid ciphertext of the form $((r,y),\cdot)$).

   Case 2: No triple of the form $((r,y),\cdot)$ has appear before as such an answer (to an encryption query). In this sub-case, the ciphertext is valid if and only if $t = f_{k'}(r,y)$. That is, in order to produce such a valid ciphertext the adversary must guessed the value of $f_{k'}$ at (r,y), when only seeing the value of $f_{k'}$ at other arguments. By the pseudorandomness of the function $f_{k'}$, the adversary may succeed in such a guess only with negligible probability, and hence our answer is wrong only with negligible probability.

Finally, note that the CPA-security of Construction 5.3.9 (see Proposition 5.4.12) implies that so is Construction 5.4.19. The proposition follows. ■

An alternative proof of Proposition 5.4.20: Building on the proof of Proposition 5.4.18, we (need to) consider here also *description queries* made after the challenge ciphertext, denoted $((r_C,y_C),t_C)$, is presented. Let us denote such a generic query by $((r,y),t)$. We consider three cases:

1. If $r \neq r_C$ then this query can be treated as in the proof of Proposition 5.4.18 (i.e., it is not more dangerous than a query made during an a-priori-CCA attack).

2. If $r = r_C$ and $(y,t) \neq (y_C,t_C)$ then except with negligible probability this query is not a valid ciphertext, because it is infeasible to guess the value of $f_{k'}(r,y)$ (which is the only value of t such that $((r,y),t)$ is valid).

3. If $r = r_C$ and $(y,t) = (y_C,t_C)$ then this query is not allowed.

The proposition follows. ■

The same construction and analysis can be applied to Construction 5.3.12. Combining this with Corollary 3.6.7, we get

Theorem 5.4.21 *If there exist (non-uniformly hard) one-way functions then there exist private-key encryption schemes that are secure under a-posteriori chosen ciphertext attacks.*

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5.4.4.4 Constructing CCA-secure public-key schemes

Using strong forms of Non-Interactive Zero-Knowledge (NIZK) proofs, we show how to transform any public-key encryption scheme that is secure in the passive (key-dependent) sense into one that is secure under a-posteriori CCA. As in case of private-key schemes, the idea underlying the transformation is to eliminate the adversary's gain from chosen ciphertext attacks.

Recall that in case of private-key schemes the adversary's gain from a CCA was eliminated by making it infeasible (for the adversary) to produce legitimate ciphertexts (other than those explicitly given to it). However, in the context of public-key schemes, the adversary can easily generate legitimate ciphertexts (by applying the keyed encryption algorithm to any plaintext of its choice). Thus, in the current context the adversary's gain from a CCA is eliminated by making it infeasible (for the adversary) to produce legitimate ciphertexts without "knowing" the corresponding plaintext. This, in turn, will be achieved by augmenting the plaintext with a non-interactive zero-knowledge "proof of knowledge" of the corresponding plaintext.

Since the notion of a *proof-of-knowledge* is quite complex in general (cf. Section 4.7), and more so in the *non-interactive* (zero-knowledge) context (let alone that we will need strengthenings of it), we will not make explicit use of this notion (of a non-interactive (zero-knowledge) proof-of-knowledge). Instead, we will use non-interactive (zero-knowledge) proofs of membership (NIZK) as defined in Section 4.10. In fact, our starting point is the definition of *adaptive* NIZK system (i.e., Definition 4.10.15). We focus on proof systems in which the prover is implemented by a probabilistic polynomial-time algorithm that is given a suitable auxiliary-input. For sake of clarity let us reproduce the resulting definition.

Definition 5.4.22 (adaptive NIZK): *An* adaptive non-interactive zero-knowledge proof system (adaptive NIZK) *for a language $L \in \mathcal{NP}$, with an NP-relation R_L, consists of a pair of probabilistic polynomial-time algorithms, denoted (P, V), that satisfy the following:*

- Syntax: *Both machines are given the same uniformly selected* reference string $r \in \{0,1\}^m$ *along with an actual input $x \in \{0,1\}^*$ such that $|x| = \text{poly}(m)$ and an auxiliary input. Specifically, on input r, x and w (supposedly, $(x,w) \in R_L$), the prover P outputs an alleged proof $\pi \leftarrow P(x,w,r)$; whereas on input r, x and π, the verifier V decides according to $V(x,r,\pi) \in \{0,1\}$.*
- Completeness: *For every $(x,w) \in R_L$ with $|x| = \text{poly}(m)$, the probability that V does not accept the input x (based on the* proof $P(x,w,U_m)$ *and the reference string U_m) is negligible; that is, $\Pr[V(x,U_m,P(x,w,U_m)) \neq 1]$ is negligible. (Typically, the error probability here is zero, in which case we say that the proof has* perfect completeness.)
- Adaptive Soundness: *For every $\Xi : \{0,1\}^m \to (\{0,1\}^{\text{poly}(m)} \setminus L)$ and every $\Pi : \{0,1\}^m \to \{0,1\}^{\text{poly}(m)}$, the probability that V accepts the input $\Xi(U_m)$*

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(based on the proof $\Pi(U_m)$ *and the reference string* U_m*) is negligible; that is,* $\Pr[V(\Xi(U_m), U_m, \Pi(U_m)) = 1]$ *is negligible.*

- Adaptive Zero-Knowledge: *There exist two probabilistic polynomial-time algorithms,* S_1 *and* S_2*, such that for every pair of functions* $\Xi : \{0,1\}^m \rightarrow (\{0,1\}^{\text{poly}(m)} \cap L)$ *and* $W : \{0,1\}^m \rightarrow \{0,1\}^{\text{poly}(m)}$ *such that* Ξ is implementable by polynomial-size circuits *and* $(\Xi(r), W(r)) \in R_L$ $(\forall r \in \{0,1\}^m)$, *the ensembles* $\{(U_m, \Xi(U_m), P(\Xi(U_m), W(U_m), U_m))\}_{m\in\mathbb{N}}$ *and* $\{S^{\Xi}(1^m)\}_{n\in\mathbb{N}}$ *are computationally indistinguishable (by non-uniform families of polynomial-size circuits), where* $S^{\Xi}(1^m)$ *denotes the output of the following randomized process:*

  1. $(r, s) \leftarrow S_1(1^m)$;
  2. $x \leftarrow \Xi(r)$;
  3. $\pi \leftarrow S_2(x, s)$;
  4. *Output* (r, x, π).

  Indeed, S *is a two-stage* simulator *that first produces (obliviously of the actual input) an alleged reference string* r *(along with auxiliary information* s*), and then given an actual input (which may depend on* r*) simulates the actual proof.*

Note that it is important that in the zero-knowledge condition the function Ξ is required to be implementable by polynomial-size circuits (as otherwise the reference string produced by S_1 would have had to be statistically close to U_m; see Exercise 34). In the rest of this subsection, whenever we refer to an adaptive NIZK, we mean the definition above. Actually, we may relax the adaptive soundness condition so that it only applies to functions Ξ and Π that are implementable by polynomial-size circuits. That is, computational-soundness will actually suffice for the rest of this subsection.

Note that (analogously to Proposition 5.4.10), in case of public-key schemes the combined chosen plaintext and ciphertext attack (as in Definitions 5.4.13 and 5.4.14) is equivalent to a "pure" chosen *ciphertext* attack. Thus, in this subsection we consider only attacks of the latter type. Another technical point is that in our construction we can use any public-key encryption scheme that is secure in the passive (key-dependent) sense, *provided that for all but a negligible measure of the key-pairs that it generates there is no decryption error.*

The general framework. Using an adaptive NIZK, (P, V) (for $\mathcal{NP}$) with simulator $S = (S_1, S_2)$, and an arbitrary public-key encryption scheme, (G, E, D), we present the following public-key encryption scheme:

Construction 5.4.23 (CCA-security construction framework): *Let* $E_e(x, s)$ *denote the ciphertext produced by* E *when given the encryption-key* e*, the plaintext* x *and coins* s*; that is,* $E_e(x) \leftarrow E_e(x, s)$*, where* s *is selected uniformly among*

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the set of $\text{poly}(|e|)$*-long bit strings. We use an adaptive NIZK* (P,V) *for the language* L_R *defined by the following NP-relation*

$$R \stackrel{\text{def}}{=} \{((e_1,e_2,y_1,y_2),(x,s_1,s_2)) : y_1 = E_{e_1}(x,s_1) \ \& \ y_2 = E_{e_2}(x,s_2)\} \quad (5.12)$$

That is, $(e_1,e_2,y_1,y_2) \in L_R$ *if both* y_i*'s are ciphertexts produced using the encryption-keys* e_1 *and* e_2*, respectively, of the same plaintext.*

key-generation: $G'(1^n) \leftarrow ((e_1,e_2,r),(d_1,d_2,r))$, *where* $(e_1,d_1) \leftarrow G(1^n)$, $(e_2,d_2) \leftarrow G(1^n)$*, and* r *is uniformly distributed in* $\{0,1\}^n$.

encrypting plaintext $x \in \{0,1\}^*$ (using the key $\overline{e} = (e_1,e_2,r)$): $E'_{\overline{e}}(x) \leftarrow (y_1,y_2,\pi)$, *where* s_1,s_2 *are uniformly selected* $\text{poly}(n)$*-long bit strings,* $y_1 = E_{e_1}(x,s_1)$, $y_2 = E_{e_2}(x,s_2)$ *and* $\pi \leftarrow P((e_1,e_2,y_1,y_2),(x,s_1,s_2),r)$.

decrypting ciphertext (y_1,y_2,π) (using the key $\overline{d} = (d_1,d_2,r)$): *If* $V((e_1,e_2,y_1,y_2),r,\pi) = 1$ *then return* $D_{d_1}(y_1)$ *else return an error symbol indicating that the ciphertext is not valid.*

Indeed, our choice to decrypt according to y_1 (in case π is a valid proof) is immaterial, and we might as well decrypt according to y_2 or decrypt according to both and return a result only if both results are identical. We stress that, here as well as in the following analysis, we rely on the hypothesis that decryption is error-free, which implies that for $x \neq x'$ the supports of $E_e(x)$ and $E_e(x')$ are disjoint. (Thus, $D_{d_1}(y_1) = D_{d_2}(y_2)$, for any $(e_1,e_2,y_1,y_2) \in L_R$, where the (e_i,d_i)'s are in the range of G.)

Clearly, Construction 5.4.23 constitute a public-key encryption scheme; that is, $D'_{\overline{d}}(E'_{\overline{e}}(x)) = x$, provided that the NIZK proof generated during the encryption stage was accepted during the decryption stage. Indeed, if the NIZK system enjoys perfect completeness (which is typically the case), then the decryption error is zero. By the zero-knowledge property, the passive security of the original encryption scheme (G,E,D) is preserved by Construction 5.4.23. Intuitively, creating a valid ciphertext seems to imply "knowledge" of the corresponding plaintext, but this appealing claim should be examined with more care (and in fact is not always valid). Furthermore, as stated above, our actual proof will not relate to the notion of "knowledge". Instead, the actual proof will proceed by showing how a chosen-ciphertext attack on Construction 5.4.23 can be transformed into a (key-dependent) passive attack on (G,E,D). In fact, we will augment the notion of (adaptive) NIZK in order to present such a transformation. Furthermore, we will do so in two steps. The first augmentation will be used to deal with a-priori CCA, and further augmentation will be used to deal with a-posteriori CCA.

Step I: a-priori CCA

Let us start by considering an a-priori CCA. Given such an adversary A, we construct a passive adversary B that attacks (G,E,D) by emulating the attack of

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A on Construction 5.4.23. One important observation is that the latter encryption scheme uses two keys of the original scheme. Thus, given an encryption-key of the original scheme, B generates another encryption-key (while storing the corresponding decryption-key), and invokes A giving it the pair of encryption-keys (along with a reference string to be generated as discussed below). When A makes a decryption query, B may answer the query by using the stored decryption-key (generated by B before). This works provided that the query ciphertext contains a pair of ciphertexts of the same plaintext according to the two keys, which is the reason we augmented this pair by a *proof of consistency*. Thus, actually, B should examine this proof and act analogously to the decryption process of Construction 5.4.23.

The next problem arises when A asks to be given a challenge. Algorithm B outputs the request as its own challenge templet, but the challenge given to B is a single ciphertext of the original scheme and so B needs to augment it into something that looks like a ciphertext of Construction 5.4.23. Here is where we rely on the zero-knowledge property of the proof of consistency (for producing the required proof that relates to a plaintext we do not know), but in order for so the reference string needs to be generated by S_1 (rather than be uniformly distributed). But this leads to the following problem: when referring (above) to the soundness of the proofs of consistency we assumed that the reference string is uniformly distributed (since soundness was stated for that case), and it is not clear whether soundness holds when the reference string is generated by the simulator (who must use a different[22] distribution). This question is addressed by the notion of (weak) simulation-soundness.

Defining and constructing adaptive NIZKs with weak simulation-soundness. The above discussion leads to the following definition.

Definition 5.4.24 (weak simulation-soundness): *Let (P,V) be an adaptive NIZK for a language L, and (S_1,S_2) be a corresponding two-stage simulator. We say that* weak simulation-soundness *holds if for every polynomial-size implementable Ξ and Π,*

$$\Pr\left[\Xi(r)\notin L \text{ and } V(\Xi(r),r,\Pi(r))=1, \text{ where } (r,s)\leftarrow S_1(1^n)\right] < \mu(n)$$

where $\mu:\mathbb{N}\to[0,1]$ is a negligible function.

Note that the computational limitation on Π is essential to the viability of the definition (see Exercise 35). It is tempting to conjecture that every adaptive NIZK (or rather its simulator) satisfies weak simulation-soundness; however, this is not true (for further discussion see Exercise 36). Nevertheless, adaptive NIZK (for $\mathcal{NP}$) with a simulator satisfying weak simulation-soundness can be constructed given any adaptive NIZK (for $\mathcal{NP}$).

[22] Prove that the distribution produced by S_1 must be far-away from uniform. See related Exercises 34 and 35.

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Construction 5.4.25 (from adaptive NIZK to weak simulation-soundness): *Let (P,V) be an adaptive NIZK for some language L, and let (S_1,S_2) be the corresponding two-stage simulator. We construct the following adaptive NIZK that works with reference string $((r_1^0,r_1^1),...,(r_n^0,r_n^1))$, where $r_i^\sigma \in \{0,1\}^n$.*

Prover P': *on common input x and auxiliary-input w* (s.t., $(x,w)\in R_L$), *(and reference string $((r_1^0,r_1^1),...,(r_n^0,r_n^1))$), uniformly select $b_1,...,b_n \in \{0,1\}$, compute $\pi_i \leftarrow P(x,w,r_i^{b_i})$ for $i=1,...,n$, and output $\overline{\pi} \stackrel{\rm def}{=} (b_1,...,b_n,\pi_1,...,\pi_n)$.*

Verifier V': *on common input x (and reference string $((r_1^0,r_1^1),...,(r_n^0,r_n^1))$), given an alleged proof $\overline{\pi}=(b_1,...,b_n,\pi_1,...,\pi_n)$, accept if and only if $V(x,r_i^{b_i},\pi_i)=1$ for all $i\in\{1,...,n\}$.*

Simulator's first stage S_1': *on input 1^n, select uniformly $c_1,...,c_n \in \{0,1\}$, generate $(r_i^{c_i},s_i) \leftarrow S_1(1^n)$, select uniformly $r_1^{1-c_1},...,r_n^{1-c_n} \in \{0,1\}^n$, and output $(\overline{r},\overline{s})$, where $\overline{r} \stackrel{\rm def}{=} ((r_1^0,r_1^1),...,(r_n^0,r_n^1))$ and $\overline{s} \stackrel{\rm def}{=} (c_1,...,c_n,s_1,...,s_n)$.*

Simulator's second stage S_2': *on input $(\overline{s},x)$, where $\overline{s}=(c_1,...,c_n,s_1,...,s_n)$, compute $\pi_i \leftarrow S_2(x,s_i)$ for $i=1,...,n$, and output $(c_1,...,c_n,\pi_1,...,\pi_n)$.*

It is easy to see that Construction 5.4.25 preserves the adaptive NIZK features of (P,V,S_1,S_2). Furthermore, as shown below, Construction 5.4.25 is weak simulation-sound.

