\textit{MAXCUT} is \textit{UGC}-hard to approximate [KKMO05]

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Abstract

We present a key result of Khot, Kindler, Mossel and O'Donnel [KKMO05] which states that if the Unique Games Conjecture [Khot02] holds then the Goemans-Williamson approximation algorithm [GW95] for \textit{MAXCUT} is optimal, unless \( P = NP \).

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These notes aim to help the students (including, most importantly, me) during the presentation, so informalities and inaccuracies are to be expected.

1 Preliminaries

1.1 Fourier analysis of Boolean functions

Recall that each \( f : \{\pm 1\}^n \to \mathbb{R} \) can be uniquely written as \( f = \sum_{S \subseteq [n]} \hat{f}(S) \chi_S \) where \( \chi_S(x) = \prod_{i \in S} x_i \), and the following definitions and facts

\textbf{Definition 1.1.} The influence of \( i \) on \( f \) is

\[ \text{Inf}_i(f) := \sum_{S \ni i} \hat{f}(S)^2 \]

The \( k \)-bounded influence of \( i \) on \( f \) is

\[ \text{Inf}^\leq_k(i,f) := \sum_{S \ni i, |S| \leq k} \hat{f}(S)^2 \]

where \( S \ni i \) is shorthand for \( S \subseteq [n] \) s.t \( i \in S \).

\textbf{Exercise 1.2.} For a boolean \( f : \{\pm 1\}^n \to \{\pm 1\} \), \( \sum_{i=1}^n \text{Inf}_i^\leq k(f) \leq k \). [Guideline: \( \sum_{i=1}^n \text{Inf}_i^\leq k(f) = \sum_{i=1}^n \sum_{j=1}^k \sum_{i \ni S, |S|=j} \hat{f}(S)^2 \). Show that \( \sum_{i=1}^n \sum_{S \ni i, |S|=j} \hat{f}(S)^2 \leq j \sum_{|S|=j} \hat{f}(S)^2 \) by counting how many time each \( S \ni i, |S|=j \) is summed.

We turn to define the notion of a \textit{stability} of a function.

\textbf{Definition 1.3.} For \( \rho \in [-1,1] \) and \( x \in \{\pm 1\}^n \) we say that (the random variable) \( y \) is \( \rho \)-\textit{correlated} to \( x \) and denote \( y \sim N_\rho(x) \) when \( y \) is drawn by independently setting each \( y_i \) as follows:

\[ y_i := \begin{cases} x_i & \text{w.p.} \frac{1}{2} + \frac{1}{2}\rho \\ -x_i & \text{w.p.} \frac{1}{2} - \frac{1}{2}\rho \end{cases} \]

The process of drawing a \( \rho \)-\textit{correlated} pair \((x,y)\) is defined as follows:

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1. Draw $x \sim U (\{\pm 1\}^n)$
2. Draw $y \sim N_\rho (x)$

Using this random variable we can define the *stability of* $f$, which measures the correlation between $f (x)$ and $f (y)$ when $(x, y)$ is a $\rho$-correlated pair.

**Definition 1.4.** Let $f : \{\pm 1\}^n \to \mathbb{R}$ and $\rho \in [-1, 1]$. The *noise stability of* $f$ at $\rho$ is defined by

$$S_\rho (f) = \mathbb{E} [f (x) f (y)]$$

where the expectation is taken over $\rho$-correlated pairs $(x, y)$.

**Remark 1.5.**

1. If $f$ is boolean then

$$S_\rho (f) = 2 \mathbb{P} [f (x) = f (y)] - 1$$

where the probability is taken over $\rho$-correlated pairs $(x, y)$.

2. When $\rho = 1$, $y \equiv x$. When $\rho = -1$, $y \equiv -x$. When $\rho = 0$, $y \sim U (\{\pm 1\}^n)$.

3. $(x, y)$ is a $\rho$-correlated pair iff for all $i \in [n]$ $x_i$ is independent of $y_i$, $\mathbb{E} [x_i] = \mathbb{E} [y_i] = 0$ and $\mathbb{E} [x_i y_i] = \rho$.

We present an alternative view of the noise stability of $f$. Let $T_\rho$ be the linear operator on the space $\mathbb{R}^{\{\pm 1\}^n}$ that is defined by $T_\rho (f) (x) = \mathbb{E}_{y \sim N_\rho (x)}$. Then

$$T_\rho (\chi_S) (x) = \mathbb{E}_{y \sim N_\rho (x)} \left[ \prod_{i \in S} y_i \right] = \prod_{i \in S} \mathbb{E}_{y_i} [y_i] = \prod_{i \in S} \rho x_i = \rho^{|S|} \chi_S (x)$$

and since $T_\rho$ is a linear operator this means that $T_\rho (f) = \sum_{S \subseteq [n]} \rho^{|S|} \tilde{f} (S) \chi_S$. On the other hand,

$$S_\rho (f) = \mathbb{E}_{(x, y) \sim \rho\text{-noisy pair}} [f (x) f (y)] = \mathbb{E}_{x \sim U (\{\pm 1\}^n)} [f (x) \mathbb{E}_{y \sim N_\rho (x)} |f (y)|] = \langle f, T_\rho f \rangle$$

$$= \sum_{S, T \subseteq [n]} \tilde{f} (S) \rho^{|T|} \tilde{f} (T) \langle \chi_S, \chi_T \rangle$$

$$= \sum_{S \subseteq [n]} \rho^{|S|} \tilde{f} (S)^2$$

### 1.2 The Long Code

We describe a highly inefficient way of representing numbers in $[n]$.

