GARLAND'S ESTIMATE REVISITED—LAPLACIANS OF SIMPLICIAL COMPLEXES

BASED ON [GW, SECTION 3.1]

Interpreted by Orr Paradise and Noam Wies - mistakes are likely ours.

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1. Reminders and definitions

Let’s start with a few reminders. Let $X$ be a pure $k$-dimensional simplicial complex, and assume that $X_0$ is linearly ordered.

**Definition 1.1.** Let $F \in X_i$ and $G \in X_{i-1}$. Denote $F = \{v_0, \ldots, v_i\}$ where $v_0 < \cdots < v_i$. Then the **oriented incidence number** of $G$ in $F$ is

$$[F : G] := \begin{cases} (-1)^j & F \setminus G = \{v_j\} \\ 0 & \text{else (}G \not\subseteq F\text{)} \end{cases}$$

The **boundary map** is the linear map $\partial_i$ defined on basis vectors by

$$\partial_i : C^{i+1} \rightarrow C^i \quad \partial_i(e_H) = \sum_{F \in X_i} [H : F] e_F$$

The **coboundary map** is the linear map $\delta_i$ defined by

$$\delta_i : C^i \rightarrow C^{i+1} \quad (\delta_i f)(H) := \sum_{F \in X_i} [H : F] \cdot f(F)$$

for $i \in \{-1, \ldots, \dim(X)\}$ and $\delta_i := 0$ otherwise.

**Definition 1.2.** The **degree** of a face $F \in X$ is defined as

$$\deg_X F := |\{H \in X | F \not\subseteq H\}|$$

**Remark 1.3.** If $X$ is pure and $k$-dimensional, then for all $i < k$, for all $F \in X_i$ we have $\deg_X F > 0$.

**Definition 1.4.** For any weight function $w : X_i \rightarrow \mathbb{R}_{\geq 0}$ we may define an **weighted inner product** on $C_i$ by

$$\langle f, g \rangle_w := \sum_{F \in X_i} w(F) \cdot f(F) \cdot g(F)$$

**Exercise 1.5.** For each $F \in X_i$ we define $e_F \in C_i$ by $e_F(G) = \begin{cases} 1 & F = G \\ 0 & F \neq G \end{cases}$. Then $\{e_F\}_{F \in X_i}$ is an orthonormal basis for $C_i$ w.r.t the weighted inner product when $w = 1$ (that is $w \equiv 1$), but it is not necessarily orthonormal w.r.t $w = \deg_X$.

**Notation 1.6.** We will be interested in the inner products derives when $w \equiv 1$ and when $w = \deg_X$, and specifically with adjoint maps with respect to these inner products. To avoid confusion, for a given $f \in C_i$ we denote its adjoint with respect to $\langle \cdot, \cdot \rangle_1$ by $f^\dagger$ and its adjoint with respect to $\langle \cdot, \cdot \rangle_{\deg_X}$ with $f^*$. 

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1
Fact 1.7. \( \delta_i^i = \partial_{i+1} \)

**Definition 1.8.** The \( i \)-th upper laplacian is the map defined
\[
L_i^{up}: C^i \to C^i \quad L_i^{up} := \partial_{i+1}\delta_i = \delta_i^i \delta_i
\]

The \( i \)-th normalized laplacian is the map defined
\[
\Delta_i^{up}: C^i \to C^i \quad \Delta_i^{up} := \delta_i^i \delta_i
\]

2. **Matrix representation of the normalized laplacian**

We now turn to compute the matrices corresponding to \( L_i^{up}, \Delta_i^{up} \) under the basis \( \{e_F\}_{F \in X_i} \) where \( e_F (G) = \begin{cases} 1 & F = G \\ 0 & F \neq G \end{cases} \)

**Definition 2.1.** \( D_i, W_i \) are the diagonal \( |X_i| \times |X_i| \) matrices defined
\[
\left( D_i \right)_{F,F} := \left| \{ H \in X_{i+1} | F \subseteq H \} \right|
\]
\[
\left( W_i \right)_{F,F} := \deg_X F
\]

For \( F, G \in X_i \) we say that \( F \sim G \) if \( F \cap G \in X_{i-1} \) and \( F \cup G \in X_{i+1} \). We then define the \( i \)-th adjacency matrix to be
\[
\left( A_i \right)_{F,G} := \begin{cases} -|F \cup G : F| |F \cup G : G| & F \sim G \\ 0 & \text{else} \end{cases}
\]

**Claim 2.2.** The matrix corresponding to \( L_i^{up} \) under \( \{e_F\}_{F \in X_i} \) is \( D_i - A_i \).

**Proof.** Since \( \{e_F\}_{F \in X_i} \) is an orthonormal basis w.r.t \( \langle \cdot, \cdot \rangle_1 \), then \( (L_i^{up})_{F,G} = (L_i^{up} e_F, e_G)_{X_i} \) for all \( F,G \in X_i \).

