A Candidate Counterexample to the Easy Cylinders Conjecture

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Abstract

We present a candidate counterexample to the easy cylinders conjecture, which was recently suggested by Manindra Agrawal and Osamu Watanabe (see *ECCC*, TR09-019). Loosely speaking, the conjecture asserts that any 1-1 function in $\mathcal{P}$/poly can be decomposed into "cylinders" of sub-exponential size that can each be inverted by some polynomial-size circuit. Although all popular one-way functions have such easy (to invert) cylinders, we suggest a possible counterexample. Our suggestion builds on the candidate one-way function based on expander graphs (see *ECCC*, TR00-090), and essentially consists of iterating this function polynomially many times.

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1 The Easy Cylinders Conjecture

Manindra Agrawal and Osamu Watanabe [2, Sec. 4] have recently suggested the following interesting conjecture. The conjecture refers to the notion of an easy cylinder, defined next, and asserts that *every 1-1 and length-increasing function in $\mathcal{P}/\mathrm{poly}$ has easy cylinders.*

Definition 1 (easy cylinders, simplified[1]): *A length function $\ell:\mathbb{N}\to\mathbb{N}$ is* admissible *if the mapping $n \mapsto \ell(n)$ can be computed in $\mathrm{poly}(n)$-time and there exists a constant $\epsilon > 0$ such that $\ell(n) \in [n^\epsilon, n - n^\epsilon]$. A function f* has easy cylinders *if for some admissible length function ℓ there exists mappings $\sigma_1, \sigma_2 : \{0,1\}^* \to \{0,1\}^*$ such that the following conditions hold:*

1. *For every x, it holds that $|\sigma_1(x)| = \ell(|x|)$ and $|\sigma_2(x)| = |x| - \ell(|x|)$.*
2. *The function $\sigma(x) = (\sigma_1(x), \sigma_2(x))$ is 1-1, polynomial-time computable and polynomial-time invertible. The* cylinders defined by σ_1 *consists of the collection of sets $\{\sigma_1^{-1}(x')|_n : x' \in \{0,1\}^{\ell(n)}\}_{n\in\mathbb{N}}$ where $\sigma_1^{-1}(x')|_n \stackrel{\rm def}{=} \{x \in \{0,1\}^n : \sigma_1(x) = x'\}$.*
3. *For every $n \in \mathbb{N}$ and $x' \in \{0,1\}^{\ell(n)}$, there exists a $\mathrm{poly}(n)$-size circuit $C = C_{x'}$ such that for every $x \in \sigma_1^{-1}(x')|_n$ it holds that $C(f(x)) = \sigma_2(x)$.*

That is, when restricted to any such cylinder, the function f is easy to invert.

Needless to say, the existence of easy cylinders is interesting only in the case that f is not polynomial-time invertible. Agrawal and Watanabe noted that all popular candidates one-way functions have easy cylinders. Generalizing their observations (and going somewhat beyond them), we first present four classes of functions that are conjectured to be one-way and still have easy cylinders. Next (in Section 3), we present our candidate counterexample.

2 Four Classes of Functions that have Easy Cylinders

The first class generalizes the multiplication function (i.e., $(x', x'') \mapsto x' \cdot x''$). This class consists of (polynomial-time computable) functions f having the form $f(x) = g(\sigma_1(x), \sigma_2(x))$, where the σ_i's satisfy the first two conditions in Definition 1 and the mapping $(x', x'') \mapsto (x', g(x', x''))$ is easy to invert (by an efficient algorithm I). Clearly, the cylinders defined by σ_1 are easy (since we can have $C_{\sigma_1(x)}(f(x)) = I(\sigma_1(x), f(x))$).