**Definition 1.6.** Set $n \in \mathbb{N}$. The *Long Code* of $i \in [n]$ is $\chi_{(i)} : \{\pm 1\}^n \to \{\pm 1\}$. One can think of the map

$$\text{LongCode} : [n] \to \{\pm 1\}^{\{\pm 1\}^n} \text{ LongCode} (i) = \chi_{(i)}$$

Notice that we map $\log n$ bits to $2^n$ bits which is a doubly-exponential blowup.

### 1.3 The Goemans-Williamson MAXCUT approximation algorithm

**Definition 1.7.** To us, the *Maximal Cut problem* (MAXCUT) is finding a cut of maximal weight in a weighted graph. Formally, the input is a weighted graph $G = (V, E, w)$ where $w : E \to \mathbb{R}_+$ and we wish to find a set $S \subseteq V$ that maximizes $\frac{1}{|E|} \sum_{e \in (S \times S)^c} w (e)$. 


The Goemans-Williamson algorithm \[GW95\] approximates \(\text{MAXCUT}\) with a ratio\(^1\) of \(\alpha_{GW}\), which is given by minimizing some trigonometric expression

\[
\alpha_{GW} := \min_{\theta \in (0, \pi)} \frac{\theta/\pi}{\frac{1}{2} - \frac{1}{2} \cos \theta} \approx 0.878567
\]

(1.1)

The above ratio is derived from the geometric nature of the GW algorithm and seems somewhat arbitrary for a combinatorial problem such as \(\text{MAXCUT}\). Surprisingly, we show that \(\text{MAXCUT}\) cannot be approximated with a better ratio, assuming certain conjectures – one is the famous \(P \neq NP\), while the other will be described in more details in 3.1.

For this talk no familiarity with the algorithm is needed, however a high-level understanding of it could provide deeper insight. Those may be obtained by glossing over [Cai03, Section 2].

2 Hardness of approximation

2.1 Approximation

To show that a problem \(\Pi\) is tractable, one needs to prove the existence of a polynomial-time algorithm that solves \(\Pi\) (e.g, by constructing said algorithm), but showing that is is hard to solve \(\Pi\) efficiently requires more sophisticated tools, namely using \(NP\)-hardness. \(NP\)-hardness aids us in proving that it is hard to solve all instances of a problem \textit{exactly}. Specifically, when dealing with an optimization problem, if \(P \neq NP\) then we cannot always find the optimal solution. The study of \textit{approximation algorithms} offers to trade the optimality of the output within the runtime of the computation.

**Definition 2.1.** Let \(\Pi \in NP\) be some maximization problem with value function\(^2\) \(v : \{0, 1\}^* \to [0, 1]\). Algorithm \(A\) is a \(\rho\)-approximation algorithm for \(\Pi\) if for every \(I \in \{0, 1\}^*\) instance to \(\Pi\) it holds that

\[
\rho v^*(I) \leq v(A(I))
\]

where \(v^*\) denotes the value of the maximal solution (w.r.t \(v\)) of \(I\).

As in the opening paragraph, showing that we can approximate a problem \(\Pi\) to a factor \(\rho\) is as simple as constructing an \(\rho\)-approximation algorithm (which is sometimes very hard!). But how can we show that it is \textit{hard} to approximate \(\Pi\) to a ratio \(\rho\)?

2.2 Gap reductions

Just as in the precise case, a framework of \textit{hardness} will be the solution to our problem. The building block of such a framework is the \textit{reduction}, which we need to adapt to our imprecise relaxation.

**Definition 2.2.** Let \(s_1 \leq c_1, s_2 \leq c_2 \in [0, 1]\). A \((s_1, c_1, s_2, c_2)\)-gap reduction from \(\Pi_1\) to \(\Pi_2\) (viewed as maximization problems with value functions \(v_1, v_2\) resp.) is a poly-time computable function \(g : \{0, 1\}^* \to \{0, 1\}^*\) that maps instances of \(\Pi_1\) to instances of \(\Pi_2\) such that the following holds:

\[
c_1 \leq v_1^*(I) \implies c_2 \leq v_2^*(g(I))
\]

\[
v_1^*(I) < s_1 \implies v_2^*(g(I)) < s_2
\]

Notice that there is no constraint on the behavior of the reduction on \(I\) is for which \(v_1^*(I) \in (s_1 | I |, c_1 | I |)\)

We say that \(\Pi\) is \(NP\)-hard to \((s, c)\)-distinguish if every (decision) problem \(\Delta \in NP\) is \((1, 1, s, c)\)-gap reducible to \(\Pi\), where \(\Delta\) is endowed with the value function \(\chi_\Delta\)\(^3\) of \(\Delta\). In words, the reduction(s) should map \(x \notin \Delta\) to instance \(I\) (of \(\Pi\)) with value at most \(s\), and \(x \in \Delta\) to instance \(I\) with value at least \(c\).

\(^1\)The reader should be familiar with the notion of an approximation algorithm, although a formal definition will be given.

\(^2\)We assume all value functions are efficiently computable w.r.t the input and the solution

\(^3\)Let \(S\) be a set. The characteristic function of \(S\) is defined by \(\chi_\delta(x) = \begin{cases} 1 & x \in S \\ 0 & x \notin S \end{cases}\)
We justify the existence of these definitions with the following theorem:

**Theorem 2.3. (Gap reduction)**

1. Assume \( \Pi_1 \) is \((s_1, c_1, s_2, c_2)\)-gap reducible to \( \Pi_2 \). If \( \Pi_1 \) is \(\text{NP}\)-hard to \((s_1, c_1)\)-distinguish then \( \Pi_2 \) is \(\text{NP}\)-hard to \((s_2, c_2)\)-distinguish.

2. Assume \( \Pi \) is \(\text{NP}\)-hard to \((s, c)\)-distinguish. If there exists a polynomial time \(\frac{s}{c}\)-approximation algorithm for \( \Pi \), then \( P = \text{NP} \).