We have
\[
(L_i^{up})_{F,G} = (L_i^{up} e_F, e_G)_{X_i}
\]
\[
= \langle \partial_{i+1} \delta_i e_F, e_G \rangle_{X_i}
\]
\[
= \langle \delta_i^i \delta_i e_F, e_G \rangle_{X_i} = \langle \delta_i e_F, \delta_i e_F \rangle_{X_i}
\]
\[
= \sum_{H \in X_{i+1}} \delta_i e_F (H) \delta_i e_F (G)
\]
\[
= \sum_{H \in X_{i+1}} \left( \sum_{F' \in X_i} \left[ H : F' \right] e_F (F') \right) \left( \sum_{G' \in X_i} \left[ H : G' \right] e_G (G') \right)
\]
\[
= \sum_{H \in X_{i+1}} \left[ H : F \right] \left[ H : G \right]
\]

- **If** \( F = G \), **since** \( F \neq F \) **always**, then \( (A_i)_{F,F} = 0 \) and therefore
\[
\sum_{H \in X_{i+1}} \left[ H : F \right] \left[ H : G \right] = \left| \{ H \in X_{i+1} | F \subseteq H \} \right| = (D_i)_{F,F} = (D_i - A_i)_{F,F}
\]

- **If** \( F \neq G \), **then** \( (D_i)_{F,F} = 0 \)
  - **If** also \( F \sim G \), **then** \( [H : F] \) and \( [H : G] \) are nonzero only when \( H = F \cup G \), and then
  \[
  \sum_{H \in X_{i+1}} [H : F] [H : G] = [F \cup G : F] [F \cup G : G]
  \]
  \[
  = (D_i - A_i)_{F,G} = (D_i - A_i)_{F,G}
  \]
  - **Else** \( F \neq G \) and **then** \( [H : F] \neq 0 \) \( \implies [H : G] = 0 \) (because otherwise \( H = F \cup G \) which implies \( F \sim G \)) and then
  \[
  \sum_{H \in X_{i+1}} [H : F] [H : G] = 0 = (D_i - A_i)_{F,G}
  \]
Claim 2.3. Denote the matrix corresponding to $\delta_i$ under the bases $\{e_G\}_{G \in X_i}, \{e_F\}_{F \in X_{i+1}}$ by $M$. Then $M_{F,G} = [F : G]$.

Proof. Since the bases are orthonormal

$$M_{F,G} = \langle \delta_i e_G, e_F \rangle_1$$

$$= \sum_{F' \in X_{i+1}} (\delta_i e_G) (F') e_F (F')$$

$$= (\delta_i e_G) (F)$$

$$= \sum_{G' \in X_i} [F : G'] e_G (G')$$

$$= [F : G]$$

Corollary 2.4. Since the bases are orthonormal w.r.t $\langle \cdot, \cdot \rangle_1$, then the matrix corresponding to $\delta_i^\dagger$ under the bases $\{e_F\}_{F \in X_{i+1}}, \{e_G\}_{G \in X_i}$ is $M^\top$.