Proposition 5.4.26 *Construction 5.4.25 is an adaptive NIZK for L and weak simulation-soundness holds with respect to the prescribed simulator.*

Proof Sketch: Completeness and soundness follow by the corresponding properties of (P,V). To see that the simulation is indistinguishable from the real execution of (P',V'), note that the two probability ensembles differ in two aspects: first, the simulation uses $r_i^{c_i}$'s generated by $S_1(1^n)$, whereas in the real execution the $r_i^{c_i}$'s are uniformly distributed; and second, the simulation uses simulated proofs produced by $S_2(x,s_i)$ rather than real proofs produced by $P(x,w,r_i^{b_i})$. Still, indistinguishability the output of the original simulator from the real execution of (P,V), can be used to prove the the current ensembles are indistinguishable too. Specifically, we consider a hybrid distribution in which all r_i^b's are generated by $S_1(1^n)$ but the individual proofs (i.e., π_i's) are produced by $P(x,w,r_i^{b_i})$. Using the fact that indistinguishability (by small circuits) is preserved under repeated sampling, we show that the hybrid is indistinguishable from each of the two ensembles (i.e., real execution of (P',V') and the simulation by (S_1',S_2')).

To establish the weak simulation-soundness property, we consider an arbitrary cheating prover $C=(\Xi,\Pi)$ that is implementable by a family of small circuits. We say that $C(\overline{r})=(\Xi(\overline{r}),\Pi(\overline{r}))$ succeeds if it holds that $\Xi(\overline{r})\not\in L$ and $V'(\Xi(\overline{r}),\overline{r},\Pi(\overline{r}))=1$. We are interested in the probability that $C(\overline{r})$ succeeds when $(\overline{r},\overline{s}) \leftarrow S_1'(1^n)$. Recall that $\overline{s}=(c_1,...,c_n,s_1,...,s_n)$, where the c_i's are selected uniformly in $\{0,1\}$, whereas $\Pi(\overline{r})$ has the form $(b_1,...,b_n,\pi_1,...,\pi_n)$. Let

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us denote the latter b_i's by $B(\overline{r})$; that is, $\Pi(\overline{r}) = (B(\overline{r}), \Pi'(\overline{r}))$. We distinguish two cases according to whether or not $B(\overline{r}) = \overline{c} \stackrel{\rm def}{=} (c_1, ..., c_n)$:

$$\begin{aligned} &\mathsf{Pr}[C(\overline{r}) = (\Xi(\overline{r}), (B(\overline{r}), \Pi'(\overline{r}))) \text{ succeeds, when } (\overline{r}, \overline{s}) \leftarrow S'_1(1^n)] \\ &\quad = \mathsf{Pr}[C(\overline{r}) \text{ succeeds and } B(\overline{r}) = \overline{c}, \text{ when } (\overline{r}, (\overline{c}, \overline{s}')) \leftarrow S'_1(1^n)] \\ &\quad\quad + \mathsf{Pr}[C(\overline{r}) \text{ succeeds and } B(\overline{r}) \neq \overline{c}, \text{ when } (\overline{r}, (\overline{c}, \overline{s}')) \leftarrow S'_1(1^n)] \end{aligned}$$

The first term must be negligible because otherwise B can distinguish a sequence of $2n$ uniformly generated r_i^b's from a sequence of r_i^b's as generated by S'_1 (since in the first case $\mathsf{Pr}[B(\overline{r}) = \overline{c}] = 2^{-n}$ by information theoretic considerations). The second term must be negligible because in case the $i^{\rm th}$ bit of $B(\overline{r})$ is different from c_i (i.e., $b_i \neq c_i$), the $i^{\rm th}$ alleged proof (i.e., π_i) is with respect to a uniformly distributed reference string (i.e., $r_i^{b_i} = r_i^{1-c_i}$, which is selected uniformly in $\{0,1\}^n$), and thus can be valid only with negligible probability (or else the (adaptive) soundness of (P, V) is violated). ■

Using adaptive NIZKs with weak simulation-soundness. Following the foregoing motivating discussion, we show that if the adaptive NIZK used in Construction 5.4.23 has the weak simulation-soundness property then the encryption scheme in the construction is secure under a-priori CCA.

Theorem 5.4.27 *Suppose that the adaptive NIZK (P, V) used in Construction 5.4.23 has the weak simulation-soundness property and that the public-key encryption scheme (G, E, D) is passively secure in the key-dependent sense. Further suppose that the probability that $G(1^n)$ produces a pair (e, d) such that $\mathsf{Pr}[D_d(E_e(x)) = x] < 1$ for some $x \in \{0,1\}^{\mathrm{poly}(n)}$, is negligible. Then Construction 5.4.23 constitutes a public-key encryption scheme that is secure under a-priori CCA.*

Combining the above with Theorem 4.10.16 and Proposition 5.4.26, it follow that public-key encryption schemes that are secure under a-priori CCA exist, provided that trapdoor permutations exist.

Proof Sketch: Assuming towards the contradiction that the scheme (G', E', D') is not secure under a-priori CCA, we show that the scheme (G, E, D) is not secure under a (key-dependent) passive attack. Specifically, we refer to the definitions of security in the sense of indistinguishable encryptions (as in Definitions 5.4.14 and 5.4.2, respectively). To streamline the proof, we reformulate Definition 5.4.2, incorporating both circuits (i.e., the one selecting message pairs and the one trying to distinguish their encryptions) into one circuit and allow this circuit to be probabilistic. (Certainly, this model of a key-dependent passive attack is equivalent to the one in Definition 5.4.2.)

Let (A'_1, A'_2) be an a-priori CCA adversary attacking the scheme (G', E', D') (as per Definition 5.4.14), and (S_1, S_2) be the two-stage simulator for (P, V). We construct a (key-dependent) passive adversary A (attacking (G, E, D)) that, given an encryption-key e (in the range of $G_1(1^n)$), behaves as follows:

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1. *Initialization*: A generates $(e_1, d_1) \leftarrow G(1^n)$, $(r, s) \leftarrow S_1(n)$, and sets $\overline{e} = (e_1, e, r)$.

2. *Emulation of* ${A'}_1^{D_{\overline{d}}}(\overline{e})$: A invokes A'_1 on input $\overline{e}$, and answers its (decryption) queries as follows. When asked to decrypt the alleged ciphertext (q_1, q_2, q_3), adversary A checks if q_3 is a valid proof of consistence of q_1 and q_2 (with respect to the reference string r). If the proof is valid, then A answers with $D_{d_1}(q_1)$ else A returns the error symbol.

   (Note that the emulation is perfect, although A only knows part of the corresponding decryption-key $\overline{d}$.)

3. *Using A'_2 for the final decision*: Let $((x^{(1)}, x^{(2)}), \sigma)$ denote the challenge templet output by A'_1. Then, given a ciphertext $y = E_e(x)$, where $x \in \{x^{(1)}, x^{(2)}\}$, adversary A form a corresponding ciphertext (y_1, y, π), by letting $y_1 \leftarrow E_{e_1}(0^{|x^{(1)}|})$ and $\pi \leftarrow S_2(s, (e_1, e, y_1, y))$. Finally, A invokes A'_2 on input $(\sigma, (y_1, y, \pi))$, and outputs whatever the latter does. Recall that, here (in case of a-priori CCA), A'_2 is an ordinary machine (rather than an oracle machine).

   (Note that this emulation is not perfect, since (typically) A invokes A'_2 with an illegal ciphertext, still we shall see that A'_2 cannot tell the difference.)

In order to analyze the performance of A, we introduce the following hybrid process as a *mental experiment*. The hybrid process behaves as A, with the only exception that (in Step 3) $y_1 \leftarrow E_{e_1}(x)$ (rather than $y_1 \leftarrow E_{e_1}(0^{|x|})$). Thus, unlike A, the hybrid process invokes A'_2 with a legal ciphertext. (The question of how the hybrid process "knows" or gets this y_1 is out of place; we merely define a mental experiment.) Let $p_A^{(j)} = p_A^{(j)}(n)$ (resp., $p_H^{(j)} = p_H^{(j)}(n)$) denote the probability that A (resp., the hybrid process) outputs 1 when $x = x^{(j)}$.

Claim 5.4.27.1: For both j's the absolute difference between $p_A^{(j)}(n)$ and $p_H^{(j)}(n)$ is a negligible function in n.

Proof: Define an auxiliary hybrid process that behaves as the hybrid process except that when emulating $D_{\overline{d}}$, the auxiliary process answers according to D_{d_2} (rather than according to D_{d_1}). (Again, this is a mental experiment.) Let $p_{HH}^{(j)}$ denote the probability that this auxiliary process outputs 1 when $x = x^{(j)}$. Similarly, define another mental experiment that behaves as A except that when emulating $D_{\overline{d}}$, the auxiliary process answers according to D_{d_2} (rather than according to D_{d_1}), and let $p_{AA}^{(j)}$ denote the probability that this process outputs 1 when $x = x^{(j)}$. Let $m \stackrel{\rm def}{=} |x|$. We establish the following facts:

Fact 1. For both j's the absolute difference between $p_H^{(j)}$ and $p_{HH}^{(j)}$ is negligible.

The reason is that by weak simulation-soundness, it is infeasible to produce triples (q_1, q_2, q_3) such that $D_{d_1}(q_1) \neq D_{d_2}(q_2)$ and yet q_3 is a valid proof (w.r.t r) that q_1 and q_2 encrypt the same plaintext. Here we rely on the hypothesis that except with negligible probability over the key-generation, the decryption is error-less (i.e., always yields the original plaintext).

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Fact 2. Similarly, for both j's the absolute difference between $p_A^{(j)}$ and $p_{AA}^{(j)}$ is negligible.

Fact 3. Finally, for both j's the absolute difference between $p_{HH}^{(j)}$ and $p_{AA}^{(j)}$ is negligible.

The reason is that the experiments AA and HH differ only in the input $(\sigma, (y_1, y, \pi))$ that they feed to A'_2: whereas AA forms $y_1 \leftarrow E_{e_1}(0^m)$ (and $\pi \leftarrow S_2(s, (e_1, e, y_1, y)))$, the process HH forms $y_1 \leftarrow E_{e_1}(x)$ (and $\pi \leftarrow S_2(s, (e_1, e, y_1, y)))$. However, A'_2 cannot distinguish the two cases because this would have violated the security of E_{e_1}.

That is, to establish Fact 3, we construct a passive attack, denoted B, that behaves similarly to A except that it switches its reference to the two basic keys (i.e., the first two components of the encryption-key $\overline{e}$) and acts very differently in Step 3 (e.g., produces a different challenge templet). Specifically, given an attacked encryption-key e, adversary B generates $(e_2, d_2) \leftarrow G(1^n)$, sets $\overline{e} = (e, e_2, \cdot)$, and emulates ${A'}_1^{D_{\overline{d}}}(\overline{e})$ using the decryption-key d_2 to answer queries. For a fixed j, when obtaining (from A'_1) the challenge templet $((x^{(1)}, x^{(2)}), \sigma)$, adversary B produces the challenge templet $((0^m, x^{(j)}), \sigma)$, and invokes A'_2 on input $(\sigma, (y, y_2, \pi))$, where $y = E_e(x)$ ($x \in \{0^m, x^{(j)}\}$) is the challenge ciphertext given to B, and B computes $y_2 \leftarrow E_{e_2}(x^{(j)})$ and $\pi \leftarrow S_2(s, (e, e_2, y, y_2))$. (Finally, B outputs the output obtained from A'_2.) Note that when given the challenge ciphertext $E_e(x^{(j)})$, the adversary B behaves as experiment HH, whereas when given $E_e(0^m)$ it behaves as experiment AA. Thus, if $p_{HH}^{(j)}$ and $p_{AA}^{(j)}$ differ in a non-negligible manner, then B violates the passive security of the encryption scheme (G, E, D).

Combining the above three facts, the current claim follows. □

Let us denote by $p_{\rm cca}^{(j)}(n)$ the probability that the CCA adversary (A'_1, A'_2) outputs 1 when given a ciphertext corresponding to the $j^{\rm th}$ plaintext in its challenge templet (see Definitions 5.4.14). Recall that by the hypothesis $|p_{\rm cca}^{(j)}(n) - p_{\rm cca}^{(j)}(n)|$ is not negligible.

Claim 5.4.27.2: For both j's the absolute difference between $p_{\rm cca}^{(j)}(n)$ and $p_H^{(j)}(n)$ is a negligible function in n.

Proof: The only difference between the output in a real attack of (A'_1, A'_2) and the output of the hybrid process is that in the hybrid process a "simulated reference string" and a "simulated proof" are used instead of a uniformly distributed reference string and a real NIZK proof. However, this difference is indistinguishable. □

Combining Claims 5.4.27.1 and 5.4.27.2, we conclude that A violates the passive security of (G, E, D). This contradicts the hypothesis, and so the theorem follows. ■

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Step II: a-posteriori CCA

In order to use Construction 5.4.23 in the context of a-posteriori CCA security, we need to further strengthen the NIZK proof in use. The reason is that, in an a-posteriori CCA, the adversary may try to generate false proofs (as part of the ciphertext queries in the second stage) after being given a (single) proof (as part of the challenge ciphertext). Specifically, when trying to extend the proof of Theorem 5.4.27, we need to argue that, given a simulated proof (to either a false or a true statement), it is infeasible to generate a false proof to a false statement (as long as one does not just copy the given simulated proof, in case it is to a false statement). The notion of weak simulation-soundness does not suffice to bound the probability of success in such attempts, because it only refer to what one can do when only given the simulated reference string. The following definition addresses the situation in which one is given a *single* simulated proof (along with the simulated reference string). (We comment that a more general notion that refers to situations in which one is given a *many* simulated proofs is not necessary for the current application.)