**Proof.**

1. Set \( \Delta \in \text{NP} \). Take \( g, h \) to be the gap reductions \( \Pi_0 \xrightarrow{g} \Pi_1 \xrightarrow{h} \Pi_2 \) with gaps as in the theorem statement. Then \( h \circ g \) is poly-time computable, and indeed

\[
\begin{align*}
x \in \Delta & \implies c_1 \leq v_1^x (g(x)) \implies c_2 \leq v_2^x ((h \circ g)(x)), \\
x \notin \Delta & \implies v_1^x (g(x)) < s_1 \implies v_2^x ((h \circ g)(x)) < s_2.
\end{align*}
\]

2. Assume there is a polynomial time \(\frac{s}{c}\)-approximation algorithm for \( \Pi \). Let \( \Delta \in \text{NP} \) and take the corresponding gap reduction \( g \). The polynomial time decider for \( \Delta \), upon receiving input \( x \), will output “Yes” iff \( s \leq v(A(g(x))) \). It is indeed polynomial, and it is correct:

- If \( x \in \Delta \) then \( c \leq v^*(g(x)) \), so

\[
s \leq \frac{s}{c} v^*(g(x)) \leq v(A(x)).
\]

- If \( x \notin \Delta \) then \( v^*(g(x)) < s \), by definition of \( v^* \) we have \( v(A(g(x))) < s \).

So gap reductions provide us with an \(\text{NP}\)-hardness theory for approximation problems. Since we know of problems that are \(\text{NP}\)-hard to approximate ([SG76]), we can develop this theory. Since we are particularly interested in \(\text{MAXCUT} \), the following observation will prove useful: \(\square\)

**Remark 2.4.** Crescenzi, Silvestri and Trevisan proved that the approximation thresholds of \(\text{MAXCUT} \) and \(\text{unweighted-MAXCUT} \) are equal, which for our interest means that there exists an \( \alpha \) such that \(\text{MAXCUT} \) and \(\text{unweighted-MAXCUT} \) are \( \alpha \)-approximable, but for any \( \varepsilon > 0 \) if there is an \( (\alpha + \varepsilon) \)-approximation algorithm for \( \text{MAXCUT} \) or \(\text{unweighted-MAXCUT} \) then \( P = \text{NP} \) [CST01].

### 3 Two tools

#### 3.1 The Unique Games Conjecture

Though there is a plethora of hardness of approximation results, some much desired results seem to be too elusive without making further assumptions. In particular, we prove a hardness of approximation result for \(\text{MAXCUT} \) assuming an important conjecture regarding the Unique Games problem which improves on the previously best-known ratio, \( 16/17 \simeq 0.941176 \) due to [Håstad01].

**Definition 3.1.** An instance to the Unique \( M \)-Label Cover problem \((ULC(M)) \) is composed of a bipartite graph \( G = (V, W, E) \) (we assume all edges are oriented from \( V \) to \( W \)) and a set of constraints \( \{\pi_e\}_{e \in E} \) such that for every \( e \in E \), \( \pi_e \) is a permutation of \( [M] \). A solution is an assignment \( \sigma : V \cup W \to [M] \) whose value is defined

\[
v(\sigma) := \Pr_{(v, w) \sim U(E)}[\pi_{v, w}(\sigma(w)) = \sigma(v)]
\]

that is, \( v(\sigma) \) is the fraction of constraints it satisfies.
Notice that for every $M$, deciding whether an instance to $ULC(M)$ is completely satisfiable (that is, whether $v^*(I) = 1$) is tractable: Assuming $G$ is connected, choose an arbitrary $v \in V$. Then for each $i \in [M]$ set $\sigma(v) = i$ — by uniqueness of the constraints, this completely decides the rest of the labels so we may iteratively label all vertices in the graph with BFS. If a contradiction is reached (that is, we attempt to relabel a vertex with a different label) then continue to the next $i \in [M]$, otherwise we found a satisfying $\sigma$.

The Unique Games Conjecture (originally formulated in [Khot02]) states that it is hard to distinguish between highly and slightly satisfiable instances, and has a fascinating history ([Klarreich11]).

**Conjecture 3.2.** (Unique Games) For every $s \leq c \in (0,1)$ there exists an $M \in \mathbb{N}$ such that $ULC(M)$ is $NP$-hard to $(s,c)$-distinguish.

Assuming the Unique Games Conjecture provides us with a family of problems that are $NP$-hard to $(s,c)$-distinguish, which will be one of two key tools used in our main result. A problem that is $NP$-hard to approximate under this assumption is sometimes called $UG$-hard to approximate but note that $UGC$ is not a class of problems so there is some abuse in this term.

We end the introduction of this key player with the following observation.

**Remark 3.3.** Without loss of generality, we assume that all instances to $ULC$ are regular on the $V$ side, that is that for all $v, v' \in V$ $\deg v = \deg v'$. This is a step in the proof that the weighted and unweighted versions of $UGC$ are equivalent, specifically [KR08, Lemma 3.4].