Claim 2.5. For $i < k$,

$$\delta_i^* (f)(G) = \sum_{F \in X_{i+1}} \frac{\deg_X F}{\deg_X G} [F : G] f (F)$$

Proof. Denote the map on the right hand side of the equation with $\rho$. To prove that $\rho = \delta_i^*$ it suffices to show that $\rho$ is adjoint to $\delta_i$, since the adjoint map is unique. Indeed

$$\langle \delta_i f, g \rangle_{\deg_X} = \sum_{F \in X_{i+1}} \frac{\deg_X F}{\deg_X G} [F : G] g (F)$$

Corollary 2.6. The $F,G$-th entry in the matrix corresponding to $\delta_i^*$ under $\{e_F\}_{F \in X_{i+1}}, \{e_G\}_{G \in X_i}$ is

$$\frac{\deg_X F}{\deg_X G} [F : G]$$

Proof. From the above

$$(\delta_i^* e_F)(G) = \sum_{F' \in X_{i+1}} \frac{\deg_X F'}{\deg_X G} [F' : G] e_F (F')$$

and therefore

$$\delta_i^* e_F = \sum_{G \in X_i} \frac{\deg_X F}{\deg_X G} [F : G] e_G$$

Claim 2.7. The matrix corresponding to $\Delta_i^{up}$ under $\{e_F\}_{F \in X_i}$ is $W_i^{-1} \delta_i^\top W_{i+1} \delta_i$
Proof. Recall that $W_i^{-1}$ is the $|X_i| \times |X_i|$ diagonal matrix where $(W_i^{-1})_{F,F} = \frac{1}{\deg_X F}$, and notice that $(\delta_i^\top)_{F,H} = [H : F]$ and so since $W_i^{-1}$, $W_i+1$ are diagonal

$$
(W_i^{-1} \delta_i^\top W_i+1)_{F,H} = \frac{1}{\deg_X F} [H : F] \deg_X H = \frac{\deg_X H}{\deg_X F} [H : F]
$$

therefore

$$
(W_i^{-1} \delta_i^\top W_i+1 \delta_i)_{F,G} = \sum_{H \in X_{i+1}} (W_i^{-1} \delta_i^\top W_i+1)_{F,H} (\delta_i)_{H,G}
$$

$$
= \sum_{H \in X_{i+1}} \frac{\deg_X H}{\deg_X F} [H : F] [H : G]
$$

$$
= \sum_{H \in X_{i+1}} (\delta_i^*)_{F,H} (\delta_i)_{H,G}
$$

$$
= (\delta_i^* \delta_i)_{F,G}
$$

\[ \square \]

Corollary 2.8. When $i = k - 1$ in the above, then the matrix is $I - D_{k-1}^{-1} A_{k-1}$

Proof. Checking the definition of $W_i$ we immediately get that $W_k = I$ and $W_{k-1} = D_{k-1}$. Therefore

$$
\Delta_{k-1}^{up} = W_{k-1}^{-1} \delta_{k-1}^\top W_k \delta_{k-1} = D_{k-1}^{-1} \delta_{k-1}^\top I \delta_{k-1}
$$

$$
= D_{k-1}^{-1} \delta_{k-1}^\top \delta_{k-1}
$$

$$
= D_{k-1}^{-1} L_{k-1}^{up}
$$

$$
= D_{k-1}^{-1} (D_{k-1} - A_{k-1})
$$

$$
= I - D_{k-1}^{-1} A_{k-1}
$$

\[ \square \]

3. Garland’s estimate revisited

From this point onwards, we are only interested in the inner product derived from $w = \deg_X$. That is, we let $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{\deg_X}$ and say that $f \perp g$ when $\langle f, g \rangle = 0$.

Definition 3.1. For a given $F \in X$, the link of $F$ is defined

$$
\lk F := \{ G \in X | F \cup G \in X \}
$$

If $F \in X_{k-2}$ then $\dim (\lk F) \leq 1$. That is, $\lk F$ is a graph. In this case we identify $A(\lk F) = A_0(\lk F)$, $D(\lk F) = D_0(\lk F)$, and $\Delta(\lk F) = \Delta_0(\lk F)$.

Exercise 3.2. Check that the definitions for $A, D, \Delta$ indeed coincide for graphs and simplicial complexes of dimension $\leq 1$.

Definition 3.3. For a pure $k$-dimensional complex $X$, let $F \in X_{k-2}$. Define $\Delta_{k-1}^{up,F}$ by nullifying all entries of $\Delta_{k-1}^{up,F}$ residing in a row or column corresponding to a face that does not contain $F$.