The second class consists of functions that are derived from collections of finite one-way functions having a dense index set and dense domains.[2] For example, consider the DLP-based collection that consists of the functions $\{f_{p,g} : \mathbb{Z}_p \to \mathbb{Z}_p\}_{(p,g)}$, where p is prime, g is a generator of the multiplicative group modulo p, and $f_{p,g}(z) = g^z \bmod p$. For simplicity, we consider collections of the form $\{f_\alpha : \{0,1\}^{|\alpha|} \to \{0,1\}^{|\alpha|}\}_{\alpha\in I}$, where the index set I is dense (i.e., $|I \cap \{0,1\}^n| > 2^n/\mathrm{poly}(n)$). The one-wayness condition means that, for a typical $\alpha \in I$, the function f_α is hard to invert, and so the "natural" cylinders defined by $\sigma_1(\alpha, z) = \alpha$ are not easy. Nevertheless, the function $F(\alpha, z) = (\alpha, f_\alpha(z))$, which is (weakly) one-way, has easy cylinders that are defined by $\sigma_1(\alpha, z) = z$;

[1]Our formulation is a special case of the formulation in [2], but we believe that our candidate counterexample also holds for the definition in [2].

[2]Indeed, we consider a restricted case of [4, Def. 2.4.3]. On the other hand, note that any collection of finite one-way functions with dense domains can be converted into a collection of finite one-way functions over the set of all strings of a fixed length. Thus, we may freely use the latter.

specifically, by virtue of the circuits C_z that (easily) extract $\alpha = \sigma_2(\alpha, z)$ from $F(\alpha, z)$ (since $F(\alpha, z) = (\alpha, f_\alpha(z))$).

The third class consists of functions that are derived from collections of trapdoor one-way permutations. Here it is essential to have an non-trivial index-sampling algorithm, denoted I, that samples the index set along with corresponding trapdoors; that is, the coins used to sample an index-trapdoor pair cannot be used as the index (because the trapdoor must be hard to recover from the index). Let $I_1(r)$ denote the index sampled on coins r, and let $I_2(r)$ denote the corresponding trapdoor (and suppose that the domains are dense as before, which indeed restricts [4, Def. 2.4.4]). Then, the function $F(r, z) = (I_1(r), f_{I_1(r)}(z))$ is (weakly) one-way, but it has easy cylinders that are defined by $\sigma_1(r, z) = r$ (using the circuit $C_r(F(r, z)) = f^{-1}_{I_1(r)}(z)$, which in turn uses the trapdoor $I_2(r)$ that corresponds to the index $I_1(r)$).

The last class consists of all functions that computable in $\mathcal{NC}_0$; that is, functions in which each output bit depends on a constant number of input bits. Recall that this class is widely conjectured to contain one-way functions (cf., the celebrated work of Applebaum, Ishai, and Kushilevitz [1]). For every such function $f : \{0,1\}^n \to \{0,1\}^n$, letting σ_1 be the projection of the n-bit input on $n - n^{1/3}$ random coordinates, with high probability, we obtain easy cylinders.[3] The reason is that, with high probability, no output bit of the function is influenced by more than one of the $n^{1/3}$ remaining coordinates (and so the residual function $f(x)$ obtained after fixing the value of $\sigma_1(x)$ is essentially a projection).

3 Our Candidate Counterexample to the Conjecture

We note that the last class of functions (i.e., $\mathcal{NC}_0$) contains the candidate one-way function suggested by us [3]. However, *we believe that iterating this function for a polynomial* (or even linear) *number of times yields a function that has no easy cylinders.* For sake of self-containment, we recall the proposal of [3], hereafter referred to as the basic function.

The basic function. We consider a collection of functions $\{f_n : \{0,1\}^n \to \{0,1\}^n\}_{n\in\mathbb{N}}$ such that f_n is based a collection of $d(n)$-subsets, $S_1, ..., S_n \subset [n] \stackrel{\rm def}{=} \{1, ..., n\}$, and a predicate $P : \{0,1\}^{d(n)} \to \{0,1\}$ (as follows).

1. The function d is relatively small; that is, $d = O(\log n)$ or even $d = O(1)$, but $d > 2$.

2. The predicate $P : \{0,1\}^d \to \{0,1\}$ should be thought of as being a random predicate. That is, it will be randomly selected, fixed, and "hard-wired" into the function. For sure, P should *not* be linear, nor depend on few of its bit locations.

3. The collection $S_1, ..., S_n$ should be expanding: specifically, for *some* k, the union of every k subsets should cover at least $k + \Omega(n)$ elements of $[n]$ (i.e., for every $I \subset [n]$ of size k it holds that $|\bigcup_{i\in I} S_i| \geq k + \Omega(n)$). Specifically, it is suggested to have S_i be the set of neighbors of the $i^{\rm th}$ vertex in a d-regular expander graph.