### 3.2 Majority is Stablest (revisited)

Another key player in our proof will be the Majority Is Stablest theorem (which was only a conjecture when [KKMO05] was originally published!). Although the below formulation is somewhat different from the one we saw a few weeks ago, we will use it as a black-box for our proof:

**Theorem 3.4.** (Majority Is Stablest) Let $\rho \in [0,1)$. For any $\varepsilon > 0$ there is $\delta > 0$ such that if $f : \{\pm 1\}^n \to [-1,1]$ satisfies

$$E[f] = 0, \quad \forall i \in [n] \quad \Inf_i(f) \leq \delta$$

then

$$S_\rho(f) \leq 1 - \frac{2}{\pi} \arccos \rho + \varepsilon$$

Defining $Dict^n, \text{Maj}_n : \{\pm 1\}^n \to \{\pm 1\}$ by

$$Dict^n(x) := x_i, \quad \text{Maj}_n(x) := \text{sign} \left( \frac{1}{2} + \sum_{i=1}^n x_i \right) = \begin{cases} 1 & |\{i | x_i = -1\}| \leq |\{i | x_i = 1\}| \\ -1 & \text{else} \end{cases}$$

it can be shown easily and less easily (resp.) be shown that

$$\forall i,n \ S_\rho(Dict^n_i) = \rho \quad \lim_{n \to \infty} S_\rho(\text{Maj}_n) = 1 - \frac{2}{\pi} \arccos \rho$$

so theorem 3.4 means that of all (zero-meaned) functions that aren’t close to being dictatorships (meaning no single coordinate has significant influence), the majority function is stabest.

In fact, we will expand our black-box slightly to include a slightly different formulation of the Majority Is Stablest theorem, tailor-made for our main proof.

**Proposition 3.5.** Let $\rho \in (-1,0)$. For any $\varepsilon > 0$ there is $\delta > 0$ and $k \in \mathbb{N}$ such that if $f : \{\pm 1\}^n \to [-1,1]$ satisfies

$$\forall i \in [n] \quad \Inf_i^{\leq k}(f) \leq \delta$$

then

$$S_\rho(f) > 1 - \frac{2}{\pi} \arccos \rho - \varepsilon$$
The differences are that we take negative $\rho$ (and switch the inequality accordingly), and generalize to all (not necessarily zero-meaned) $f$s that have small $k$-bounded influence. The proposition follows from the two following rather technical claims.

**Claim 3.6.** For all $\rho \in [0,1), \varepsilon > 0$ there are $\delta > 0, k \in \mathbb{N}$ such that if $f : \{\pm 1\}^n \to [-1,1]$ satisfies

$$\mathbb{E}[f] = 0, \ \forall i \in [n] \ \text{Inf}_i(f) \leq \delta$$

then

$$S_\rho(f) \leq 1 - \frac{2}{\pi} \arccos \rho + \varepsilon$$

**Proof.** Let $\rho \in [0,1), \varepsilon > 0$. Take\(^4\) $\gamma \in (0,1)$ small enough such that such that for all $k \rho^k \left(1 - (1 - \gamma)^{2k}\right) < \frac{\varepsilon}{4}$. Take $\delta' > 0$ from 3.4 applied to $\rho$ and $\frac{\varepsilon}{4}$. Choose $\delta = \frac{\delta'}{2}$, and $k$ such that $(1 - \gamma)^{2k} < \delta$.

Now, take $f$ that satisfies $\mathbb{E}[f] = 0$ and $\text{Inf}_i^{\leq k}(f) \leq \delta$ for all $i$, and let $T_{1-\gamma}f$ where $T_{1-\gamma}f$ is the noise operator with noise $1 - \gamma$. Then

$$\text{Inf}_i(T_{1-\gamma}f) = \sum_{i \in S} (1 - \gamma)^{2|S|} \hat{f}(S)^2$$

since $\mathbb{E}[T_{1-\gamma}f] = 0$, from 3.4 it must holds that $S_\rho(T_{1-\gamma}f) \leq 1 - \frac{2}{\pi} \arccos \rho + \frac{\varepsilon}{4}$. Finally, notice that

$$S_\rho(T_{1-\gamma}f) = \sum_{S \subseteq [n]} \rho^{|S|} \hat{T_{1-\gamma}f}(S)^2 = \sum_{S \subseteq [n]} \rho^{|S|} (1 - \gamma)^{2|S|} \hat{f}(S)^2$$

which gives us

$$\text{Inf}_i(f) = S_\rho(T_{1-\gamma}f) - A \leq S_\rho(T_{1-\gamma}f) \leq 1 - \frac{2}{\pi} \arccos \rho + \frac{1}{2} \varepsilon + \frac{1}{4} \varepsilon$$

\[\square\]

**Claim 3.7.** For all $\rho \in (-1,0], \varepsilon > 0$ there is $\delta > 0$ such that if $f : \{\pm 1\}^n \to [-1,1]$ satisfies for all $i \in [n] \ \text{Inf}_i(f) \leq \delta$ then $S_\rho(f) \geq 1 - \frac{2}{\pi} \arccos \rho - \varepsilon$.

\(^4\)For all $\gamma \in (0,1)$ the LHS tends to 0 as $k \to \infty$. Fix an arbitrary $\gamma$, then there is a $K$ s.t for every $k \geq K$ the inequality holds. Now shrink $\gamma$ so that the inequality holds also for $k < K$. 

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Proof. Let \( \rho, \varepsilon \) and take \( \delta \) corresponding to \( -\rho, \varepsilon \) in 3.4. Let \( g(x) := \frac{f(x)-f(-x)}{2} = \sum_{|S| \text{ odd}} \hat{f}(S) \chi_S(x) \), where the latter inequality can be seen by noticing that \( (f(-x), \chi_S(x)) = (-1)^{|S|} \hat{f}(S) \). Then, \( E[g] = 0 \) and \( \inf_i (g) \leq \inf_i (f) \leq \delta \) so

\[
S_{-\rho}(g) \leq 1 - \frac{2}{\pi} \arccos \rho + \varepsilon = 1 - \frac{2}{\pi} (\pi - \arccos \rho) + \varepsilon = - \left(1 + \frac{2}{\pi} \arccos \rho - \varepsilon\right)
\]

Therefore

\[
S_\rho(f) = \sum_{S \subseteq [n]} \rho^{|S|} \hat{f}(S) \geq \sum_{S \subseteq [n], |S| \text{ odd}} \rho^{|S|} \hat{f}(S) S_\rho(g) = -S_{-\rho}(g) \geq 1 + \frac{2}{\pi} \arccos \rho - \varepsilon
\]

\[\square\]

4 Main result

Without further ado, we present and prove the main result

Theorem 4.1. Assume the Unique Games Conjecture 3.2. For any \( \rho \in (-1,0) \) and \( \varepsilon > 0 \), MAXCUT is \( \text{NP-hard to} \ (\frac{\arccos \rho}{\pi} + \varepsilon, \frac{1}{2} - \frac{1}{2} \rho) \)-distinguish.

