Information Theory: Exercise IV

1) Let $G$ be the complete (undirected) graph with $m = 4n$ vertices. That is, $G = (V,E)$, where $V$ is a set of size $4n$, and $E$ contains all pairs $(i,j)$, s.t., $i, j \in V$ and $i \neq j$.

Let $H$ be the complete (undirected) tripartite graph with parts of sizes $2n, n, n$. That is, $H = (V_1 \cup V_2 \cup V_3, E')$, where $V_1, V_2, V_3$ are disjoint sets of sizes $2n, n, n$ (respectively), and $E'$ contains all pairs $(i,j)$, s.t., $i \in V_a, j \in V_b$, where $a \neq b$.

We say that a sequence of graphs $H_1, \ldots, H_k$ covers a graph $G$ if the nodes of $H_1, \ldots, H_k$ can be placed on the nodes of $G$ (i.e., the nodes of every $H_i$ are mapped one-to-one to the nodes of $G$), such that, every edge of $G$ is covered by at least one edge of $H_1, \ldots, H_k$.

Show that at least $(2/3) \cdot \log_2 m$ copies of $H$ are needed to cover $G$.

2) Let $G$ be the graph with set of nodes $\{1,2,3\}^n$, where two nodes $x, y \in \{1,2,3\}^n$ are connected by an edge iff they differ in exactly one coordinate. State and prove an isoperimetric inequality for the graph $G$. (Similar to the isoperimetric inequality for the discrete cube $\{0,1\}^n$).

3) Prove or give a counter example: For every $X_1, X_2, X_3, X_4$,

$$H(X_1, X_2, X_3) + H(X_1, X_2, X_4) + H(X_1, X_3, X_4) \leq H(X_1, X_2) + H(X_1, X_3) + H(X_1, X_4) + H(X_2, X_3) + H(X_2, X_4) + H(X_3, X_4).$$

4) Prove or give a counter example: For every $X_1, X_2, X_3, X_4$,

$$H(X_1, X_2, X_3) + H(X_2, X_3, X_4) + H(X_3, X_4, X_1) + H(X_4, X_1, X_2) \leq 1.5 \cdot [H(X_1, X_2) + H(X_2, X_3) + H(X_3, X_4) + H(X_4, X_1)].$$

5) Prove or give a counter example: For every $X_1, X_2, X_3, X_4$,

$$H(X_1, X_2, X_3) + H(X_1, X_2, X_4) + H(X_1, X_3, X_4) + H(X_2, X_3, X_4) \leq 3 \cdot [H(X_1, X_2) + H(X_3, X_4)].$$