[3] In fact, the argument remain intact as long as $\ell(n) = n - o(n^{1/2})$ (rather than $\ell(n) = n - n^{1/3}$). Actually, using $n - o(n^{2/3})$ random coordinates would work too, since then (w.h.p.) no output bit of the function is influenced by more than two of the $o(n^{2/3})$ remaining coordinates (and so a 2SAT solver can invert the residual function on each of the individual cylinders).

For $x = x_1 \cdots x_n \in \{0,1\}^n$ and $S \subset [n]$, where $S = \{i_1, i_2, ..., i_t\}$ and $i_j < i_{j+1}$, we denote by x_S the projection of x on S; that is, $x_S = x_{i_1} x_{i_2} \cdots x_{i_t}$. Fixing P and $S_1, ..., S_n$ as above, we define

$$f_n(x) \stackrel{\text{def}}{=} P(x_{S_1}) P(x_{S_2}) \cdots P(x_{S_n}). \tag{1}$$

Note that we think of d as being relatively small (i.e., $d = O(\log n)$), and hope that the complexity of inverting f_n is related to $2^{n/O(1)}$. Indeed, the hardness of inverting f_n cannot be due to the hardness of inverting P, but is rather supposed to arise from the combinatorial properties of the collection of sets $\{S_1, ..., S_n\}$ (as well as from the combinatorial properties of predicate P). In general, the conjecture is that the complexity of the inversion problem (for f_n constructed based on such a collection) is exponential in the "net expansion" of the collection (i.e., the cardinality of the union minus the number of subsets).

We note that a non-uniform complexity version of this basic function (or rather the sequence of f_n's) may use possibly different predicates (i.e., different P_i's) for the different n applications of P in Eq. 1.

The iterated function – the vanilla version. The candidate counterexample, F, is defined by $F(x) = f_{|x|}^{p(|x|)}(x)$, where p is some fixed polynomial (e.g., $p(n) = n$) and $f_n^{i+1}(x) = f_n(f_n^i(x))$ (and $f_n^1(x) = f_n(x)$). We conjecture that this function has no easy cylinders.

The iterated function, revisited. One possible objection to the foregoing function F as a counterexample to the easy cylinder conjecture is that F is unlikely to be 1-1. Although we believe that the essence of the easy cylinder conjecture is unrelated to the 1-1 property, we point out that this property may be obtained by suitable modifications. One possible modification that may yield a 1-1 function is obtained by prepending the application of F with an adequate expanding function (e.g., a function that stretches n-bit long strings to $m(n)$-bit long strings, where m is a polynomial or even a linear function). Specifically, for a function $m : \mathbb{N} \to \mathbb{N}$ such that $m(n) \in [2n, \text{poly}(n)]$, we define $g_n : \{0,1\}^n \to \{0,1\}^{m(n)}$ analogously to Eq. 1 (i.e., here we use an expanding collection of $m(n)$ subsets), and let $F'(x) = F(g_{|x|}(x))$; that is, for every $x \in \{0,1\}^n$, we have $F'(x) = f_{m(n)}^{p(m(n))}(g_n(x))$.

4 Conclusion

Starting with the aforementioned non-uniform complexity version of the basic function f_n, and applying different incarnations of this function in the different iterations, we actually obtain a rather generic counterexample. Alternatively, we may directly consider functions $F_n : \{0,1\}^n \to \{0,1\}^{m(n)}$ such that the function F_n has a poly(n)-sized circuit. Note that such a circuit may be viewed as a composition of polynomially many circuits in $\mathcal{NC}_0$, which in tern may be viewed as basic functions. Furthermore, a random poly(n)-sized circuit is likely to be decomposed to $\mathcal{NC}_0$ circuits that correspond to basic functions in which the collection of sets (of input bits that influence individual output bits) are expanding. Needless to say, we believe that generic polynomial-size circuits have no easy cylinders.

It seems that the existence of easy cylinders in all popular candidate one-way functions is due to the structured nature of these candidates. Such a structure will not exist in the generic case, and so *we conjecture that the Easy Cylinders Conjecture is false.*

References

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