Corollary 4.2. Assuming the Unique Games Conjecture, if MAXCUT can be approximated with ratio greater than \( \alpha_{GW} \) then \( P = NP \).

First, let’s see how 4.2 follows from theorem 4.1.

Proof. Assume UGC and theorem 4.1, and let \( \varepsilon' > 0 \). Let

\[ r : [-1,0] \to \mathbb{R} \quad r(\rho) := \frac{\arccos \rho}{\frac{1}{2} - \frac{1}{2} \rho} \]

Notice that \( r \) obtains a unique minimum at \( \rho^* := \arg\min r \approx -0.689 \) – either by some analysis (Weierstrass tells us it gets a minimum, find roots of the derivative) or by picture (figure 4.1). Furthermore, \( r(\rho^*) = \alpha_{GW} \) – again, either because minimizing \( r \) over \( \rho \in [-1,0] \) is equivalent to minimizing it the the fraction in equation (1.1) over \( \theta \in \left[ \frac{\pi}{2}, \pi \right] \) which in turn is equivalent to minimizing that fraction over \( \theta \in [0,\pi] \), or because of a figure 4.1.

theorem 4.1 with \( \rho = \rho^* \in (-1,0) \) and \( \varepsilon = (\frac{1}{2} - \frac{1}{2} \rho^*) \varepsilon' > 0 \) tells us that MAXCUT is \( \text{NP-hard to} \ (s,c) \)-distinguish, where \( s = \frac{\arccos \rho^*}{\frac{1}{2} - \frac{1}{2} \rho^*} + (\frac{1}{2} - \frac{1}{2} \rho^*) \varepsilon' \) and \( c = \frac{1}{2} - \frac{1}{2} \rho^* \). Notice that

\[
\frac{s}{c} = \frac{\arccos \rho^* + (\frac{1}{2} - \frac{1}{2} \rho^*) \varepsilon'}{\frac{1}{2} - \frac{1}{2} \rho^*} = \frac{\arccos \rho^*}{\frac{1}{2} - \frac{1}{2} \rho^*} + \varepsilon' = \alpha_{GW} + \varepsilon'
\]

so by theorem 2.3, if there exists a poly-time algorithm that achieves approximation ratio \( \alpha_{GW} + \varepsilon' \) then \( P = NP \).

\[\square\]

Before moving on to the proof, we introduce one final piece of notation: For a vector \( x \in \{\pm 1\}^n \) and a permutation \( \pi \in \text{Sym}_n \), we obtain the vector \( x^\pi \in \{\pm 1\}^n \) is defined by \( x_i^\pi := x_{\pi(i)} \) for all \( i \in [n] \).

4.1 The reduction

Let \( \rho \in (-1,0), \varepsilon > 0 \). We construct a \( (\gamma,1-\eta,\frac{\arccos \rho}{\pi} + \varepsilon, \frac{1}{2} - \frac{1}{2} \rho) \)-gap reduction from ULC(M) to MAXCUT, where \( M \in \mathbb{N} \) is the one corresponding to \( (\gamma,1-\eta) \) in 3.2, and we will choose \( \gamma, \eta \) to be sufficiently small later.
4.1.1 Consistency Test

First, we define the (probabilistic) Consistency Test that takes a weighted instance to $ULC(M)$ denoted by graph $G = (V, W, E, m)$ and constraints $\{\pi_e\}_{e \in E}$, and a set $\{f_w\}_{w \in W}$ where for all $w \in W$, $f_w : \{\pm 1\}^M \rightarrow \{\pm 1\}$.

1. Pick $v \sim U(V)$, and then pick two of its neighbours $w, w'$ independently and uniformly from $\Gamma(v)$.
   
   (a) Let $\pi := \pi_{(v,w)}$ and $\pi' := \pi_{(v,w')}$ be the respective constraints on these edges.

2. Choose $x \sim U(\{\pm 1\}^M)$ and $\mu \sim N_\rho(1)$ independently, where $1 \in \{\pm 1\}^M$ is the all-ones vector.

3. Accept iff $f(x^\pi) \neq f'(x^\pi \mu)$.

Notice that $(x, x\mu)$ is a $\rho$-correlated pair so $(x\mu, x)$ is the same, therefore the test is symmetric in the inputs $f, f'$.

Exercise 4.3. If $f = f' = \text{LongCode}(i)$ for some $i \in [M]$ then the above test accepts with probability $\left(\frac{1}{2} - \frac{1}{2}\rho\right)$.

4.1.2 The actual reduction

The reduction itself takes as input an instance to $ULC(M)$ as in 4.1.1 and outputs a weighted complete graph $G' = (W', W' \times W', m')$, where $W'$ and $m'$ are defined as follows:

- For each $w \in W$ we construct $2^M$ vertices in $W'$ that correspond to the truth table of a function $f_w : \{\pm 1\}^M \rightarrow \{\pm 1\}$. Formally,
  $$W' = \left\{f_w(x) | w \in W, x \in \{\pm 1\}^M\right\}$$

- The weight of the edge $\{f_w(x), f_w'(y)\}$ is the probability that the test $f_w(x) \neq f_w'(y)$ is performed in the run of the Consistency Test on input $G = (V, W, E)$. 