Additionally, for $f \in C^{k-1}(X)$ we define $f_F \in C^0(\lk F)$ by

$$
f_F (\{u\}) := [F \cup \{u\} : F] f (F \cup \{u\})
$$

Remark 3.4. Formally, $\Delta_{k-1}^{up,F}$ is defined by defining the $|X_{k-1}| \times |X_{k-1}|$ diagonal matrix $(\rho_F)_{G,G} = \begin{cases} 1 & F \subseteq G \\ 0 & \text{else} \end{cases}$ and letting

$$
\Delta_{k-1}^{up,F} := \rho_F \Delta_{k-1}^{up,F} \rho_F
$$
3.1. Lemmata (Lemma 10 in the paper). Before proving the theorem we wish to explore the relation between the local and global properties of the normalized laplacian.

Lemma 3.5. \[ \sum_{F \in X_{k-2}} \Delta_{k-1}^{up,F} (X) = \Delta_{k-1}^{up} (X) + (k-1) I \]

Proof. Let \( G, H \in X_{k-1} \).

If \( G = H \), since \((A_{k-1})_{G,G} = 0\) then \((\Delta_{k-1}^{up})_{G,G} = 1\) and therefore \((\Delta_{k-1}^{up} (X) + (k-1) I)_{G,G} = k\). On the other hand, \(|\{F \in X_{k-2}|F \subseteq G\}| = |G| = k\), since each \( F \) is obtained by removing an element of \( G \). If \( F \not\subset G \) we get that \((\Delta_{k-1}^{up,F})_{G,G} = 0\) and otherwise \((\Delta_{k-1}^{up,F})_{G,G} = 1\) (because as stated above the diagonal is always all 1s), we get that \( (\sum_{F \in X_{k-2}} \Delta_{k-1}^{up,F} (X))_{G,G} = k \).

If \( G \neq H \) then \((\Delta_{k-1}^{up} (X) + (k-1) I)_{G,H} = (\Delta_{k-1}^{up})_{G,H}\) and one of the following holds:

- If \((\Delta_{k-1}^{up})_{G,H} = 0\) then \( \forall F \in X_{k-2} \) \((\Delta_{k-1}^{up,F})_{G,H} = 0\) since \(\Delta_{k-1}^{up,F}\) is obtained by nullifying some elements of \(\Delta_{k-1}^{up}\). So the equality holds.
- If \((\Delta_{k-1}^{up})_{G,H} \neq 0\), since \((\Delta_{k-1}^{up})_{G,H} = (I - D_{k-1}^{-1}A_{k-1})_{G,H} = (-D_{k-1}^{-1}A_{k-1})_{G,H}\), we have \((A_{k-1})_{G,H} \neq 0\), therefore \( G \sim H \) which means that \( G \cap H \in X_{k-2} \). For all \( F \neq G \cap H \), \( F \not\subset G \) or \( F \not\subset H \) and so \((\Delta_{k-1}^{up,F})_{G,H} = 0\). Also \((\Delta_{k-1}^{up,G\cap H})_{G,H} = (\Delta_{k-1}^{up})_{G,H}\) and therefore \( (\sum_{F \in X_{k-2}} \Delta_{k-1}^{up,F} (X))_{G,H} = \sum_{F \in X_{k-2}} (\Delta_{k-1}^{up,F} (X))_{G,H} = (\Delta_{k-1}^{up})_{G,H}\).

\[ \square \]

Lemma 3.6. For \( u, v \in \text{lk}F \) let \( F_u := F \cup \{u\}, F_v := F \cup \{v\} \). Then \( (\Delta_{k-1}^{up,F})_{F_u,F_v} = [F_u : F][F_v : F](\Delta(\text{lk}F))_{u,v} \).

Proof. Notice that for all \( w \in \text{lk}F \) \( \deg F_w = \deg_{\text{lk}F} w \), because \( F \in X_{k-2} \) and so \( \dim_X(F_w) = k - 1 \) so it is \( \subseteq \) only in maximal faces, so \( F_w \subseteq G \) for \( G \in X_k \) iff \( G = F_u \cup \{w\} \in X_k \) iff \( \{w, w'\} \in \text{lk}F \).

Assume that \( u \neq v \), let \( F_{u,v} := F \cup \{u, v\} \) and assume \( F_{u,v} \in X \). We claim that \( [F_{u,v} : F_u][F_{u,v} : F_v] = -[F_u : F][F_v : F] \).