Definition 5.4.28 (1-proof simulation-soundness): *Let (P, V) be an adaptive NIZK for a language L, and (S_1, S_2) be a corresponding two-stage simulator. We say that* 1-proof simulation-soundness *holds if for every triplet of polynomial-size circuit families (Ξ^1, Ξ^2, Π^2), the probability of the following event is negligible:*

The event**:** *for (x^1, π^1, x^2, π^2) generated as described below the following three conditions hold: $x^2 \notin L$, $(x^2, \pi^2) \neq (x^1, \pi^1)$, and $V(x^2, r, \pi^2) = 1$.*

The generation process**:** *First $(r, s) \leftarrow S_1(1^n)$, then $x^1 \leftarrow \Xi^1(r)$, next $\pi^1 \leftarrow S_2(s, x^1)$, and finally $(x^2, \pi^2) \leftarrow (\Xi^2(r, \pi^1), \Pi^2(r, \pi^1))$.*

Note that weak simulation-soundness is obtained as a special case (by letting $\Xi(r) = \Xi^2(r, \lambda)$ and $\Pi(r) = \Pi^2(r, \lambda)$).

Theorem 5.4.29 *Suppose that the adaptive NIZK (P, V) used in Construction 5.4.23 has the 1-proof simulation-soundness property and that the encryption scheme (G, E, D) is as in Theorem 5.4.27. Then Construction 5.4.23 constitutes a public-key encryption scheme that is secure under a-posteriori CCA.*

Proof Sketch: The proof follows the structure of the proof of Theorem 5.4.27. Specifically, given an a-posteriori CCA (A'_1, A'_2) (attacking (G', E', D')), we first construct a passive adversary A (attacking (G, E, D)). The construction is as in the proof of Theorem 5.4.27 with the exception that in Step 3 we need to emulate the decryption oracle (for A'_2). This emulation is performed exactly as the one performed in Step 2 (for A'_1). Next, we analyze this passive adversary as in the proof of Theorem 5.4.27, while referring (in the current analysis unlike in the previous one) to an A'_2 that may make decryption queries. The analysis of the handling of these queries relies on the 1-proof simulation-soundness property.

In particular, when proving a claim analogous to Claim 5.4.27.1, we have to establish two facts (corresponding to Facts 1 and 2) that refer to the difference

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in the process's output when decrypting according to D_{d_1} and D_{d_2}, respectively. Both facts follow from the fact (established below) that, except with negligible probability, neither A'_1 nor A'_2 can produce a query (q_1, q_2, q_3) such that q_3 is a *valid proof that* q_1 *and* q_2 *are consistent* and yet $D_{d_1}(q_1) \neq D_{d_2}(q_2)$. (We stress that in the current context we refer also to A'_2, which may try to produce such a query based on the challenge ciphertext given to it.)

Fact 5.4.29.1: The probability that A'_1 produces a query (q_1, q_2, q_3) such that q_3 is a valid proof (w.r.t reference string r) that (supposedly) there exists x, s_1, s_2 such that $q_i = E_{e_i}(x, s_i)$ (for $i = 1, 2$), and yet $D_{d_1}(q_1) \neq D_{d_2}(q_2)$ is negligible. The same holds for A'_2 as long as the query is different from the challenge ciphertext given to it. This holds regardless of whether the challenge ciphertext (given to A'_2) is produced as in A (i.e., $y_1 = E_{e_1}(0^m)$) or as in the hybrid process H (i.e., $y_1 = E_{e_1}(x)$).

Proof: Recall that one of our hypotheses is that the encryption (G, E, D) is error-free (except for a negligible measure of the key-pairs). Thus, the current fact refers to a situation that either A'_1 or A'_2 produces a valid proof for a false statement. The first part (i.e., referring to A'_1) follows from the weak simulation-soundness of the NIZK, which in turn follows from its 1-proof simulation-soundness property. We focus on the second part, which refers to A'_2.

Let (y_1, y_2, π) denote the challenge ciphertext given to A'_2 (i.e., $y_2 = y$ is the challenge ciphertext given to $A(e)$ (or to $H(e)$), which augments it with y_1 and $\pi \leftarrow S_2(s, (e_1, e_2, y_1, y_2)))$. Recall that $(r, s) \leftarrow S_1(1^n)$ and that $e_2 = e$. Suppose that A'_2 produces a query (q_1, q_2, q_3) as in the claim; that is, $(q_1, q_2, q_3) \neq (y_1, y_2, \pi)$, the encryptions q_1 and q_2 are not consistent (w.r.t e_1 and e_2 respectively), and yet $V((e_1, e_2, q_1, q_2), r, q_3) = 1$. Specifically, it holds that $x^2 \stackrel{\text{def}}{=} (e_1, e_2, q_1, q_2) \notin L_R$, where L_R is as in Construction 5.4.23 (see Eq. (5.12)), and yet $V(x^2, r, q_3) = 1$ (i.e., $\pi^2 \stackrel{\text{def}}{=} q_3$ is a valid proof of the false statement regarding x^2). Since (y_1, y_2, π) is produced by letting $\pi \leftarrow S_2(s, (e_1, e_2, y_1, y_2))$, it follows that $\pi^1 \stackrel{\text{def}}{=} \pi$ is a simulated proof (w.r.t the reference string r) for the alleged membership of $x^1 \stackrel{\text{def}}{=} (e_1, e_2, y_1, y_2)$ in L_R, where $(r, s) \leftarrow S_1(1^n)$. Furthermore, given such a proof (along with the reference string r), A'_2 produces a query (q_1, q_2, q_3) that yields a pair (x^2, π^2) such that $x^2 = (e_1, e_2, q_1, q_2) \notin L_R$ and yet $V(x^2, r, \pi^2) = 1$ (where $\pi^2 = q_3$). Thus, using A'_1 and A'_2 (along with (G, E, D)), we obtain circuits Ξ^1, Ξ^2, Π^2 that violate the hypothesis that (S_1, S_2) is 1-proof simulation-sound (i.e., $\Xi^1(r) = (e_1, e_2, y_1, y_2)$, $\pi^1 = \pi \leftarrow S_2(s, (e_1, e_2, y_1, y_2))$, $\Xi^2(r, \pi^1) = (e_1, e_2, q_1, q_2)$ and $\Pi^2(r, \pi^1) = q_3$, where (y_1, y_2, π) and (q_1, q_2, q_3) are derived from the input and output to A'_2). □

Fact 5.4.29.1 implies (adequate extension of) the first two facts in the proof of a claim analogous to Claim 5.4.27.1. The third fact in that proof as well as the proof of Claim 5.4.27.2 do not refer to the soundness of the NIZK-proofs, and are established here exactly as in the proof of Theorem 5.4.27. The current theorem follows. ■

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Constructing adaptive NIZK with 1-proof simulation-soundness property. Using a standard NIZK proof, a weak form of a signature scheme, and a specific commitment scheme, we construct the desired NIZK. Since all ingredients can be implemented using trapdoor permutations, we obtain:

Theorem 5.4.30 *If there exist collections of (non-uniformly hard) trapdoor permutations then every language in $\mathcal{NP}$ has an adaptive NIZK with 1-proof simulation-soundness property.*

Proof Sketch: Let $L \in \mathcal{NP}$. We construct a suitable NIZK for L using the following three ingredients:

1. An *adaptive Non-Interactive Witness-Indistinguishable (NIWI)* proof, denoted $(P^{\rm wi}, V^{\rm wi})$, for a suitable language in $\mathcal{NP}$. We stress that we mean a proof system that operates with a reference string of length n and can be applied to prove (adaptively chosen) statements of length $\text{poly}(n)$, where the adaptivity refers both to the soundness and witness-indistinguishability requirements.

   By Theorem 4.10.16, the existence of trapdoor permutations implies that every language in $\mathcal{NP}$ has an adaptive NIZK that operates with a reference string of length n and can be applied to prove statements of length $\text{poly}(n)$. Indeed, in analogy to discussions in Section 4.6, any NIZK is a NIWI.

2. A *super-secure one-time signature scheme*, denoted $(G^{\rm ot}, S^{\rm ot}, V^{\rm ot})$. Specifically, one-time security (see Section 6.4.1) means that we consider only attacks in which the adversary may obtain a signature to a single document of its choice (rather than signatures to polynomially-many documents of its choice). On the other hand, super-security (see Section 6.5.2) means that the adversary should fail to produce a valid document-signature that is different from the query-answer pair that appeared in the attack. (We stress that, unlike in ordinary security, the adversary may succeed even in case it produces a different signature to the same document for which it has obtained a signature during the attack.) By Theorem 6.5.2, super-secure one-time signature scheme can be constructed based on any one-way function. (If we were willing to assume the existence of collision-free hashing functions then we could have used the easier-to-establish Theorem 6.5.1 instead.)

3. A *perfectly-binding commitment scheme*, denoted C, as defined in Section 4.4.1. Furthermore, we require that the commitment strings are pseudorandom; that is, the ensembles $\{C(x)\}_{x\in\{0,1\}^*}$ and $\{U_{|C(x)|}\}_{x\in\{0,1\}^*}$ are computationally indistinguishable. Additionally, we require that the support of $C(U_n)$ is a negligible portion of $\{0,1\}^{|C(U_n)|}$. (The latter requirement may be omitted if we are willing to settle for (ordinary) computational-soundness rather than (ordinary) information-theoretic soundness.)

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Using any collection of one-way permutations (e.g., the one in the hypothesis), we may obtain the desired commitment scheme. Specifically, Construction 4.4.2 satisfies the pseudorandomness property required above. To obtain the additional ("negligible portion") requirement, we merely let $C(x)$ equal a pair of two independent commitments to x (and it follows that the support of $C(U_n)$ is at most a $2^n \cdot (2^{-n})^2 = 2^{-n}$ fraction of $\{0,1\}^{|C(U_n)|}$).[23] We denote by $C(x,r)$ the commitment produced to value x while using coins r; that is, $C(x) = C(x,r)$, where r is uniformly chosen in $\{0,1\}^{\ell(|x|)}$, for some polynomial ℓ.

Given the above ingredients, we construct an adaptive NIZK for L (with witness relation R) as follows. The NIZK proof uses a *reference string* of the form $\overline{r} = (r_1, r_2)$, where $n \stackrel{\rm def}{=} |r_2|$ and $m \stackrel{\rm def}{=} |r_1| = \text{poly}(n)$.

Prover P: On common input x and auxiliary-input w (and reference string $\overline{r} = (r_1, r_2)$), where supposedly $(x,w) \in R$, the prover behaves as follows

1. Generates a key-pair for the one-time signature scheme; that is, $(s,v) \leftarrow G^{\rm ot}(1^n)$.
2. Compute a pre-proof $p \leftarrow P^{\rm wi}((x,r_1,v),w,r_2)$, where $(V^{\rm wi}, V^{\rm wi})$ is a proof system (using r_2 as reference string) for the following NP-language L':

$$L' \stackrel{\rm def}{=} \{(x,y,v) : (x \in L) \vee (\exists w' \; y = C(v,w'))\} \tag{5.13}$$

The corresponding NP-relation is

$$R' \stackrel{\rm def}{=} \{((x,y,v),w') : ((x,w') \in R) \vee (y = C(v,w'))\} \tag{5.14}$$

Note that P indeed feeds $P^{\rm wi}$ with an adequate NP-witness (i.e., $((x,r_1,v),w) \in R'$ since $(x,w) \in R$). The first part of the reference string of P is part of the statement fed to $P^{\rm wi}$, whereas the second part of P's reference string serves as a reference string for $P^{\rm wi}$. The behavior of V (w.r.t $V^{\rm wi}$) will be analogous.

3. The prover computes a signature σ to (x,p) relative to the signing-key s (generated in Step 1). That is, P computes $\sigma \leftarrow S^{\rm ot}{}_s(x,p)$.

The prover outputs the triplet (v,p,σ).

Verifier V: On common input x and an alleged proof (v,p,σ) (and reference string $\overline{r} = (r_1,r_2)$), the verifier accepts if and only if the following two conditions hold

[23] This presupposes that in the original commitment scheme the support of $C(x)$ is at most a $2^{-|x|}$ fraction of $\{0,1\}^{|C(x)|}$, which does hold for Construction 4.4.2. Alternatively, using any collection of one-way functions, we may also obtain the desired commitment scheme. Specifically, Construction 4.4.4 will do, except that it uses two messages. However, since the first message (i.e., sent by the receiver) is a random string, we may incorporate it in the reference string (of the scheme presented below).

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1. σ is a valid signature with respect to the verification-key v to the pair (x,p). That is, $V^{\rm ot}{}_v((x,p),\sigma)=1$.
2. p is a valid proof with respect to the reference string r_2 to the statement $(x,r_1,v)\in L'$. That is, $V^{\rm wi}((x,r_1,v),r_2,p)=1$.

Simulator's first stage S_1: On input 1^{m+n} (from which S_1 determines n and m), the first stage produces a reference string and auxiliary information as follows.

1. As the real prover, $S_1(1^{m+n})$ starts by generating a key-pair for the one-time signature scheme; that is, $(s,v)\leftarrow G^{\rm ot}(1^n)$.
2. Unlike in the real setting, $S_1(1^{m+n})$ selects s_1 uniformly in $\{0,1\}^{\ell(|v|)}$, and set $r_1=C(v,s_1)$. (Note that in the real setting, r_1 is uniformly distributed independently of v, and thus r_1 is unlikely to be in the support of $C(v)$.)
3. As in the real setting, $S_1(1^{m+n})$ selects r_2 uniformly in $\{0,1\}^n$.

$S_1(1^{m+n})$ outputs the pair $(\overline{r},\overline{s})$, where $\overline{r}=(r_1,r_2)$ and $\overline{s}=(v,s,s_1,r_2)$.

Simulator's second stage S_2: On input a statement x and auxiliary input $\overline{s}=(v,s,s_1,r_2)$ (as generated by S_1), S_2 proceeds as follows:

1. Using (the NP-witness) s_1, computes a pre-proof $p\leftarrow P^{\rm wi}((x,C(v,s_1),v),s_1,r_2)$. Note that indeed, $((x,C(v,s_1),v),s_1)\in R'$.
2. Using (the signing-key) s, computes a signature σ to (x,p) relative to s, where p is as computed in the first step. That is, $\sigma\leftarrow S^{\rm ot}{}_s(x,p)$.

$S_2(\overline{s},x)$ outputs (v,p,σ).

As we will see below, the above (two-stage) simulator produces output that is indistinguishable from the output of the real execution. Intuitively, the first stage of the simulator enables cheating to entities (such as the second stage of the simulator) that can produce signatures with respect to the verification-key committed to in the string r_1 (which is part of the reference string generated by S_1). This allows the simulation (which gets the signing-key) to cheat, but does not allow cheating by an adversary that sees only the verification-key and one valid signature (which are both part of the single proof given to the adversary in the definition of 1-proof simulation-soundness). Thus, one-time signatures yield 1-proof simulation-soundness, and indeed using general signature schemes (as well as some technical modifications) yield "many-proofs simulation-soundness" (which is none of our concern here). We now turn to the actual proof of the above properties.

Claim 5.4.30.1: (P,V) satisfies completeness and adaptive soundness.