Figure 4.1: $(r(\rho) - \alpha_{GW})$ and its minimum
4.2 Runtime

The reduction seems immensely inefficient, but the trick is that $M$ is constant. $G'$ is a graph on $|W| \cdot 2^M$ vertices, and to compute the weights the reduction needs to simulate all possible "coins" (randomness) of the Consistency Test on the given input. Notice that there are at most $|V| \cdot |W|^2 \cdot 2^{M+1}$ possible outcomes for the random choices, and the run of the test on each choice is polynomial in $|G|$. So, the reduction is polynomial in its input.

4.3 Correctness

We prove that for all $\rho \in (-1,0), \varepsilon, \eta > 0$ there is a $\gamma$ such that a reduction from $(\gamma, 1-\eta)$-gap-$ULC(M)$ to $(\frac{\arccos \rho}{\pi} + \varepsilon, (\frac{1}{2} - \frac{1}{2}\rho))$-gap-$MAXCUT$ exists. We can get rid of the $(1-2\eta)$ factor in the completeness by "trading off soundness for completion". More (but not entirely) formally, for a given $\varepsilon, \rho$, take $\rho', \varepsilon'$ such that $(\frac{1}{2} - \frac{1}{2}\rho') (1-2\eta) = (\frac{1}{2} - \frac{1}{2}\rho)$ and $\frac{\arccos \rho'}{\pi} + \varepsilon' = \frac{\arccos \rho}{\pi} + \varepsilon$, and take $\eta > 0$ sufficiently small such that $\rho' \in (-1,0)$ and $\varepsilon > 0$. Applying the reduction with $\rho', \varepsilon', \eta$ gives us the gap

$$\left(\frac{\arccos \rho'}{\pi} + \varepsilon', \left(\frac{1}{2} - \frac{1}{2}\rho'\right) (1-2\eta)\right) = \left(\frac{\arccos \rho}{\pi} + \varepsilon, \left(\frac{1}{2} - \frac{1}{2}\rho\right)\right)$$

4.3.1 Completeness

Assume the $ULC(M)$ instance has a labeling $\sigma$ of value at least $(1-\eta)$, that is it satisfies a $(1-\eta)$-fraction of constraints. The cut in $G'$ is obtained by assigning $f_w$ the truth table of the long code of $\sigma(w)$. Formally, the cut in $G'$ is $W' = W'_1 \sqcup W'_2$ where

$$W'_1 = \{ f_w(x) | \chi_{\sigma(w)}(x) = 1 \}, \quad W'_2 = \{ f_w(x) | \chi_{\sigma(w)}(x) = -1 \}$$

We argue that this cut has value $(1-2\eta) (\frac{1}{2} - \frac{1}{2}\rho)$. We say that $\sigma$ satisfies $(v, w) \in E$ if $\pi_{v,w}(\sigma(w)) = \sigma(v)$. Well, taking probability over $v \sim U(V)$ and $w, w' \sim U(\Gamma(v))$,

$$\mathbb{P}[\sigma \text{ satisfies } (v, w) \text{ and } (v, w')] = 1 - \mathbb{P}[\sigma \text{ doesn't satisfy } (v, w) \text{ or } (v, w')] \geq 1 - 2\mathbb{P}[\sigma \text{ doesn't satisfy } (v, w)] = 1 - 2\eta$$

where the last equality uses the fact that $G$ is regular on the $V$ side (3.3), so that choosing $v$ uniformly and then choosing $w$ uniformly from $\Gamma(v)$ is akin to choosing uniform edge from $E$.

If $\sigma$ satisfies $(v, w)$ and $(v, w')$ then

$$f_w(x^{\pi_{v,w}}) = x_{\pi_{v,w}(\sigma(w))} = x_{\sigma(v)}$$

Combining the above, the Consistency Test chooses satisfied $v, w, w'$ with probability at least $1 - 2\eta$, and for these vertices the probability that $f_w(x^{\pi_{v,w}}) \neq f_{w'}(x^{\pi_{v,w'}})$ is precisely the probability that $\mu_{\sigma(v)} = -1$, which is $(\frac{1}{2} - \frac{1}{2}\rho)$. Since $\mu$ is chosen independently of $v, w, w'$ we have that the probability acceptance of the Consistency Test is at least $(1 - 2\eta) (\frac{1}{2} - \frac{1}{2}\rho)$. Since an edge $(f_w(x), f_{w'}(x'))$ is in the cut iff $f_w(x) \neq f_{w'}(x')$ and the weight of such edge is the probability that the test $f_w(x) \neq f_{w'}(x')$ is performed, the weight of the cut is the probability that the Consistency Test accepts, giving us the required lower bound on the value. \[\square\]
4.3.2 Soundness

We prove that contrapositive. Assume that we have a graph $G$ with a cut of weight at least $\frac{\arccos \rho}{\pi} + \varepsilon$, and we show that it was obtained from a ULC $(M)$ instance that has an assignment satisfying at least a $\gamma'(\varepsilon, \rho) = \gamma'$-fraction of constraints. Then, since $\gamma'$ does not depend on $M$ we can take $\gamma < \gamma'$ (enlarging $M$) to obtain the required result.