Indeed, assume WLOG \( u < v \) and let \( F_{u,v} = \{w_0 < \cdots < w_i = u < \cdots < w_j = v < \cdots < w_k\} \), then direct computation shows that \( [F_{u,v} : F_u] = (-1)^j, [F_{u,v} : F_v] = (-1)^i \), \( [F_u : F] = (-1)^i \), \( [F_v : F] = (-1)^{j-1} \), and so \( [F_{u,v} : F_u][F_{u,v} : F_v] = (-1)^j(-1)^i = -(1)^i(-1)^j(-1)^{i-1} = -[F_u : F][F_v : F] \).
And now, noticing that \((\Delta lkF)_{u,v} = (I - D^{-1}(lkF)A(lkF)) = -\frac{1}{\deg_{lkF} u}\) and that \(F_u \sim F_v\),

\[
\left(\Delta^\text{up,F}_{k-1}(X)\right)_{F_u,F_v} = (I - D^{-1}_k(X)A_{k-1}(X))_{F_u,F_v} = (-D^{-1}_k(X)A_{k-1}(X))_{F_u,F_v}
\]

\[
= -\frac{1}{\deg_X F_u} (A_{k-1}(X))_{F_u,F_v}
\]

\[
= \frac{[F_{u,v} : F_{u}][F_{u,v} : F_{v}]}{\deg_X F_u}
\]

\[
= \frac{[F_{u,v} : F_{u}][F_{u,v} : F_{v}]}{\deg_{lkF} u} = [F_{u} : F_{u}][F_{v} : F_{v}](\Delta lkF)_{u,v}
\]

For the case when \(u \neq v\) and \(F_{u,v} \notin X_k\), we have that \(\{u,v\} \notin lkF\) and thus \((A(lkF))_{u,v} = 0\) so \((\Delta lkF)_{u,v} = 0\). On the left side, \((\Delta^\text{up,F}_{k-1}(X))_{F_u,F_v} = 0\) since \(F_u \not\sim F_v\).

The final case is that \(u = v\). Then \((\Delta^\text{up,F}_{k-1}(X))_{F_u,F_u} = 1\) because we are on the diagonal and \(F \subseteq F_u\), and on the right side we similarly have \((\Delta lkF)_{u,u} = 1\) and so

\[
[F_{u} : F_{u}][F_{v} : F_{v}](\Delta lkF)_{u,u} = 1 \cdot 1 = 1
\]

\quad □

**Corollary 3.7.** For \(f \in C^{k-1}(X)\),

\[
\left\langle \Delta^\text{up,F}_{k-1}(X)f, f \right\rangle = \left\langle \Delta (lkF)f, f \right\rangle
\]

**Proof.** Identifying \(f \in C^{k-1}\) with its matrix representation under \(\{e_F\}_{F \in X_{k-1}}\), for all \(G \in X_{k-1}\) we have

\[
\left(\Delta^\text{up,F}_{k-1}(X)f\right)(G) = \sum_{H \in X_{k-1}} \left(\Delta^\text{up,F}_{k-1}\right)_{G,H}(f) = \sum_{H \in X_{k-1}} \left(\Delta^\text{up,F}_{k-1}\right)_{G,H} f(H)
\]

and similarly

\[
(\Delta lkF f)_{F_u}(u) = \sum_{v \in lkF} (\Delta lkF)_{u,v} f_{F_v}(v)
\]

and so

\[
\left\langle \Delta^\text{up,F}_{k-1}(X)f, f \right\rangle = \sum_{G \in X_{k-1}} \deg_X G \left(\Delta^\text{up,F}_{k-1}(X)f\right)(G) \cdot f(G)
\]

\[
= \sum_{G \in X_{k-1}} \deg_X G \sum_{H \in X_{k-1}} \left(\Delta^\text{up,F}_{k-1}\right)_{G,H} f(H) \cdot f(G)
\]

\[
(*) = \sum_{u \in lkF} \deg_{lkF} F_u \sum_{v \in lkF} \left(\Delta^\text{up,F}_{k-1}\right)_{F_u,F_v} \cdot [F_{u} : F_{v}] f_{F_v}(v) \cdot [F_{u} : F_{v}] f_{F_v}(u)
\]

de of \(f\) and since \(\deg_{lkF} F_u = \deg_{lkF} w\)