Proof: Completeness follows by combining the syntactic properties of the one-time signature scheme, the completeness property of the proof system $(P^{\rm wi},V^{\rm wi})$ and the definition of R'. Adaptive computational-soundness follows from the fact

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that, given only the (uniformly distributed) reference string $\overline{r} = (r_1, r_2)$, it is infeasible to find v such that r_1 is in the support of $C(v)$. Using the additional property by which $C(G^{\text{ot}}{}_2(1^n))$ covers a negligible portion of $\{0,1\}^m$, it follows that for a uniformly selected $r_1 \in \{0,1\}^m$ there exist no v such that r_1 is in the support of $C(v)$. Thus, using also the (adaptive) soundness of $(P^{\text{wi}}, V^{\text{wi}})$, except with negligible probability, when presented with a valid proof (v, p, σ) for x, it must be the case that $(x, r_1, v) \in L'$, and so $x \in L$. □

Claim 5.4.30.2 (adaptive zero-knowledge): For every efficient way of selecting inputs Ξ, the output produced by the two-stage simulator (S_1, S_2) is indistinguishable from the one produced by P. That is, the ensembles $\{S^{\Xi}(1^{m+n})\}$ and $R^{\Xi,W} \stackrel{\text{def}}{=} \{(U_{m+n}, \Xi(U_{m+n}), P(\Xi(U_{m+n}), W(U_{m+n}), U_{m+n}))\}$ are computationally indistinguishable, where S^{Ξ} is defined as in Definition 5.4.22.

Proof: Consider a hybrid distribution $H^{\Xi}(1^{m+n})$, in which everything except the pre-proof is produced as by $S^{\Xi}(1^{m+n})$, and the pre-proof is computed as by the real prover. That is, $(\overline{r}, \overline{s}) \leftarrow S_1(1^{m+n})$ (where $\overline{r} = (r_1, r_2)$ and $\overline{s} = (v, s, s_1, r_2)$) is produced as by S^{Ξ}, but then for $(x, w) = (\Xi(\overline{r}), W(\overline{r}))$, the pre-proof is computed using the witness w (i.e., $p \leftarrow P^{\text{wi}}((x, r_1, v), w, r_2)$ rather than $p \leftarrow P^{\text{wi}}((x, r_1, v), s_1, r_2)$). The final proof $\pi = (v, p, \sigma)$ is obtained (as in both cases) by letting $\sigma \leftarrow S^{\text{ot}}{}_s(x, p)$. We now relate the hybrid ensemble to both ensembles in the claim.

1. By the (adaptive) witness indistinguishability of P^{wi}, the ensembles H^{Ξ} and S^{Ξ} are computationally indistinguishable. (Recall that these ensembles differ only in the way the pre-proof is produced; specifically, they differ only in the NP-witness used by P^{wi} to prove the very same claim.)
2. By the pseudorandomness of the commitments produced for any fixed value, H^{Ξ} and $R^{\Xi,W}$ are computationally indistinguishable. (Recall that these ensembles differ only in the way the first part of the reference string (i.e., r_1) is produced.)

The claim follows. □

Claim 5.4.30.3 (1-proof simulation-soundness): For every triplet of polynomial-size circuit families (Ξ^1, Ξ^2, Π^2), consider the following process: First $(\overline{r}, \overline{s}) \leftarrow S_1(1^{m+n})$, then $x^1 \leftarrow \Xi^1(\overline{r})$, next $\pi^1 \leftarrow S_2(\overline{s}, x^1)$, and finally $(x^2, \pi^2) \leftarrow (\Xi^2(\overline{r}, \pi^1), \Pi^2(\overline{r}, \pi^1))$. Then, the probability that the following three conditions hold simultaneously is negligible: (1) $x^2 \notin L$, (2) $(x^2, \pi^2) \neq (x^1, \pi^1)$, and (3) $V(x^2, \overline{r}, \pi^2) = 1$.

Proof: Recall that $\overline{r} = (r_1, r_2)$ and $\overline{s} = (v, s, s_1, r_2)$, where $(s, v) \leftarrow G^{\text{ot}}(1^n)$ and $r_1 = C(v, s_1)$ for a uniformly chosen $s_1 \in \{0,1\}^{\ell(|v|)}$ (and r_2 is selected uniformly in $\{0,1\}^n$). Also recall that $\pi^1 = (v^1, p^1, \sigma^1)$, where $v^1 = v$, $p^1 \leftarrow P^{\text{wi}}((x, C(v, s_1), v), s_1, r_2)$ and $\sigma^1 \leftarrow S^{\text{ot}}{}_s(x^1, p^1)$. Let us denote $(v^2, p^2, \sigma^2) \stackrel{\text{def}}{=} \pi^2$. Using the definition of V, we need to upper bound

$$\mathsf{Pr}\left[\begin{array}{l} (x^2 \notin L) \wedge ((x^2, \pi^2) \neq (x^1, \pi^1)) \\ \wedge (V^{\text{ot}}{}_{v^2}((x^2, p^2), \sigma^2) = 1) \wedge (V^{\text{wi}}((x^2, r_1, v^2), r_2, p^2) = 1) \end{array}\right] \tag{5.15}$$

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We consider two cases (in which the event in Eq. (5.15) may hold):

$v^2 = v^1$: In this case, either $(x^2, p^2) \neq (x^1, p^1)$ or $\sigma^2 \neq \sigma^1$ must hold (because otherwise $(x^2, \pi^2) = (x^2, (v^2, p^2, \sigma^2)) = (x^1, (v^1, p^1, \sigma^1)) = (x^1, \pi^1)$ follows). But this means that (Ξ^2, Π^2), given a single valid signature σ^1 (to the document (x^1, p^1)) with respect to a randomly generated verification-key $v = v^1 = v^2$, is able to produce a valid document-signature pair $((x^2, p^2), \sigma^2)$ (with respect to the same verification-key) such that $((x^2, p^2), \sigma^2) \neq ((x^1, p^1), \sigma^1)$, in contradiction to the super-security of the one-time signature scheme.

> Details: It suffices to upper bound
>
> $$\Pr\left[\begin{array}{l} (v^2 = v^1) \ \wedge \ ((x^2, \pi^2) \neq (x^1, \pi^1)) \\ \wedge \ (V^{\rm ot}{}_{v^2}((x^2, p^2), \sigma^2) = 1) \end{array}\right] \qquad (5.16)$$
>
> As explained above, the first two conditions in Eq. (5.16) imply that $((x^2, p^2), \sigma^2) \neq ((x^1, p^1), \sigma^1)$. Using (S_1, S_2) and (Ξ^1, Ξ^2, Π^2), we derive an attacker, A, that violates the super-security of the (one-time) signature scheme. The attacker just emulates the process described in the claim's hypothesis, except that it obtains v as input (rather than generating the pair (s, v) by invoking $G^{\rm ot}$) and uses oracle access to $S^{\rm ot}{}_s$ in order to produce the signature σ^1. Note that A uses its oracle only once and that the probability that A produces a document-signature pair different from the (single) query-answer pair is lower bounded by Eq. (5.16).

$v^2 \neq v^1$: Since $r_1 = C(v^1, s_1)$, it follows (by the perfect binding property of C) that $r_1 \neq C(v^2, w')$ for every w'. Thus, if $(x^2, r_1, v^2) \in L'$ then $x^2 \in L$, and so assuming $x^2 \notin L$ it follows that $(x^2, r_1, v^2) \notin L'$. Now, by the adaptive soundness of $(P^{\rm wi}, V^{\rm wi})$ and the fact that r_2 was selected uniformly in $\{0,1\}^n$, it follows that except with negligible probability p^2 is not a valid proof (w.r.t the reference string r_2) of the false statement "$(x^2, r_1, v^2) \in L'$".

> Details: It suffices to upper bound
>
> $$\Pr\left[\begin{array}{l} (v^2 \neq v^1) \ \wedge \ (x^2 \notin L) \\ \wedge \ (V^{\rm wi}((x^2, r_1, v^2), r_2, p^2) = 1) \end{array}\right] \qquad (5.17)$$
>
> As explained above, the first two conditions in Eq. (5.17) imply $(x^2, r_1, v^2) \notin L'$. The key observation is that r_2 (generated by S_1) is uniformly distributed in $\{0,1\}^n$, and thus the adaptive soundness of the NIWI system applies. We conclude that Eq. (5.17) is upper bounded by the soundness error of the NIWI system, and the claim follows also in this case.

Combining both cases, the claim follows. □

The current theorem follows. ■

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Conclusion: Combining Theorems 5.4.6, 5.4.30 and 5.4.29, we get:

Theorem 5.4.31 *If there exist collections of (non-uniformly hard) trapdoor permutations then there exist public-key encryption schemes that are secure under a-posteriori chosen ciphertext attacks.*

5.4.5 Non-malleable encryption schemes

So far, our treatment has referred to an adversary that, when given a ciphertext, tries to gain explicit information about the plaintext. A less explicit gain, captured by the so-called notion of *malleability*, is the ability to generate an encryption of a related plaintext (possibly without learning anything about the original plaintext). Loosely speaking, an encryption scheme is called *non-malleable* if given a ciphertext it is infeasible (for an adversary) to produce a (different) valid ciphertext for a related plaintext. For example, given a ciphertext of a plaintext of the form $1x$, for an unknown x, it should be infeasible to produce a ciphertext to the plaintext $0x$.

Non-malleability may relate to any of the types of attacks considered above (e.g., passive attacks, chosen ciphertext attacks, etc). Thus, we have a "matrix" of adversaries, with one dimension (parameter) being the *type of attack* and the second being its *purpose*. So far, we have discussed the first dimension (i.e., the type of the attack), when focusing on a particular purpose (i.e., of violating the secrecy of the plaintext). We now turn to the second dimension (i.e., the purpose of the attack), and consider also the purpose of malleability. That is, we make a distinction between the following two notions (or purposes of attack):

1. Standard *security*: the infeasibility of *obtaining information regarding the plaintext*. As defined above, such information is captured by a function (or a randomized process) applied to the bare plaintext, and it may not depend on the encryption-key (or decryption-key).

2. In contrast, the notion of *non-malleability* refers to generating a string depending on both the plaintext and the current encryption-key. Specifically, one requires that it should be infeasible for an adversary, given a ciphertext, to produce a valid ciphertext (under the same encryption-key) for a related plaintext.

We shall show below that, with the exception of passive attacks on private-key schemes, non-malleability always implies security against attempts to obtain information on the plaintext. We shall also show that security and non-malleability are equivalent under a-posteriori chosen ciphertext attack. Thus, the results of the previous sections imply that non-malleable (under a-posteriori chosen ciphertext attack) encryption schemes can be constructed based on the same assumptions used to construct passively-secure encryption schemes.

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5.4.5.1 Definitions

For sake of brevity, we present just a couple of definitions. Specifically, focusing on the public-key model, we consider only (key-oblivious) passive attacks and chosen ciphertext attacks. The definitions refer to an adversary that given a ciphertext tries to generate a different ciphertext to a plaintext related to the original one. That is, given $E_e(x)$, the adversary tries to output $E_e(y)$ such that $(x, y) \in R$ with respect to some (efficiently recognizable)[24] relation R. Loosely speaking, the adversary's success probability in such an attempt is compared to the success probability of generating such $E_e(y)$ when not given $E_e(x)$. As in case of semantic security, we strengthen the definition by consider all possible partial information functions h.

Definition 5.4.32 (passive non-malleability) *A public-key encryption scheme (G, E, D) is said to be* non-malleable under passive attacks *if for every probabilistic polynomial-time algorithm A there exists a probabilistic polynomial-time algorithm A' such that for every ensemble $\{X_n\}_{n\in\mathbb{N}}$, with $|X_n| = \text{poly}(n)$, every polynomially-bounded $h:\{0,1\}^* \to \{0,1\}^*$, every polynomially-bounded relation R that is recognizable by a (non-uniform) family of polynomial-size circuits, every polynomial $p(\cdot)$ and all sufficiently large n*

$$\begin{aligned}
&\mathsf{Pr}\left[\begin{array}{ll} (x,y)\in R & \text{where} \\ & (e,d)\leftarrow G(1^n) \text{ and } x\leftarrow X_n \\ & c\leftarrow E_e(x) \text{ and } c'\leftarrow A(e,c,1^{|x|},h(x)) \\ & y\leftarrow D_d(c') \text{ if } c'\neq c \text{ and } y\leftarrow 0^{|x|} \text{ otherwise} \end{array}\right] \\
&< \quad \mathsf{Pr}\left[\begin{array}{ll} (x,y)\in R & \text{where} \\ & (e,d)\leftarrow G(1^n) \text{ and } x\leftarrow X_n \\ & c'\leftarrow A'(e,1^{|x|},h(x)) \text{ and } y\leftarrow D_d(c') \end{array}\right] + \frac{1}{p(n)}
\end{aligned}$$

We stress that the definition effectively prevents the adversary A from just outputting the ciphertext given to it (because in this case the output is treated as if it were $E_e(1^{|x|})$). This provision is important because otherwise no encryption scheme could have satisfied the definition (see Exercise 37). Note that, since A' is given the encryption-key, it (i.e., A') can certainly produce ciphertexts, but its information regarding X_n is restricted to $h(X_n)$ (and $1^{|X_n|}$). Thus, if given $h(X_n)$ and $1^{|X_n|}$ it is hard to generate y such that $(X_n, y) \in R$ then it will be hard for A' to produce an encryption of such y. We comment that an equivalent definition may be obtained by requiring A' to output the plaintext (i.e., y) rather than its encryption under a randomly generated encryption-key.

Definition 5.4.32 cannot be satisfied by encryption schemes in which one can modify bits in the ciphertext without changing the corresponding plaintext (i.e., consider the identity relation). We stress that such encryption schemes may be semantically secure under passive attacks (e.g., given a semantically secure encryption scheme (G, E, D), consider $E'_e(x) = E_e(x)\sigma$, for randomly chosen

[24] The computational restriction on R is essential here; see Exercise 15 that refers to a related definition of semantic security.

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$\sigma \in \{0,1\}$). However, such encryption schemes may not be (semantically) secure under a-posteriori-CCA.

Turning to the definition of non-malleability under chosen ciphertext attacks, we adopt the definitional framework of Section 5.4.4.1. Specifically, analogously to Definition 5.4.13, the challenge templet produced by A_1 (and A'_1) is a triplet of circuits representing a distribution S (represented by a sampling circuit), a function h (represented by an evaluation circuit), and a relation R (represented by an membership recognition circuit). The goal of A_2 (and A'_2) will be to produce a ciphertext of a plaintext that is R-related to the challenge plaintext $S(U_{\mathrm{poly}(n)})$.