From a cut $W' = W'_1 \cup W'_2$ of weight at least $\frac{\arccos \rho}{\pi} + \varepsilon$ obtain functions $\{ f_w : \{ \pm 1 \}^M \rightarrow \{ \pm 1 \} \}_{w \in W}$ by letting $f_w(x) = 1$ iff $f_w(x) \in W'_1$. Say that $v \in V$ is good if $P[ \text{acc} | \text{v} ] \geq \frac{\arccos \rho}{\pi} + \frac{\varepsilon}{2}$, that is if the test accepts with probability higher than the r.h.s when $v$ is drawn. Since$^5$ $P[ \text{acc} | \text{v} ]$ is not good $< \frac{\arccos \rho}{\pi} + \frac{\varepsilon}{2}$, using the law of total probability we have

$$\frac{\arccos \rho}{\pi} + \varepsilon \leq P[ \text{acc} ] \leq P[ v \text{ is good} ] + P[ \text{acc} | \text{v} ] \leq P[ v \text{ is good} ] + \frac{\arccos \rho}{\pi} + \frac{\varepsilon}{2}$$

and so $P[ v \text{ is good} ] \geq \frac{\varepsilon}{2}$. Second, for fixed $v \in V$ we have the below arithmetization

$$\mathbb{E}[ f_w(x) f_{w'}(x) ] = 1 \cdot P[ f_w(x) = f_{w'}(x) ] = (-1) \cdot P[ f_w(x) \neq f_{w'}(x) ]$$

$$= P[ \text{rej} | \text{v} ] - P[ \text{acc} | \text{v} ]$$

$$= 1 - 2 P[ \text{acc} | \text{v} ]$$

Where $\mathbb{E}$ and $P$ are taken over $w, w' \sim U(\Gamma(v))$ and $x, \mu$ are drawn as in the Consistency Test. This implies that

$$P[ \text{acc} | \text{v} ] = \frac{1}{2} - \frac{1}{2} \mathbb{E}_{w, w', x, \mu} [ f_w(x) f_{w'}(x) ]$$

$$\mu\text{'s elements are independent} = \frac{1}{2} - \frac{1}{2} \mathbb{E}_{w, w', x, \mu} [ f_w(x) f_{w'}(x) ]$$

$$\text{Law of Total Exp.} = \frac{1}{2} - \frac{1}{2} \mathbb{E}_{x, \mu} [ \mathbb{E}_{w, w'} [ f_w(x) f_{w'}(x) ] ]$$

$$w, w' \text{ are independent} = \frac{1}{2} - \frac{1}{2} \mathbb{E}_{x, \mu} [ \mathbb{E}_{w} [ f_w(x) ] \mathbb{E}_{w'} [ f_{w'}(x) ] ]$$

$$= \frac{1}{2} - \frac{1}{2} \mathbb{E}_{x, \mu} [ g_v(x) g_v(x) ]$$

$$\text{Law of Total Exp.} = \frac{1}{2} - \frac{1}{2} \mathbb{E}_{\rho} [ g_v(x) ]$$

where $g_v(z) = \mathbb{E}_{w \sim U(\Gamma(v))} [ f_w(z) ]$ and the expectation is taken over $w, w' \sim U(\Gamma(v))$ and $x, \mu$ as in the Consistency Test. So for good $v$'s we have

$$\frac{\arccos \rho}{\pi} + \varepsilon \leq P[ \text{acc} | \text{v} ] = \frac{1}{2} - \frac{1}{2} \mathbb{E}_{\rho} [ g_v(x) ]$$

therefore if $v$ is good then

$$\mathbb{E}_{\rho} [ g_v(x) ] \leq 1 - 2 \frac{\arccos \rho}{\pi} - \varepsilon$$

Finally, from the counterpositive to 3.5, there is a large enough $k$ such that for each good $v$ there exists $\sigma (v) \in [n]$ with $\operatorname{Inf}_{\sigma(v)}^k (g_v) > \delta$ (if there is more than one $\sigma (v)$, we fix one arbitrarily). This completes the task of labeling good $v$'s, and we label the rest of $V$ arbitrarily.

What's left is to find labels $\sigma (w)$. Let the candidate set of $w \in W$ be

$$\text{Cand} (w) := \left\{ j \in [M] | \operatorname{Inf}_j^k (f_w) \geq \frac{\delta}{2} \right\}$$

$^5$If $c \geq 0$ and $A, \{ B_i \}$ are events s.t $\{ B_i \}$ are pairwise disjoint and $P[B_i] > 0$ and $P[A | B_i] < c$ for all $i$, then $P[A \bigcup_{i} B_i] < c$
From 1.2 we have
\[
|\text{Cand}(w)| \cdot \frac{\delta}{2} \leq \sum_{i \in \text{cand}(w)} \text{Inf}_{\tau_{i}^{k}}(f_{w}) \leq k \implies |\text{Cand}(w)| \leq \frac{2\delta}{k}
\]
and on the other hand, notice that for all good \(v\)
\[
\delta \leq \text{Inf}_{\rho}^{\leq k}(g_{v}) = \sum_{S:|S| \leq k} \widehat{g}_{v}(S)^{2} = \sum_{S:|S| \leq k} \mathbb{E}_{w} \left[ \widehat{f}_{w} \left( \pi_{v,w}^{-1}(S) \right) \right]^{2}
\]
Jensen’s ineq. \[
\mathbb{E}_{w} \left[ \sum_{S:|S| \leq k} \widehat{f}_{w} \left( \pi_{v,w}^{-1}(S) \right)^{2} \right] = \mathbb{E}_{w} \left[ \text{Inf}_{\rho}^{\leq k}(\pi_{v,w}^{-1}(\sigma(v))) (f_{w}) \right]
\]
where \((*)\) is obtained by noticing that \(\widehat{g}_{v}(S) = \mathbb{E}_{w} \left[ \widehat{f}_{w} \left( \pi_{v,w}^{-1}(S) \right) \right],\) by definition of the fourier coefficient and perhaps Fubini’s theorem\(^6\). It follows\(^7\) that for any good \(v\), at least a \(\frac{\delta}{2}\)-fraction of \(w \in \Gamma(v)\) satisfy \(\text{Inf}_{\rho}^{\leq k}(\pi_{v,w}^{-1}(\sigma(v))) (f_{w}) \geq \frac{\delta}{2}\) so for those \(w\) it holds that \(\pi_{v,w}^{-1}(\sigma(v)) \in \text{Cand}(w)\).