\[
= \sum_{u \in lkF} \deg_{lkF} F_u \left\langle F_{u} : F_{u} \right\rangle \sum_{v \in lkF} [F_{u} : F_{v}] \left\langle F_{v} : F_{v} \right\rangle (\Delta (lkF)_{u,v} f_{F_v}(v))
\]

\[
= \sum_{u \in lkF} \deg_{lkF} F_u f_{F}(u) \left\langle\Delta (lkF)\right\rangle_{u,v} f_{F}(v)
\]

\[
= \sum_{u \in lkF} \deg_{lkF} F_u f_{F}(u) (\Delta (lkF) f_{F})(u)
\]

\[
= \left\langle \Delta (lkF) f_{F}, f_{F} \right\rangle
\]
Claim 3.10. \( \sum_{F \in X_{k-2}} \langle f_F, f_F \rangle = k \langle f, f \rangle \)

Proof.

\[
\sum_{F \in X_{k-2}} \langle f_F, f_F \rangle = \sum_{F \in X_{k-2}} \sum_{u \in \text{lk} F} (\deg_{\text{lk} F} u) f_F (u) f_F (u)
\]

\[
= \sum_{F \in X_{k-2}} \sum_{u \in \text{lk} F} \deg_X F_u [F_u : F] f(F_u) f(F_u)
\]

\[
= \sum_{F \in X_{k-2}} \sum_{u \in \text{lk} F} \deg_X F_u f(F_u) f(F_u)
\]

\[
= \sum_{G \in X_{k-1}} k \deg_X G f(G) f(G)
\]

\[
= k \langle f, f \rangle
\]

where the penultimate equality is because each \( G \in X_{k-1} \) appears exactly \( k \) times in the two \( \sum \)'s, since for every \( u \in G \) there is exactly one \( F = G \setminus \{u\} \) such that \( G = F_u \), and on the other hand each \( G \) may be obtained in such a way. Since \( |G| = k \), we have that for each \( G \), \( \deg_X G f(G)^2 \) is summed over exactly \( k \) times.

\[\Box\]

**Theorem 3.11.** Let \( X \) be a pure \( k \)-dimensional complex and let \( \Delta_{k-1}^{up} = \Delta_{k-1}^{up} (X) \) be its normalized Laplacian. Assume that there exist \( \lambda_{\min}, \lambda_{\max} \) such that for all \( F \in X_{k-2} \)

\[
\lambda_{\min} \leq \lambda_2 (\Delta (kF)) \leq \lambda_{n-k+1} (\Delta (kF)) \leq \lambda_{\max}
\]
Then for all $f \in B^{k-1}(X)^\perp$

$$(1 + k\lambda_{\min} - k) \langle f, f \rangle \leq \langle \Delta_{k-1}^{up} f, f \rangle \leq (1 + k\lambda_{\max} - k) \langle f, f \rangle$$

and therefore all nontrivial eigenvalues of $\Delta_{k-1}^{up}$ on $B^{k-1}(X)^\perp$ lie in

$$[1 + k\lambda_{\min} - k, 1 + k\lambda_{\max} - k]$$

**Proof.** Let $f \in B^{k-1}(X)^\perp$. Define

$$S_f := \{ F \in X_{k-2} | \exists G \in X_{k-1} \text{ } \subseteq G \wedge f(G) \neq 0 \}$$

Let $F \in S_f$. We prove that

$$\lambda_{\min} \leq \frac{\langle \Delta (lkF) f_F, f_F \rangle}{\langle f_F, f_F \rangle} \leq \lambda_{\max}$$

- By definition of $S_f$ $f_F \neq 0$ so $f_F \in C^0(lkF) \setminus \{0\}$ so from 3.9

$$\lambda_{\max} \geq \lambda_{n-k+1} (\Delta (lkF)) = \max_{h \in C^0(lkF) \setminus \{0\}} \frac{\langle \Delta (lkF) h, h \rangle}{\langle h, h \rangle} \geq \frac{\langle \Delta (lkF) f_F, f_F \rangle}{\langle f_F, f_F \rangle}$$

- Observe that $D^{-1}(lkF) A (lkF) \mathbf{1} = \mathbf{1}$ and therefore

$$\Delta (lkF) \mathbf{1} = (I - D^{-1}(lkF) A (lkF)) \mathbf{1} = I - D^{-1}(lkF) A (lkF) \mathbf{1} = \mathbf{1} - \mathbf{1} = 0$$