Definition 5.4.33 (non-malleability under chosen ciphertext attacks): *A public-key encryption scheme is said to be* non-malleable under a-priori chosen ciphertext attacks *if for every pair of probabilistic polynomial-time oracle machines, A_1 and A_2, there exists a pair of probabilistic polynomial-time algorithms, A'_1 and A'_2, such that the following two conditions hold:*

1. *For every positive polynomial $p(\cdot)$, and all sufficiently large n and $z \in \{0,1\}^{\mathrm{poly}(n)}$:*

$$\Pr\left[\begin{array}{ll} (x,y)\in R & \text{where} \\ & (e,d)\leftarrow G(1^n) \\ & ((S,h,R),\sigma)\leftarrow A_1^{E_e,D_d}(e,z) \\ & (c,v)\leftarrow (E_e(x),h(x))\,,\text{where } x\leftarrow S(U_{\mathrm{poly}(n)}) \\ & c'\leftarrow A_2^{E_e}(\sigma,c,v) \\ & y\leftarrow D_d(c') \text{ if } c'\neq c \text{ and } y\leftarrow 0^{|x|} \text{ otherwise.} \end{array}\right]$$

$$< \Pr\left[\begin{array}{ll} (x,y)\in R & \text{where} \\ & ((S,h,R),\sigma)\leftarrow A'_1(1^n,z) \\ & x\leftarrow S(U_{\mathrm{poly}(n)}) \\ & y\leftarrow A'_2(\sigma,1^{|x|},h(x)) \end{array}\right] + \frac{1}{p(n)}$$

2. *For every n and z, the first element* (i.e., the (S,h,R) part) *in the random variables $A'_1(1^n,z)$ and $A_1^{E_{G_1(1^n)},D_{G_2(1^n)}}(G_1(1^n),z)$ are identically distributed.*

Non-malleability under a-posteriori chosen ciphertext attacks *is defined analogously, except that A_2 is given oracle access to both E_e and D_d with the restriction that when given the challenge (c,v), machine A_2 is not allowed to make the query c to the oracle D_d.*

5.4.5.2 Relation to semantic security

With the exception of passive attacks on private-key schemes, for any type of attack, non-malleability under this type of attack implies semantic security under the same type. For example, we show the following:

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Proposition 5.4.34 *Let (G, E, D) be a public-key encryption scheme that is non-malleable under passive attacks* (resp., under a-posteriori chosen ciphertext attacks). *Then, (G, E, D) is semantically secure under passive attacks* (resp., under a-posteriori chosen ciphertext attacks).

Proof Sketch: For clarity, the reader may consider the case of passive attacks, but the same argument holds also for each of the other types of attacks considered above.

Suppose (towards the contradiction) that (G, E, D) is not semantically secure (under the relevant type of attacks). Using the equivalence to indistinguishability of encryptions, it follows that under such attacks one can distinguish encryption to x_n from encryption to y_n. Consider the relation $R = \{(x, \bar{x}) : x \in \{0,1\}^*\})$, where $\bar{x}$ is the complement of x, and the uniform distribution Z_n on $\{x_n, y_n\}$. We construct an algorithm than given a ciphertext (as well as an encryption-key e) runs the above distinguisher, and produces $E_e(\bar{x}_n)$ in case the distinguisher "votes" for x_n (and produces $E_e(\bar{y}_n)$ otherwise). Indeed, given $E_e(Z_n)$, our algorithm outputs $E_e(\bar{Z}_n)$ (and so hit R) with probability that is non-negligibly higher than $1/2$. This performance cannot be met by any algorithm that is not given $E_e(Z_n)$. Thus, we derive a contradiction to the hypothesis that (G, E, D) is non-malleable. ■

We stress that the above argument only relies on the fact that, in the public-key model, we can produce the encryption of any string, since we are explicitly given the encryption-key. In fact, it suffices to have access to an encryption oracle, and thus the argument extends also to active attacks in the private-key model (in which the attacker is allowed encryption queries).

On the other hand, under most types of attacks considered above, non-malleability is strictly stronger than semantic security. Still, in the special case of a-posteriori chosen ciphertext attacks, the two notions are equivalent. Specifically, we prove that, in case of a-posteriori-CCA, semantic security implies non-malleability.

Proposition 5.4.35 *Let (G, E, D) be a public-key encryption scheme that is semantically secure under a-posteriori chosen ciphertext attacks. Then, (G, E, D) is non-malleabable under a-posteriori chosen ciphertext attacks. The same holds for private-key encryption schemes.*

Proof Sketch: Suppose towards the contradiction that (G, E, D) is not non-malleabable under a-posteriori chosen ciphertext attacks, and let $A = (A_1, A_2)$ be an adversary demonstrating this. We construct a (semantic security) adversary $B = (B_1, B_2)$ that invokes A, and at the very end uses its own decryption oracle to decrypt the ciphertext output by A, and outputs the response. Intuitively, B violates semantic security (with respect to relations, as can be defined analogously to Exercise 15). Details follow.

Given an encryption-key e, algorithm B_1 invokes $A_1(e)$, while answering A_1's queries by querying its own oracles, and obtains the challenge templet (S, h, R) (and state σ), which it outputs. Algorithm B_2, is given a ciphertext c along

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with some auxiliary information, and invokes A_2 on the very same input, while answering A_2's queries by querying its own oracles. When A_2 halts with output $c' \neq c$, algorithm B_2 forwards c' to its decryption oracle, and outputs the answer. Thus, the plaintext output by B hits the relation R with the same probability that the plaintext corresponding to (the decryption of) A's output hits R. We have to show that this hitting probability cannot be met by an algorithm that does not get the ciphertext; but this follows from the hypothesis regarding A (and the fact that in both cases the corresponding algorithm (i.e., A' or B') outputs a plaintext (rather than a ciphertext)). Finally, we have to establish, analogously to Exercise 15, that semantic security with respect to relations holds (in our current context of chosen ciphertext attacks) if and only if semantic security (with respect to functions) holds. The latter claim follows as in Exercise 15 by relying on the fact that in the current context the relevant relations have polynomial-size circuits. (A similar argument holds for private-key schemes.) ■

Conclusion: Combining Theorem 5.4.31 and Proposition 5.4.35 we get:

Theorem 5.4.36 *If there exist collections of (non-uniformly hard) trapdoor permutations then there exist public-key encryption schemes that are non-malleable under a-posteriori chosen ciphertext attacks.*

Analogously, using Theorem 5.4.21, we get:

Theorem 5.4.37 *If there exist (non-uniformly hard) one-way functions then there exist private-key encryption schemes that are non-malleable under a-posteriori chosen ciphertext attacks.*

5.5 Miscellaneous

5.5.1 On Using Encryption Schemes

Once defined and constructed, encryption schemes may be (and are actually) used as building blocks towards various goals that are different from the original motivation. Still, the original motivation (i.e., secret communication of information) is of great importance, and in this subsection we discuss several issues regarding the use of encryption schemes towards achieving it.

Using private-key schemes – the key exchange problem. As discussed in Section 5.1.1, using a private-key encryption scheme requires the communicating parties to share a secret key. This key can be generated by one party and secretly communicated to the other party by an alternative (expensive) secure channel. Often, a preferable solution consists of employing a *key-exchange* (or rather *key-generation*) protocol, which is executed over the standard (insecure) communication channel. An important distinction refers to the question

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of whether the insecure communication channel between the legitimate parties is tapped by a *passive* adversary or may even be subject to *active* attacks in which an adversary may modify the messages sent over the channel (and even delete and insert such messages). Protocols secure against passive (resp., active) adversaries are often referred to by the term *authenticated key-exchange* (resp., unauthenticated key-exchange), because in the passive case one refers to the messages received over the channel as being authentic (rather than possibly modified by the adversary).

A simple (generic) authenticated key-exchange protocol consists of using a public-key encryption scheme in order to secretly communicate a key (for the private-key encryption scheme, which is used in the actual communication).[25] Specifically, one party generates a random instance of a public-key encryption scheme, sends the encryption-key to the other party, which generates a random key (for the private-key encryption scheme), and sends an encryption (using the received encryption-key) of the newly generated key to the first party. A famous alternative is the so-called *Diffie-Hellman Key-Exchange* [78]: for a (large) prime P and primitive element g, which are universal or generated on-the-fly (by one party that openly communicates them to the other), the first (resp., second) party uniformly selects $x \in \mathbb{Z}_P$ (resp., $y \in \mathbb{Z}_P$) and sends $g^x \bmod P$ (resp., $g^y \bmod P$) to the other party, and both parties determined $g^{xy} \bmod P$ as their common key, relying on the fact that $g^{xy} \equiv (g^x \bmod P)^y \equiv (g^y \bmod P)^x \pmod P$. (The security of this protocol relies on the assumption that given a prime P, a primitive element g, and the triplet $(P, g, (g^x \bmod P), (g^y \bmod P), (g^z \bmod P))$, it is infeasible to decide whether $z \equiv xy \pmod{P-1}$, for $x, y, z \in \mathbb{Z}_P$.) The construction of unauthenticated key-exchange protocols is far more complex, and the interested reader is referred to [30, 31, 18].

Using state-dependent private-key schemes. In many communication settings it is reasonable to assume that the encryption device may maintain (and modify) a state (e.g., a counter). In such a case, the stream ciphers discussed in Section 5.3.1 become relevant. Furthermore, using a stream cipher is particularly appealing in applications where decryption is performed in the same order as encryption (e.g., in FIFO communication). In such applications, the stream cipher of Construction 5.3.3 is preferable to the (pseudorandom function based) encryption scheme of Construction 5.3.9 for a couple of reasons. First, applying an on-line pseudorandom generator is likely to be more efficient than applying a pseudorandom function. Second, for a ℓ-bit long counter (or random value), Construction 5.3.3 allows to securely encrypt 2^ℓ messages (or bits), whereas Construction 5.3.9 definitely becomes insecure when $\sqrt{2^\ell}$ messages (or bits) are encrypted. For small values of ℓ (e.g., $\ell = 64$), this difference is crucial.

Using public-key schemes – public-key infrastructure. As in the case of private-key schemes, an important distinction refers to the question of whether

[25] One reason not to use the public-key encryption scheme itself for the actual (encrypted) communication is that private-key encryption schemes tend to be much faster.

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the insecure communication channel between the legitimate parties is tapped by a *passive* adversary or may even be subject to *active* attacks. In typical applications of public-key encryption schemes, the parties communicate through a communication network (and not via a point-to-point channel), in which case active attacks are very realistic (e.g., it is easy to send mail over the internet pretending to be somebody else). Thus, the standard use of public-key encryption schemes in real-life communication requires a mechanism for providing the sender with the receiver's authentic encryption-key (rather than trusting an "unauthenticated" incoming message to specify an encryption-key). In small systems, one may assume that each user holds a local record of the encryption-keys of all other users. However, this is not realistic in large-scale systems, and so the sender must obtain the relevant encryption-key on-the-fly in a "reliable" way (i.e., typically, certified by some trusted authority). In most theoretical work, one assumes that the encryption-keys are posted and can be retrieved from a public-file that is maintained by a trusted party (which makes sure that each user can post only encryption-keys bearing its own identity). In abstract terms, such trusted party may provide each user with a (signed) certificate stating the authenticity of the user's encryption-key. In practice, maintaining such a public-file (and handling such certificates) is a major problem, and mechanisms that implement this abstraction are typically referred to by the generic term "public-key infrastructure (PKI)". For a discussion of the practical problems regarding PKI deployment see, e.g., [180, Chap. 13].

5.5.2 On Information Theoretic Security

In contrast to the bulk of our treatment, which focuses on computationally-bounded adversaries, in this section we consider computationally-unbounded adversaries. We stress that also here the length (and number) of the plaintexts is still bounded (as usual, by an unknown polynomial). The resulting notion of security is the one suggested by Shannon: a (private-key or public-key) encryption scheme is called perfectly-secure (or information-theoretically secure) if the ciphertext yields no information regarding the plaintext. That is, perfect-security is derived from Definitions 5.2.1 and 5.2.2 by allowing computationally-unbounded algorithms (in the roles of A and A').

It is easy to see that no public-key encryption scheme may be perfectly-secure: a computationally-unbounded adversary that is given a encryption-key can find a corresponding decryption-key, which allows it to decrypt any ciphertext.

In contrast, restricted types of private-key encryption schemes may be perfectly-secure. Specifically, the traditional "one-time pad" yields such a (private-key) scheme that can be used to securely communicate an a-priori bounded number of bits. Furthermore, multiple-messages may be handled provided that their total length is a-priori bounded and that we use a state (as in Construction 5.3.3). We stress that this state-based private-key perfectly-secure encryption scheme uses a key of length equal to the total length of plaintexts to be encrypted. Indeed, the key must be at least that long (to allow perfect-security), and a state is essential for allowing several plaintexts to be securely encrypted.

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Partial information models. Note that, in case of private-key encryption scheme, the limitation of perfect-security hold only if the adversary has full information of the communication over the channel (i.e., holds the full contents of all ciphertexts sent). On the other hand, perfectly-secure private channels can be implemented on top of channels to which the adversary has limited access. We mention three types of channels of the latter type, which have received a lot of attention.

- The *bounded-storage model*, where the adversary can freely tap the communication channel but is restricted in the amount of data it can store (cf., [56]).
- The *noisy channel model* (which generalizes the *wiretap channel* of [237]) where both the communication between the legitimate parties and the tapping channel of the adversary are subjected to noise (cf., [179, 71] and the references therein).
- *Quantum Channels* where an adversary is (supposedly) prevented from obtaining full information by the (currently believed) laws of quantum mechanics (cf., [51] and the references therein).

Following are the author's subjective opinions regarding these models (as a possible basis for actual secure communication). The bounded-storage model is very appealing, because it clearly states its *reasonable assumptions* regarding the the *abilities of the adversary*. In contrast, making absolute assumptions about the noise level at any point in time seems (overly) optimistic, and thus not adequate in the context of cryptography. Basing cryptography on quantum mechanics sounds as a very appealing idea, but attempts to implement this idea have often stumbled over unjustified hidden assumptions (which are to be expected given the confusing nature of quantum mechanics and the discrepancy between its scientific culture and cryptography).

5.5.3 On Popular Schemes

The reader may note that we have avoided the presentation of several popular encryption schemes. We regret to say that most of these schemes are proposed without any reference to a satisfactory notion of security. That is, not only that no reason is given to believe that these schemes are (semantically) secure (which is often clearly not the case), but it seems that the proposal does not even consider such a property to be desirable (not to say necessary). It is thus not surprising that we have nothing to say about the contents of such proposals. In contrast, we highlight a few things that we have said about other popular schemes and common practices:

- The common practice of using "pseudorandom generators" as a basis for private-key stream ciphers (i.e., Construction 5.3.3) is sound, provided that one actually uses pseudorandom generators (rather than programs that are

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called "pseudorandom generators" but actually produce sequences that are easy to predict).[26]

- Whereas the *plain* RSA public-key encryption scheme (which employs a deterministic encryption algorithm) is not secure, the *randomized* RSA encryption scheme (i.e., Construction 5.3.16) is secure, provided that the *large hard-core conjecture* holds (see Section 5.3.4.1). Some support for the latter (clearly stated) conjecture may be derived from the fact that a related function (i.e., much fewer least significant bits) constitutes a hard-core of the RSA.
- Assuming the intractability of factoring, there exists a *secure* public-key encryption scheme with efficiency comparable to that of plain RSA: we refer to the Blum-Goldwasser public-key encryption scheme (i.e., Construction 5.3.20).

Finally, we warn that encryption schemes *proved to be secure in the random oracle model* are not necessarily secure (in the standard sense). For further discussion of the Random Oracle Methodology, we refer the reader to Section 6.6.3.