Consider the random process that labels \(\sigma(w) \sim U(\text{cand}(w))\) if \(\text{Cand}(w) \neq \emptyset\), else labels arbitrarily, and keep in mind the desired outcome \(\pi_{v,w}(\sigma(w)) = \sigma(v)\) which is iff \(\sigma(w) = \pi_{v,w}^{-1}(\sigma(v))\). Then we have an assignment satisfying a \(\gamma' = \frac{\varepsilon}{2} \cdot \frac{\delta}{2} \cdot \frac{\delta}{2}\) fraction of constraints of \(G\) – details follow.

From 3.3, drawing \(\{v, w\} \sim U(E)\) is equivalent to first choosing \(v \sim U(V)\) and then choosing \(w \in \Gamma(v)\). Taking \(\mathbb{P}\) over \(\{v, w\} \sim U(E)\), from the foregoing analysis we have

\[
\mathbb{P} \left[ \{v, w\} \text{ satisfied by } \sigma \right] \geq \frac{\varepsilon}{2} \mathbb{P} \left[ \{v, w\} \text{ satisfied by } \sigma | v \text{ is good} \right]
\]
\[
\geq \frac{\varepsilon}{2} \mathbb{P} \left[ \pi_{v,w}^{-1}(\sigma(v)) \in \text{Cand}(w) | v \text{ is good} \right] \cdot \mathbb{P} \left[ \{v, w\} \text{ satisfied by } \sigma | v \text{ is good, } \pi_{v,w}^{-1}(\sigma(v)) \in \text{Cand}(w) \right]
\]
\[
\geq \frac{\varepsilon}{2} \cdot \frac{\delta}{2} \cdot \frac{\delta}{2} \mathbb{P} \left[ \{v, w\} \text{ satisfied by } \sigma | v \text{ is good, } \pi_{v,w}^{-1}(\sigma(v)) \in \text{Cand}(w) \right]
\]
\(^6\)Let \(X \leq 1\) be some RV of expectation at least \(\delta\), and denote \(p = \mathbb{P} \left[ X \geq \frac{\delta}{2} \right].\) Then
\[
\delta \leq \mathbb{E}[X] \leq 1 \cdot p + \frac{\delta}{2}(1 - p) \leq p + \frac{\delta}{2} \implies \frac{\delta}{2} \leq p
\]
And now taking $E$ over $\sigma$ as in the above random process and $P$ over $\{v, w\} \sim U(E)$ we have

$$E \left[ P \left( \{v, w\} \text{ satisfied by } \sigma \right) \right] \geq \frac{\varepsilon}{2} \cdot \delta \cdot E \left[ P \left( \{v, w\} \text{ satisfied by } \sigma | v \text{ is good, } \pi_{v,w}^{-1}(\sigma(v)) \in \text{Cand}(w) \right) \right]$$

so all that’s left to show is that $E_{\sigma} \left[ P_{v,w}[\{v, w\} \text{ satisfied by } \sigma | v \text{ is good, } \pi_{v,w}^{-1}(\sigma(v)) \in \text{Cand}(w) \right] \geq \frac{\delta}{2k}$, since then we are guaranteed the existence of a $\sigma$ for which $P_{v,w}[\{v, w\} \text{ satisfied by } \sigma | v \text{ is good, } \pi_{v,w}^{-1}(\sigma(v)) \in \text{Cand}(w) \right] \geq \frac{\varepsilon}{2} \cdot \frac{\delta}{2k} = \gamma'$. Say that $(v, w)$ are great if $v$ is good and $\pi_{v,w}^{-1}(\sigma(v)) \in \text{Cand}(w)$, we have

$$E_{\sigma} \left[ P_{v,w}[\{v, w\} \text{ satisfied by } \sigma | (v, w) \text{ are great} \right]$$

Law of Total Prob. = $E_{\sigma} \left[ \sum_{v_0, w_0 \text{ great}} P_{v,w}[v = v_0, w = w_0 | v, w \text{ are great}] P_{v,w}[\{v, w\} \text{ satisfied by } \sigma | v = v_0, w = w_0] \right]$ Linearity, def of indicator RV = $\sum_{v_0, w_0 \text{ great}} P_{v,w}[v = v_0, w = w_0 | v, w \text{ are great}] E_{\sigma} [1_{v_0, w_0 \text{ sat by } \sigma}] = \sum_{v_0, w_0 \text{ great}} P_{v,w}[v = v_0, w = w_0 | v, w \text{ are great}] P_{\sigma}[\{v_0, w_0\} \text{ sat by } \sigma] \geq \sum_{v_0, w_0 \text{ great}} P_{v,w}[v = v_0, w = w_0 | v, w \text{ are great}] \frac{\delta}{2k} = \frac{\delta}{2k} \sum_{v_0, w_0 \text{ great}} P_{v,w}[v = v_0, w = w_0 | v, w \text{ are great}] \frac{\delta}{2k} = \frac{\delta}{2k} \cdot 1$

where the inequality is because $\sigma \sim U(\text{Cand}(w_0))$, $|\text{Cand}(w_0)| \geq \frac{\delta}{2}$ and $\pi_{v_0,w_0}^{-1}(\sigma(v)) \in \text{Cand}(w_0)$ for great $v_0, w_0$.

References


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