From 3.8 $f_F \perp \mathbf{1}$, so from 3.9

$$\lambda_{\min} \leq \lambda_2 (\Delta (lkF)) = \max_{g \in C^0(lkF) \setminus \{0\}} \min_{0 \neq h \perp g} \frac{\langle \Delta (lkF) h, h \rangle}{\langle h, h \rangle}$$

Notice that if $g \perp \mathbf{1}$ then

$$\min_{0 \neq h \perp g} \frac{\langle \Delta (lkF) h, h \rangle}{\langle h, h \rangle} \leq \frac{\langle \Delta (lkF) \mathbf{1}, \mathbf{1} \rangle}{\langle \mathbf{1}, \mathbf{1} \rangle} = 0$$

If $\lambda_{\min} \leq 0$ the result is trivial so assuming $\lambda_{\min} > 0$ this implies that any $g \perp \mathbf{1}$ cannot be the maximizer. So the maximum is obtained at some $g \not\perp \mathbf{1}$, but then

Summing over all $F \in S_f$ we get

$$\lambda_{\min} \sum_{F \in S_f} \langle f_F, f_F \rangle \leq \sum_{F \in S_f} \langle \Delta (lkF) f_F, f_F \rangle \leq \lambda_{\max} \sum_{F \in S_f} \langle f_F, f_F \rangle$$

Notice that if $F \notin S_f$ then

$$G \in X_{k-1} \wedge F \subseteq G \implies f(G) = 0$$

therefore $f_F = 0$. Similarly if $F \in S_f$ then $f_F \neq 0$. In other words, $F \in S_f \iff f_F \neq 0$. Therefore

$$\lambda_{\min} \sum_{F \in X_{k-2}} \langle f_F, f_F \rangle \leq \sum_{F \in X_{k-2}} \langle \Delta (lkF) f_F, f_F \rangle \leq \lambda_{\max} \sum_{F \in X_{k-2}} \langle f_F, f_F \rangle$$

so from 3.10

$$k \lambda \langle f, f \rangle \leq \sum_{F \in X_{k-2}} \langle \Delta (lkF) f_F, f_F \rangle \leq k \lambda_{\max} \langle f, f \rangle$$

and since $F \in S_f \iff f_F \neq 0$ we have

$$\left\langle \sum_{F \in X_{k-2}} \Delta_{k-1}^{up,F} f, f \right\rangle = \sum_{F \in X_{k-2}} \left\langle \Delta_{k-1}^{up,F} f, f \right\rangle = \sum_{F \in X_{k-2}} \left\langle \Delta (lkF) f_F, f_F \right\rangle = \sum_{F \in S_f} \langle \Delta (lkF) f_F, f_F \rangle$$

and in the context of our inequalities that means that

$$k \lambda_{\min} \langle f, f \rangle \leq \left\langle \sum_{F \in X_{k-2}} \Delta_{k-1}^{up,F} f, f \right\rangle \leq k \lambda_{\max} \langle f, f \rangle$$

From 3.5

$$\left\langle \sum_{F \in X_{k-2}} \Delta_{k-1}^{up,F} f, f \right\rangle = \left\langle \left( \Delta_{k-1}^{up} (X) + (k-1) I \right) f, f \right\rangle = \left\langle \Delta_{k-1}^{up} (X) f, f \right\rangle + (k-1) \langle f, f \rangle$$
so

\[ k\lambda_{\min} \langle f, f \rangle \leq \langle \Delta_{k-1}^{up} (X) f, f \rangle + (k - 1) \langle f, f \rangle \leq k\lambda_{\max} \langle f, f \rangle \]

rearranging we get

\[ (1 + k\lambda_{\min} - k) \langle f, f \rangle \leq \langle \Delta_{k-1}^{up} (X) f, f \rangle \leq (1 + k\lambda_{\max} - k) \langle f, f \rangle \]

which is the first required result.

To conclude the proof, the above implies that for \( f \neq 0 \),

\[ \frac{\langle \Delta_{k-1}^{up} (X) f, f \rangle}{\langle f, f \rangle} \in [1 + k\lambda_{\min} - k, 1 + k\lambda_{\max} - k] \]

and this holds when min/max-imizing over all \( f \neq 0 \). \[3.9\] then yields the required result. \[\Box\]

References