5.5.4 Historical Notes

The notion of private-key encryption scheme seems almost as ancient as the alphabet itself. Furthermore, it seems that the development of encryption methods went along with the development of communication media. As the amounts of communication grow, more efficient and sophisticated encryption methods were required. Computational complexity considerations were explicitly introduced into the arena by Shannon [226]: In his work, Shannon considered the classical setting where no computational considerations are present. He showed that in this information theoretic setting, secure communication of information is possible only as long as its entropy is lower than the entropy of the key. He thus concluded that if one wishes to have an encryption scheme that is capable of handling messages with total entropy exceeding the length of the key then one must settle for a computational relaxation of the secrecy condition. That is, rather than requiring that the ciphertext yields no information on the plaintext, one has to require that such information cannot be efficiently computed from the ciphertext. The latter requirement indeed coincides with the above definition of semantic security.

The notion of public-key encryption scheme was introduced by Diffie and Hellman [78]. First concrete candidates were suggested by Rivest, Shamir and Adleman [216] and by Merkle and Hellman [185]. However, satisfactory definitions of security were presented only a few years afterwards, by Goldwasser

[26] The linear congruential generator is easy to predict [49]. The same holds for some modifications of it that output a constant fraction of the bits of each resulting number [99]. We warn that sequences having large linear-complexity (LFSR-complexity) are *not* necessarily hard to predict.

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and Micali [141]. The two definitions presented in Section 5.2 originate in [141], where it was shown that ciphertext-indistinguishability implies semantic security. The converse direction is due to [186].

Regarding the seminal paper of Goldwasser and Micali [141], a few additional comments are due. Arguably, this paper is the basis of the entire rigorous approach to cryptography (presented in the current book): It introduced general notions such as computational indistinguishability, definitional approaches such as the simulation paradigm, and techniques such as the hybrid argument. The paper's title ("Probabilistic Encryption") is due to the authors' realization that public-key encryption schemes in which the encryption algorithm is deterministic cannot be secure in the sense defined in their paper. Indeed, this led the authors to (explicitly) introduce and justify the paradigm of "randomizing the plaintext" as part of the encryption process. Technically speaking, the paper only presents security definitions for public-key encryption schemes, and furthermore some of these definitions are syntactically different from the ones we have presented above (yet, all these definitions are equivalent). Finally, the term "ciphertext-indistinguishability" used here replaces the (generic) term "polynomial-security" used in [141]. Many of our modifications (to the definitions in [141]) have already appeared in [110], which is also the main source of our uniform-complexity treatment.

The first construction of a secure public-key encryption scheme based on a simple complexity assumption was given by Goldwasser and Micali [141].[27] Specifically, they constructed a public-key encryption scheme assuming that deciding Quadratic Residiousity modulo composite numbers is intractable. The condition was weaken by Yao [238] who prove that any trapdoor permutation will do. The efficient public-key encryption scheme of Construction 5.3.20 is due to Blum and Goldwasser [46]. The security is based on the fact that the least significant bit of the modular squaring function is a hard-core predicate, provided that factoring is intractable, a result mostly due to [5].

For decades, it has been common practice to use "pseudorandom generators" in the design of stream ciphers. As pointed out by Blum and Micali [47], this practice is sound *provided* that one uses pseudorandom generators (as defined in Chapter 3). The construction of private-key encryption schemes based on pseudorandom functions is due to [119].

We comment that it is indeed peculiar that the rigorous study of (the security of) private-key encryption schemes has legged behind the corresponding study of public-key encryption schemes. This historical fact may be explained by the very thing that makes it peculiar; that is, private-key encryption schemes are less complex than public-key ones, and hence the problematics of their security (when applied to popular candidates) is less obvious. In particular, the need for a rigorous study of (the security of) public-key encryption schemes arose from observations regarding some of their concrete applications (e.g., doubts raised

[27] Recall that plain RSA is not secure, whereas Randomized RSA is based on the Large Hard-Core Conjecture for RSA (which is less appealing that the standard conjecture referring to the intractability of inverting RSA).

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by Lipton concerning the security of the "mental poker" protocol of [225], which used "plain RSA" as an encryption scheme). In contrast, the need for a rigorous study of (the security of) private-key encryption schemes arose later and by analogy to the public-key case.

Credits for the advanced section (i.e., Section 5.4)

Definitional issues. The definitional treatment of Goldwasser and Micali [141] actually refer to key-dependent passive attacks (rather than to key-oblivious passive attacks). Chosen ciphertext attacks (of the a-priori and a-posteriori type) were first considered in [199] (and [213], respectively). However, these papers focused on the formulation in terms of indistinguishability of encryptions, and formulations in terms of semantic security have not appeared before. Section 5.4.4.2 is based on [128]. The study of the *non-malleability* of the encryption schemes was initiated by Dolev, Dwork and Naor [79].

Constructions. The framework for constructing public-key encryption schemes that withstand Chosen Ciphertext Attacks (i.e., Construction 5.4.23) is due to Naor and Yung [199], who used it to construct public-key schemes that withstand a-priori CCA (under suitable assumptions). This framework was applied to the setting of a-posteriori CCA by Sahai [218, 219], who followed and improved ideas of Dolev, Dwork and Noar [79] (which were the first to construct public-key schemes that withstand a-posteriori CCA and prove Theorem 5.4.31). Our presentation of the proof of Theorem 5.4.31 follows subsequent simplification due to Lindell [173]. The key role of non-interactive zero-knowledge proofs in this context was suggested by Blum, Feldman and Micali [45]. The fact that security and non-malleability are equivalent under a-posteriori chosen ciphertext attack was proven in [79, 19].

5.5.5 Suggestion for Further Reading

For discussion of Non-Malleable Cryptography, which actually transcends the domain of encryption, see [79]. Specifically, we wish to highlight the notion of non-malleable commitment scheme, which is arguably the most appealing instantiation of the "non-malleability paradigm": it is infeasible for a party that is given a non-malleable commitment to produce a commitment to a related string. Note that ability to produce related commitments may endanger some applications (cf. [127]) even if this ability is not decoupled with the ability to properly decommit (to the produced commitment) once a decommitment to the original commitment is obtained.

Recall that there is a gap between the assumptions currently required for the construction of private-key and public-key encryption schemes: whereas the former can be constructed based on any one-way functions, the latter seem to require a trapdoor permutation (or, actually, a "trapdoor predicate" [141]). A partial explanation to this gap was provided by Impagliazzo and Rudich, who

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showed that generic (black-box) constructions of public-key encryption schemes cannot rely on one-way functions [151] (or even on one-way permutations [158]).

For a detailed discussion of the relationship among the various notions of secure private-key and public-key encryption schemes, the reader is referred to [160] and [19], respectively.

5.5.6 Open Problems

Secure public-key encryption schemes exist if there exist collections of (non-uniformly hard) trapdoor permutations (cf. Theorem 5.3.15). It is not known whether the converse holds (although secure public-key encryption schemes easily imply one-way function). (The few-to-1 feature of the function collection is important; see [25].)

Randomized RSA (i.e., Construction 5.3.16) is commonly believed to be a secure public-key encryption scheme. It would be of great practical importance to gain additional support for this belief. As shown in Proposition 5.3.17, the security of Randomized RSA follows from the *Large Hard-Core Conjecture for RSA*, but the latter is not known to follow from a more standard assumption such as that RSA is hard to invert. This is indeed the third place in this book where we suggest the establishment of the latter implication as an important open problem.

The constructions of *public-key* encryption schemes *secure against chosen ciphertext attacks* (presented in Section 5.4) are to be considered as plausibility results (which also offer some useful construction paradigms). Presenting "reasonably-efficient" public-key encryption schemes that are secure against (a-posteriori) chosen ciphertext attacks, under widely believed assumptions, is an important open problem. (We comment that the "reasonably-efficient" scheme of [70] is based on a very strong assumption regarding the *Diffie-Hellman Key Exchange*. Specifically, it is assumed that for a prime P and primitive element g, given $(P, g, (g^x \bmod P), (g^y \bmod P), (g^z \bmod P))$, it is infeasible to decide whether $z \equiv xy \pmod{P-1}$.)

5.5.7 Exercises

Exercise 1: *Encryption schemes imply one-way function* [149]: Show that the existence of a secure private-key encryption scheme (i.e., as in Definition 5.2.1) implies the existence of one-way functions.

> **Guideline:** Recall that, by Exercise 11 of Chapter 3, it suffices to prove that the former implies the existence of a pair of polynomial-time constructible probability ensembles that are statistically far apart and still are computationally indistinguishable. To prove the existence of such ensembles consider the encryption of $n+1$-bit plaintexts relative to a random n-bit long key, denoted K_n. Specifically, let the first ensemble be $\{(U_{n+1}, E(U_{n+1}))\}_{n\in\mathbb{N}}$, where $E(x) = E_{K_n}(x)$, and the second ensemble be $\{(U_{n+1}^{(1)}, E(U_{n+1}^{(2)}))\}_{n\in\mathbb{N}}$, where $U_{n+1}^{(1)}$ and $U_{n+1}^{(2)}$ are independently distributed. It is easy to show that these ensembles are computationally

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indistinguishable and are both polynomial-time constructible. The more interesting part is to show that these ensembles are statistically far apart. To prove this fact, assume towards the contradiction that for all but a negligible fraction of the 2^{n+1} possible x's, the distribution of $E(x)$ is statistically close to a single distribution Y, and show that this does not allow correct decryption (since there are only 2^n possible keys).

Exercise 2: *Encryption schemes with unbounded-length plaintext*: Suppose that the definition of semantic security is modified so that no bound is placed on the length of plaintexts. Prove that in such a case there exists no semantically secure public-key encryption scheme. (Hint: A plaintext of length exponential in the security parameter allows the adversary to find the decryption key by exhaustive search.)

Exercise 3: *Encryption schemes must leak information about the length of the plaintext*: Suppose that the definition of semantic security is modified so that the algorithms are not given the length of the plaintext. Prove that in such a case there exists no semantically secure encryption scheme.

Guideline: First show that for some polynomial p, $|E(1^n)| < p(n)$, whereas for some $x \in \{0,1\}^{p(n)}$ it holds that $\Pr[|E(x)|<p(n)] < 1/2$.

Exercise 4: *Hiding partial information about the length of the plaintext*: Using an arbitrary secure encryption scheme, construct a correspondingly secure encryption scheme that hides the exact length of the plaintext. In particular, construct an encryption scheme that reveals only the following function h' of the length of the plaintext:

1. $h'(m) = \lceil m/n \rceil \cdot n$, where n is the security parameter.
2. $h'(m) = 2^{\lceil \log_2 m \rceil}$

(Hint: Just use an adequate padding convention, making sure that it always allows correct decoding.)

Exercise 5: *Length parameters*: Assuming the existence of a secure public-key (resp., private-key) encryption scheme, prove the existence of such scheme in which the length of keys equal the security parameter. Furthermore, show that (without loss of generality) the length of ciphertexts may be a fixed polynomial in the length of the plaintext.

Exercise 6: *On the distribution of public-keys*: Let (G,E,D) be a secure public-key encryption scheme. Prove that for every polynomial p, and all sufficiently large n, it holds that $\max_e\{\Pr[G_1(1^n)=e]\} < 1/p(n)$.

Guideline: Show that for any encryption-key e in the range of $G_1(1^)$, one can find a corresponding decryption-key in expected time $1/\Pr[G_1(1^n)=e]$.

Exercise 7: *Deterministic encryption schemes*: Prove that in order to be semantically secure a public-key encryption scheme must have a probabilistic encryption algorithm. (Hint: Otherwise, one can distinguish the encryptions of two candidate plaintexts by computing the unique ciphertext for each of them.)

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Exercise 8: *An alternative formulation of Definition 5.2.1*: Prove that the following definition, in which we use non-uniform families of polynomial-size circuits (rather than probabilistic polynomial-time algorithms) is equivalent to Definition 5.2.1.

> There exists a probabilistic polynomial-time transformation T such that for every polynomial-size circuit family $\{C_n\}_{n\in\mathbb{N}}$, and for every $\{X_n\}_{n\in\mathbb{N}}$, $f,h:\{0,1\}^*\to\{0,1\}^*$, $p(\cdot)$ and n as in Definition 5.2.1
>
> $$\begin{aligned}&\Pr\left[C_n(E_{G_1(1^n)}(X_n),1^{|X_n|},h(X_n))=f(X_n)\right]\\ &\quad< \Pr\left[C'_n(1^{|X_n|},h(X_n))=f(X_n)\right]+\frac{1}{p(n)}\end{aligned}$$
>
> where $C'_n \leftarrow T(C_n)$ and the probability is also taken over the internal coin tosses of T.

Same for public-key encryption.

Guideline: The alternative view of non-uniformity, discussed in Section 1.3, is useful here. That is, we can view a circuit family as a sequence of advices given to a universal machine. Thus, the above definition states that advices for a machine that gets the ciphertext can be efficiently transformed into advices for a machine that does not get the ciphertext. However, we can incorporate the (probabilistic) transformation program into the second universal algorithm (which then become probabilistic). Consequently, the advices are identical for both machines (and can be incorporated in the auxiliary input $h(X_n)$ used in Definition 5.2.1). Viewed this way, the above definition is equivalent to asserting that for some (universal) deterministic polynomial-time algorithm U there exists a probabilistic polynomial-time algorithm U' and for every $\{X_n\}_{n\in\mathbb{N}}$, $f,h:\{0,1\}^*\to\{0,1\}^*$, $p(\cdot)$ and n as in Definition 5.2.1

$$\begin{aligned}&\Pr\left[U(E_{G_1(1^n)}(X_n),1^{|X_n|},h(X_n))=f(X_n)\right]\\ &\quad< \Pr\left[U'(1^{|X_n|},h(X_n))=f(X_n)\right]+\frac{1}{p(n)}\end{aligned}$$

Still, a gap remains between the above definition and Definition 5.2.1: the above condition refers only to one possible deterministic algorithm U, whereas Definition 5.2.1 refers to all probabilistic polynomial-time algorithms. To close the gap, we first observe that (by Propositions 5.2.7 and 5.2.6) Definition 5.2.1 is equivalent to a form in which one only quantifies over deterministic polynomial-time algorithms A. We conclude by observing that one can code any algorithm A (and polynomial time-bound) referred to by Definition 5.2.1 in the auxiliary input (i.e., $h(X_n)$) given to U.

Exercise 9: *In continuation to Exercise 8*, consider a definition in which the transformation T (of the circuit family $\{C_n\}_{n\in\mathbb{N}}$ to the circuit family $\{C'_n\}_{n\in\mathbb{N}}$) is not required to (even) be computable.[28] Clearly, the new

[28] Equivalently, one may require that for any polynomial-size circuit family $\{C_n\}_{n\in\mathbb{N}}$ there exists a polynomial-size circuit family $\{C'_n\}_{n\in\mathbb{N}}$ satisfying the above inequality.

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definition is not stronger than the one in Exercise 8. Show that the two definitions are in fact equivalent.

Guideline: Use the furthermore-clause of Proposition 5.2.7 to show that the new definition implies indistinguishability of encryptions, and conclude by applying Proposition 5.2.6 and invoking Exercise 8.

Exercise 10: *An alternative formulation of Definition 5.2.3*: Prove that Definition 5.2.3 remains unchanged when supplying the circuit with auxiliary-input. That is, an encryption scheme satisfies the modified Definition 5.2.3 if and only if

for every polynomial-size circuit family $\{C_n\}$, every polynomial p, all sufficiently large n and every $x, y \in \{0,1\}^{\text{poly}(n)}$ (i.e., $|x| = |y|$) and $z \in \{0,1\}^{\text{poly}(n)}$,

$$|\mathsf{Pr}\left[C_n(z, E_{G_1(1^n)}(x))=1\right] - \mathsf{Pr}\left[C_n(z, E_{G_1(1^n)}(y))=1\right]| < \frac{1}{p(n)}$$

(Hint: incorporate z in the circuit C_n.)

Exercise 11: *Equivalence of the security definitions in the public-key model*: Prove that a public-key encryption scheme is semantically secure if and only if it has indistinguishable encryptions.

Exercise 12: *The technical contents of semantic security*: The following explains the lack of computational requirements regarding the function f, in Definition 5.2.1. Prove that an encryption scheme, (G, E, D), is (semantically) secure (in the private-key model) if and only if the following holds:

There exists a probabilistic polynomial-time algorithm A' so that for every $\{X_n\}_{n\in\mathbb{N}}$ and $h : \{0,1\}^* \to \{0,1\}^*$ as in Definition 5.2.1, the following two ensembles are computationally indistinguishable.

1. $\{E_{G_1(1^n)}(X_n), 1^{|X_n|}, h(X_n)\}_{n\in\mathbb{N}}$.
2. $\{A'(1^{|X_n|}, h(X_n))\}_{n\in\mathbb{N}}$.

Formulate and prove an analogous claim for the public-key model.

Guideline: We care mainly about the fact that the above definition implies semantic security. The other direction can be proven analogously to the proof of Proposition 5.2.7.

Exercise 13: *Equivalent formulations of semantic security*:

1. Prove that Definition 5.2.1 remains unchanged if we restrict the function h to depend only on the length of its input (i.e., $h(x) = h'(|x|)$ for some $h' : \mathbb{N} \to \{0,1\}^*$).

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2. Prove that Definition 5.2.1 remains unchanged if we may restrict the function h and the probability ensemble $\{X_n\}_{n\in\mathbb{N}}$ so that they are computable (resp., sampleable) by polynomial-size circuits.

   Guideline (Part 1): Prove that this special case (i.e., obtained by the restriction on h) is equivalent to the general one. This follows by combining Propositions 5.2.7 and 5.2.6. Alternatively, this follows by considering all possible probability ensembles $\{X'_n\}_{n\in\mathbb{N}}$ obtained from $\{X_n\}_{n\in\mathbb{N}}$ by conditioning that $h(X_n) = a_n$ (for every possible sequence of a_n's).

   Guideline (Part 2): The claim regarding h follows from Part 1. To establish the claim regarding X_n, observe that (by Propositions 5.2.7 and 5.2.6) we may consider the case in which X_n ranges over two strings.

Exercise 14: *A variant on Exercises 12 and 13.1*: Prove that an encryption scheme, (G, E, D), is (semantically) secure (in the private-key model) if and only if the following holds.

> For every probabilistic polynomial-time algorithm A there exists a probabilistic polynomial-time algorithm A' such that for every ensemble $\{X_n\}_{n\in\mathbb{N}}$, with $|X_n| = \text{poly}(n)$, and polynomially bounded h' the following two ensembles are computationally indistinguishable.
>
> 1. $\{A(E_{G_1(1^n)}(X_n), 1^{|X_n|}, h'(|X_n|))\}_{n\in\mathbb{N}}$.
> 2. $\{A'(1^{|X_n|}, h'(|X_n|))\}_{n\in\mathbb{N}}$.

(Indeed, since $|X_n|$ is constant, so is $h'(|X_n|)$. So an equivalent form is obtained by replacing $h'(|X_n|)$ with a $\text{poly}(n)$-bit long string v_n.)

Formulate and prove an analogous claim for the public-key model.

Guideline: Again, we care mainly about the fact that the above implies semantic security. The easiest proof of this direction is by applying Propositions 5.2.7 and 5.2.6. A more interesting proof is obtained by using Exercise 12: Indeed, the current formulation is a special case of the formulation in Exercise 12, and so we need to prove that it implies the general case. The latter is proven by observing that otherwise – using an averaging argument – we derive a contradiction in one of the residual probability spaces defined by conditioning on $h(X_n)$ (i.e., $(X_n|h(X_n) = v)$ for some v).

Exercise 15: *Semantic security with respect to relations*: The formulation of semantic security in Definition 5.2.1 refers to computing a function of the plaintext. Here we present a (related) definition that refers to finding strings that are in a certain relation to the plaintext. Note that unlike in Definition 5.2.1, here we consider only efficiently recognizable relations. Specifically, we require the following:

> For every probabilistic polynomial-time algorithm A there exists a probabilistic polynomial-time algorithm A' such that for every

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ensemble $\{X_n\}_{n\in\mathbb{N}}$, with $|X_n| = \text{poly}(n)$, every polynomially-bounded function $h : \{0,1\}^* \to \{0,1\}^*$, every polynomially-bounded relation R that is recognizable by a (non-uniform) family of polynomial-size circuits, every polynomial $p(\cdot)$ and all sufficiently large n

$$\begin{aligned}&\mathsf{Pr}\left[(X_n, A(E_{G_1(1^n)}(X_n), 1^{|X_n|}, h(X_n))) \in R\right] \\ &\quad < \mathsf{Pr}\left[(X_n, A'(1^{|X_n|}, h(X_n))) \in R\right] + \frac{1}{p(n)}\end{aligned}$$

1. Prove that the above definition is in fact equivalent to the standard definition of semantic security.
2. Show that if the computational restriction on the relation R is removed then so encryption scheme can satisfy the resulting definition.

Formulate and prove analogous claims for the public-key model.

Guideline (for Part 1): Show that the new definition is equivalent to indistinguishability of encryptions. Specifically, follow the proofs of Propositions 5.2.6 and 5.2.7, using the circuits guaranteed for R in the first proof, and noting that the second proof holds intact.

Guideline (for Part 2): Consider the relation $R = \{(x, E_e(x)) : |x| = 2|e|\}$, and the distribution $X_n = U_{2n}$. (Note that if the encryption scheme is semantically secure then this R is not recognizable by small circuits.)

Exercise 16: *Another equivalent definition of security*: The following exercise is interesting mainly for historical reasons. In the definition of semantic security appearing in [141], the term $\max_{u,v}\{\mathsf{Pr}[f(X_n) = v|h(X_n) = u]\}$ appears instead of the term $\mathsf{Pr}[A'(1^{|X_n|}, h(X_n)) = f(X_n)]$. That is, it is required that

for every probabilistic polynomial-time algorithm A every ensemble $\{X_n\}_{n\in\mathbb{N}}$, with $|X_n| = \text{poly}(n)$, every pair of polynomially-bounded functions $f, h : \{0,1\}^* \to \{0,1\}^*$, every polynomial $p(\cdot)$ and all sufficiently large n

$$\begin{aligned}&\mathsf{Pr}\left[A(E_{G_1(1^n)}(X_n), 1^{|X_n|}, h(X_n)) = f(X_n)\right] \\ &\quad < \max_{u,v}\{\mathsf{Pr}\left[f(X_n) = v|h(X_n) = u\right]\} + \frac{1}{p(n)}\end{aligned}$$

Prove that the above formulation is in fact equivalent to Definition 5.2.1.

Guideline: First, note that the above definition implies Definition 5.2.1 (since $\max_{u,v}\{\mathsf{Pr}[f(X_n) = v|h(X_n) = u]\} \geq \mathsf{Pr}[A'(h(X_n), 1^n, |X_n|) = f(X_n)]$, for every algorithm A'). Next note that in the *special case*, in which X_n satisfies $\mathsf{Pr}[f(X_n) = 0|h(X_n) = u] = \mathsf{Pr}[f(X_n) = 1|h(X_n) = u] = \frac{1}{2}$, for all u's,

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the above terms are equal (since A' can easily achieve success probability 1/2 by simply always outputting 1). Finally, combining Propositions 5.2.7 and 5.2.6. infer that it suffices to consider only the latter special case.

Exercise 17: *Semantic security with a randomized h*: The following syntactic strengthening of semantic security is important in some applications. Its essence is in considering information *related* to the plaintext, in the form of a related random variable, rather than partial information about the plaintext (in the form of a function of it). Prove that an encryption scheme, (G, E, D), is (semantically) secure (in the private-key model) if and only if the following holds.

For every probabilistic polynomial-time algorithm A there exists a probabilistic polynomial-time algorithm A' such that for every $\{(X_n, Z_n)\}_{n\in\mathbb{N}}$, with $|(X_n, Z_n)| = \text{poly}(n)$, where Z_n may dependent arbitrarily on X_n, and f, $p(\cdot)$ and n as in Definition 5.2.1

$$\begin{aligned}\Pr&\left[A(E_{G_1(1^n)}(X_n), 1^{|X_n|}, Z_n) = f(X_n)\right]\\ &< \Pr\left[A'(1^{|X_n|}, Z_n) = f(X_n)\right] + \frac{1}{p(n)}\end{aligned}$$

That is, the auxiliary input $h(X_n)$ of Definition 5.2.1 is replaced by the random variable Z_n. Formulate and prove an analogous claim for the public-key model.

Guideline: Definition 5.2.1 is clearly a special case of the above. On the other hand, the proof of Proposition 5.2.6 extends easily to the above (seemingly stronger) formulation of semantic security.

Exercise 18: *Semantic Security w.r.t Oracles* (suggested by Boaz Barak): Consider an extended definition of semantic security in which, in addition to the regular inputs, the algorithms have oracle access to a function $H_x : \{0,1\}^* \to \{0,1\}^*$ (instead of being given the value $h(x)$). The H_x's have to be restricted to have polynomial (in $|x|$) size circuit. That is, *an encryption scheme, (G, E, D), is* extended-semantically secure (in the private-key model) *For every probabilistic polynomial-time algorithm A there exists a probabilistic polynomial-time algorithm A' such that for every ensemble $\{X_n\}_{n\in\mathbb{N}}$, with $|X_n| = \text{poly}(n)$, every polynomially-bounded function $f : \{0,1\}^* \to \{0,1\}^*$, every family of polynomial-sized circuits $\{H_x\}_{x\in\{0,1\}^*}$, every polynomial $p(\cdot)$ and all sufficiently large n*

$$\begin{aligned}\Pr&\left[A^{H_{X_n}}(E_{G_1(1^n)}(X_n), 1^{|X_n|}) = f(X_n)\right]\\ &< \Pr\left[A'^{H_{X_n}}(1^{|X_n|}) = f(X_n)\right] + \frac{1}{p(n)}\end{aligned}$$

The definition of public-key security is analogous.

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1. Show that if (G,E,D) has indistinguishable encryptions then it is extended-semantically secure.
2. Show that if no restriction are placed on the H_x's then no scheme can be extended-semantically secure (in this unrestricted sense).

Guideline (for Part 1): The proof is almost identical to the proof of Proposition 5.2.6: The algorithm A' forms an encryption of $1^{|X_n|}$, and invokes A on it. Indistinguishability of encryptions is used in order to establish that $A'^{H_{X_n}}(1^{|X_n|})$ performs essentially as well as $A^{H_{X_n}}(E(X_n))$. Otherwise, we obtain a distinguisher of $E(x_n)$ from $E(1^{|x_n|})$, for some infinite sequence of x_n's. In particular, the oracle H_{x_n} (being implementable by a small circuit) can be incorporated into a distinguisher.

Guideline (for Part 2): In such a case, H_x may be defined so that, when queried about a ciphertext, it reveals the decryption-key in use. This is obvious in case of public-key schemes, but is also doable in some private-key schemes (e.g., suppose that the ciphertext always contains a commitment to the private-key). Such an oracle allows A (which is given a ciphertext) to recover the corresponding plaintext, but does not help A' (which is given $1^{|X_n|}$) to find any information about the value of X_n.

Exercise 19: *Multiple messages of varying lengths*: In continuation to Section 5.2.4, generalize the treatment to encryption of multiple messages of varying lengths. Provide adequate definitions, and analogous results.

Guideline: For example, a generalization of the first item of Definition 5.2.8 postulates that for every pair of polynomials $t(\cdot)$ and $\ell(\cdot)$, and every probabilistic polynomial-time algorithm A, there exists a probabilistic polynomial-time algorithm A' such that for every ensemble $\{\overline{X}_n = (X_n^{(1)},...,X_n^{(t(n))})\}_{n\in\mathbb{N}}$, with $|X_n^{(i)}| \leq \ell(n)$, every pair of functions $f,h : \{0,1\}^* \to \{0,1\}^*$, every polynomial $p(\cdot)$ and all sufficiently large n

$$\Pr\left[A(\overline{E}_{G_1(1^n)}(\overline{X}_n),(1^{|X_n^{(1)}|},...,1^{|X_n^{(t(n))}|}),h(\overline{X}_n))=f(\overline{X}_n)\right]$$
$$< \Pr\left[A'((1^{|X_n^{(1)}|},...,1^{|X_n^{(t(n))}|}),h(\overline{X}_n))=f(\overline{X}_n)\right] + \frac{1}{p(n)}$$

Exercise 20: *Private-key encryption secure w.r.t exactly t messages*: In continuation to Proposition 5.2.12, show that if secure private-key encryption schemes exist then for every t there are such scheme that are secure with respect to the encryption of t messages but not with respect to the encryption of $t+1$ messages.

Guideline: Given an arbitrary private-key encryption scheme (G,E,D), consider the following private-key encryption scheme (G',E',D'):

- $G'(1^n) = (\overline{k},\overline{k})$, where $\overline{k} = (k_0,k_1,...,k_t)$ such that $(k_0,k_0) \leftarrow G(1^n)$ and $k_1,...,k_t$ are uniformly and independently selected in $\{0,1\}^n$ (w.l.o.g., $n = |k_0|$);
- $E'_{(k_0,k_1,...,k_t)}(x) = (E_{k_0}(x), r, \sum_{i=0}^{t} k_i r^i)$, where r is uniformly selected in $\{0,1\}^n$, and the arithmetics is of the field $GF(2^n)$;
- and $D'_{(k_0,k_1,...,k_t)}(y,r,v) = D_{k_0}(y)$.

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Exercise 21: *Known plaintext attacks*: Loosely speaking, in a known plaintext attack on a private-key (resp., public-key) encryption scheme the adversary is given some plaintext/ciphertext pairs in addition to some extra ciphertexts (without corresponding plaintexts). Semantic security in this setting means that whatever can be efficiently computed about the missing plaintexts, can be also efficiently computed given only the length of these plaintexts.

1. Provide formal definitions of security for private-key/public-key in both the single-message and multiple-message settings.
2. Prove that any secure public-key encryption scheme is also secure in the presence of known plaintext attack.
3. Prove that any private-key encryption scheme that is secure in the multiple-message setting is also secure in the presence of known plaintext attack.

Guideline (for Part 3): Consider a function h in the multiple-message setting that reveals some of the plaintexts.

Exercise 22: *On the standard notion of block-cipher*: A standard block-cipher is a triple, (G, E, D), of probabilistic polynomial-time algorithms that satisfies Definition 5.3.5 as well as $|E_e(\alpha)| = \ell(n)$ for every pair (e, d) in the range of $G(1^n)$ and every $\alpha \in \{0,1\}^{\ell(n)}$.

1. Prove that a standard block-cipher cannot be semantically secure (in the multiple-message model). Furthermore, show that any semantically secure encryption scheme must employ ciphertexts that are longer than the corresponding plaintexts.
2. Present a state-based version of the definition of a standard block-cipher, and note that Construction 5.3.3 satisfies it.

Guideline (for Part 1): Consider the encryption of a pair of two identical messages versus the encryption of a pair of two different messages, and use the fact that E_e must be a permutation of $\{0,1\}^{\ell(n)}$. Extend the argument to any encryption scheme in which plaintexts of length $\ell(n)$ are encrypted by ciphertexts of length $\ell(n)+O(\log n)$, observing that otherwise most plaintexts have only poly(n)-many ciphertexts under E_e.

Exercise 23: *A secure private-key encryption scheme*: Assuming that F is pseudorandom with respect to polynomial-size circuits, prove that Construction 5.3.12 constitutes a private-key encryption scheme.

Guideline: Adapt the proof of Proposition 5.3.10. This requires bounding the probability that for t uniformly selected $r^{(j)}$'s there exists $j_1, j_2 \in \{1,...,t\}$ and $k_1, k_2 \in \{1,...,t\}$ such that $r^{(j_1)} + k_1 \equiv r^{(j_2)} + k_2 \pmod{2^n}$.

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Exercise 24: *The Blum-Goldwasser public-key encryption scheme* was presented in Construction 5.3.20 as a block-cipher (with arbitrary block length). Provide an alternative presentation of this scheme as a full-fledged encryption scheme (rather than a block-cipher), and prove its security (under the factoring assumption).

> **Guideline:** In the alternative presentation, the values of d_P and d_Q cannot be determined at key-generation time, but are rather computed by the decryption process. (This means that decryption requires two additional modular exponentiations.)

Exercise 25: *Restricting the ensembles* $\{h_e\}_{e\in\{0,1\}^*}$ *and* $\{X_e\}_{e\in\{0,1\}^*}$ *in Definition 5.4.1*:

1. Show that if one allows arbitrary function ensembles $\{h_e\}_{e\in\{0,1\}^*}$ in Definition 5.4.1 then no encryption scheme can satisfy it.
2. Show that if one allows arbitrary function ensembles $\{X_e\}_{e\in\{0,1\}^*}$ in Definition 5.4.1 then no encryption scheme can satisfy it, even if one uses only a single function h that is polynomial-time computable.

> **Guideline:** For Part 1, consider the functions $h_e(x) = d$, where d is a decryption-key corresponding to the encryption-key e. For Part 2, consider the random variable $X_e = (d, U_{|e|})$, where d is as before, and the function $h(x', x'') = x'$.

Exercise 26: *An alternative formulation of Definition 5.4.1*: Show that the following formulation of the definition of admissible ensembles $\{h_e\}_e$ and $\{X_e\}_e$ is equivalent to the one in Definition 5.4.2:

- There is a non-uniform polynomial-time algorithm (i.e., a non-uniform family of polynomial-size circuits) that maps a string $e \in \{0,1\}^*$ into a circuit that computes the corresponding function h_e. That is, on input e, the algorithm outputs a circuit C_e such that $C_e(x) = h_e(x)$ holds for all strings of length $\leq \text{poly}(|e|)$.
- There is a non-uniform polynomial-time algorithm that maps a string $e \in \{0,1\}^*$ into a circuit that samples the corresponding distributions X_e. That is, on input e, the algorithm outputs a circuit S_e such that $S_e(U_m)$ is distributed identically to X_e, where U_m denotes the uniform distribution over the set of strings of length $m = m(e)$.

Note that the above formulation is in greater agreement with the motivating discussion preceding Definition 5.4.2. The formulation in Definition 5.4.2 was preferred because of its greater simplicity.

> **Guideline:** Consider for example, the condition regarding $\{h_e\}$. The formulation in Definition 5.4.2 is shown to imply the one above by using a circuit family $\{A_n\}$ such that on input e (in the range of $G_1(1^n)$) the circuit A_n outputs the circuit $C_e(\cdot) \stackrel{\text{def}}{=} H_n(e, \cdot)$; that is, A_n has H_n hard-wired

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and just outputs it while fixing its first input to be e. On the other hand, given a circuit family $\{A_n\}$ that maps $e \mapsto C_e$ as above, we obtain a circuit H_n as required in the formulation of Definition 5.4.2 as follows. The circuit H_n has A_n hard-wired, and so on input (e,x), the circuit H_n first computes $C_e \leftarrow A_n(e)$, and the outputs $C_e(x)$.

Exercise 27: *Multiple-message security in context of key-dependent passive attacks*: Formulate multiple-message security generalizations of Definitions 5.4.1 and 5.4.2, and prove that both are equivalent to the single-message definitions.

Guideline: Note that admissibility for the multiple-message generalization of Definition 5.4.2 means that given an encryption key e, one can compute (via a polynomial-size circuit that depends only on $|e|$) a corresponding pair of sequences $((x_e^{(1)},...,x_e^{(t(|e|))}),(y_e^{(1)},...,y_e^{(t(|e|))}))$. Thus, ability to distinguish corresponding sequences of encryptions yields ability to distinguish encryptions to $x_e^{(i)}$ from encryptions to $y_e^{(i)}$, where the latter distinguisher generates the corresponding x-y hybrid (using the circuit guaranteed by the admissibility condition) and invokes the former distinguisher on the resulting sequence of encryptions.

Exercise 28: *Key-oblivious versus key-dependent passive attacks*: Assuming the existence of secure public-key encryption schemes, show that there exist one that satisfies the basic definition (i.e., as in Definition 5.2.2) but is insecure under key-dependent passive attacks (i.e., as in Definition 5.4.1).

Guideline: Given a scheme (G,E,D), define (G,E',D') such that $E'_e(x) = (1,E_e(x))$ if $x \neq e$ and $E'_e(x) = (0,x)$ otherwise (i.e., for $x = e$). Using Exercise 6 (which establishes that each encryption-key is generated with negligible probability), show that (G,E',D') satisfies Definition 5.2.2. Alternatively, use $G'(1^n) = ((r,G_1(1^n)),G_2(1^n))$, where r is uniformly distributed in $\{0,1\}^n$, which immediately implies that each encryption-key is generated with negligible probability.

Exercise 29: *Passive attacks versus Chosen Plaintext Attacks*: Assuming the existence of secure private-key encryption schemes, show that there exist one that is secure in the standard (multi-message) sense (i.e., as in Definition 5.2.8) but is insecure under a chosen plaintext attack (i.e., as in Definition 5.4.8).

Guideline: Given a scheme (G,E,D), define (G',E',D') such that

1. $G'(1^n) = ((k,r),(k,r))$, where $(k,k) \leftarrow G(1^n)$ and r is selected uniformly in $\{0,1\}^n$.
2. $E'_{(k,r)}(x) = (1,r,E_k(x))$ if $x \neq r$ and $E'_{(k,r)}(x) = (0,k,x)$ otherwise (i.e., for $x = r$).

Show that (G',E',D') is secure in the standard sense, and present a (simple but very "harmful") chosen message attack on it.

Exercise 30: *Chosen Plaintext Attacks versus Chosen Ciphertext Attacks*: Assuming the existence of secure private-key (resp., public-key) encryption

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schemes that are secure under a chosen plaintext attack, show that there exist one that is secure in the former sense but is not secure under a chosen ciphertext attack (even not in the a-priori sense).

Guideline: Given a scheme (G,E,D), define (G',E',D') such that $G'=G$ and

1. $E'_e(x)=(1,E_e(x))$ with probability $1-2^{-|e|}$ and $E'_e(x)=(0,x)$ otherwise.
2. $D'_d(1,y)=D_d(y)$ and $D'_d(0,y)=(d,y)$.

Recall that decryption is allowed to fail with negligible probability, and note that the construction is adequate for both public-key and private-key schemes. Alternatively, to obtain error-free decryption, define $E'_e(x)=(1,E_e(x))$, $D'_d(1,y)=D_d(y)$ and $D'_d(0,y)=(d,y)$. In case of private-key schemes, we may define $E'_k(k)=(0,1^{|k|})$ and $E'_k(x)=(1,E_k(x))$ for $x\neq k$.

Exercise 31: *The two versions of Chosen Ciphertext Attacks*: Assuming the existence of secure private-key (resp., public-key) encryption schemes that are secure under an a-priori chosen plaintext attack, show that there exist one that is secure in the former sense but is not secure under an a-posteriori chosen ciphertext attack.

Guideline: Given a scheme (G,E,D), define (G',E',D') such that $G'=G$ and

1. $E'_e(x)\leftarrow(b,E_e(x))$, where b is uniformly selected in $\{0,1\}$.
2. $D'_d(b,y)=D_d(y)$.

Exercise 32: *Multiple-challenge CCA security implies a-posteriori-CCA security*: Show that Definition 5.4.16 implies security under a-posteriori CCA,

Guideline: It is tempting to claim that it is immediate that Definition 5.4.13 is a special case of Definition 5.4.16, obtained when allowing only one challenge query. However, things are not so simple, because in Definition 5.4.13 the challenges are required to be identically distributed whereas in Definition 5.4.16 only computational indistinguishability is required. Instead, we suggest to show that Definition 5.4.14 (which is equivalent to Definition 5.4.13) is a special case of the (very) restricted case of Definition 5.4.16 discussed following the definition (i.e., a canonical adversary that makes a single query of the form $(S,\mathtt{0})$).

Exercise 33: *Equivalent forms of multiple-challenge CCA security*:

1. Consider a modification of Definition 5.4.16 in which challenge queries of the form (S,h) are answered by $(E_e(S(r)),h(r))$, rather than by $(E_e(S(r)),h(S(r)))$. Prove that the original definition is equivalent to the modified one.
2. Consider a modification of Definition 5.4.16 in which the challenge queries of the form (S,h) are replaced by two type of queries: partial-information queries of the form $(\mathtt{leak},h)$ that are answered by $h(r)$, and partial-encryption queries of the form $(\mathtt{enc},S)$ that are answered by $E_e(S(r))$. Prove that the original definition is equivalent to the modified one.

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> **Guideline:** Show how the modified model of Part 1 can emulate the original model (that's easy), and how the original model can emulate the modified model of Part 1 (e.g., replace the query (S, h) by the pair of queries $(S, 0)$ and $(\mathtt{id}, h)$). Next relate the models in Parts 1 and 2.

Exercise 34: *Computational restriction on the choice of input in the definition of adaptive NIZK*: Show that if Definition 5.4.22 is strengthened by waiving the computational bounds on Ξ then only trivial NIZKs (i.e., for languages in $\mathcal{BPP}$) can satisfy it.

> **Guideline:** Show that allowing a computationally-unbounded Ξ forces S_1 to generate a reference string that is statistically close to the uniform distribution. Thus, soundness implies weak simulation-soundness in the sense of Exercise 35.

Exercise 35: *Weak simulation-soundness can hold only with respect to computationally-bounded cheating provers*: Show that if Definition 5.4.24 is strengthened by waiving the computational bounds on Π then only trivial NIZKs (i.e., for languages in $\mathcal{BPP}$) can satisfy it.

> **Guideline:** Show that otherwise the two-stage simulation procedure can distinguish inputs in the language from inputs outside the language, because in the first case it produces an valid proof whereas in the second one it cannot do so. The latter fact is proved by showing that if S_2 (which also gets an auxiliary input s produced by S_1 along with the reference string) generates a valid proof then a computationally-unbounded prover may do the same by first generating s according to the conditional distribution induced by the reference string (and then invoking S_2).

Exercise 36: *Does weak simulation-soundness hold for all adaptive NIZKs?*

1. Detect the flaw in the following argument towards an affirmative answer: If weak simulation-soundness does not hold then we can distinguish a uniformly selected reference string (for which soundness holds) from a reference string generated by S_1 (for which soundness does not hold).
2. Assuming the existence of one-way permutations (and adaptive NIZKs), show an adaptive NIZK with a suitable simulator such that weak simulation-soundness does not hold.
3. (By Boaz Barak and Yehuda Lindell): For languages of pairs (α, x) such that one can generate α's along with suitable trapdoors $t(\alpha)$'s that allow to determine whether or not inputs of the form $(\alpha, \cdot)$ are in the language, define a weaker notion of simulation-soundness in which a random α is generated and then one is required to produce valid proofs for a no-instance of the form $(\alpha, \cdot)$ with respect to a reference-string generated by S_1. Provide a clear definition, prove that it is satisfied by any adaptive NIZK for the corresponding language, and show that this definition suffices for proving Theorem 5.4.27.

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Guideline (Part 1): The existence of an efficient $C = (\Xi, \Pi)$ that violates weak simulation-soundness only means that for reference string generated by S_1 the cheating Π generates a valid proof for a no-instance selected by Ξ. When C is given a uniformly selected reference string it may either fail to produce a valid proof or may produce a valid proof for a yes-instance. However, we cannot necessarily distinguish no-instances from yes-instances (see, for example, Part 2). This gap is eliminated in Part 3.

Guideline (Part 2): Given a one-way permutation f with a corresponding hard-core b, consider the pseudorandom generator $G(s) \stackrel{\rm def}{=} b(s)b(f(s)) \cdots b(f^{2|s|-1}(s))f^{2|s|}(s)$ (see proof of Proposition 5.3.14). Let L denote the set of strings that are not images of G, and note that L is in $\mathcal{NP}$. Given any adaptive NIZK for L, denoted (P, V), consider the modification (P', V') such that $P'(x, w, (r_1, r_2)) = P(x, w, r_1)$ and $V'(x, (r_1, r_2), \pi) = 1$ if either $V(x, \pi, r_1) = 1$ or $x = r_2$. The modified simulator is derived by $S'_1(1^n) \leftarrow ((r_1, r_2), s)$, where $(r_1, s) \leftarrow S_1(1^n)$ and $r_2 \leftarrow G(U_n)$ (and $S'_2(x, s) = S_2(x.s)$). Verify that the modified algorithms satisfy the definition of an adaptive NIZK, and note that weak simulation-soundness is easily violated by $\Xi(r_1, r_2) = r_2$ (and any Π).

Guideline (Part 3): For an encryption scheme (G, E, D), we are interested in the "consistency language" of pairs (α, x) such that $\alpha = (e_1, e_2)$ is a pair of encryption-keys (with corresponding trapdoor being the corresponding pair of decryption-keys) and $x = (y_1, y_2)$ is a pair of corresponding encryptions of the same plaintext (i.e., $\exists s, s_1, s_2$ such that $E_{e_i}(s, s_i) = y_i$ for $i = 1, 2$).

Exercise 37: *On defining non-malleability*: Show that when defining non-malleability (i.e., in Definitions 5.4.32 and 5.4.33) it is essential to prevent A from outputting the ciphertext that is given to it.

Guideline: Consider the identity relation, a constant function h, and let X_n be uniform over $\{0,1\}^n$. Note that A gets $(e, E_e(X_n), 1^n)$, whereas A' only gets $(e, 1^n)$.

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