

Tensor-Rank and Lower Bounds for Arithmetic Formulas

Ran Raz*
Weizmann Institute

Abstract

We show that any explicit example for a tensor $A : [n]^r \rightarrow \mathbb{F}$ with tensor-rank $\geq n^{r \cdot (1 - o(1))}$, (where $r = r(n) \leq \log n / \log \log n$), implies an explicit super-polynomial lower bound for the size of general arithmetic formulas over \mathbb{F} . This shows that strong enough lower bounds for the size of arithmetic formulas of depth 3 imply super-polynomial lower bounds for the size of general arithmetic formulas.

One component of our proof is a new approach for homogenization and multilinearization of arithmetic formulas, that gives the following results:

We show that for any n -variate homogenous polynomial f of degree r , if there exists a (fanin-2) formula of size s and depth d for f then there exists a homogenous formula of size $O\left(\binom{d+r+1}{r} \cdot s\right)$ for f . In particular, for any $r \leq \log n$, if there exists a polynomial size formula for f then there exists a polynomial size homogenous formula for f . This refutes a conjecture of Nisan and Wigderson [NW95] and shows that super-polynomial lower bounds for homogenous formulas for polynomials of small degree imply super-polynomial lower bounds for general formulas.

We show that for any n -variate set-multilinear polynomial f of degree r , if there exists a (fanin-2) formula of size s and depth d for f then there exists a set-multilinear formula of size $O((d+2)^r \cdot s)$ for f . In particular, for any $r \leq \log n / \log \log n$, if there exists a polynomial size formula for f then there exists a polynomial size set-multilinear formula for f . This shows that super-polynomial lower bounds for set-multilinear formulas for polynomials of small degree imply super-polynomial lower bounds for general formulas.

*ran.raz@weizmann.ac.il, Research supported by the Israel Science Foundation (ISF), the Binational Science Foundation (BSF) and the Minerva Foundation.

1 Introduction

1.1 Arithmetic Formulas

Let \mathbb{F} be a field and let $\{x_1, \dots, x_n\}$ be a set of input variables. An *arithmetic formula* is a directed tree whose edges are directed from the leaves to the root. Every leaf of the tree is labeled with either an input variable or a field element. Every other node of the tree is labeled with either $+$ or \times ; in the first case the node is a *sum gate* and in the second case a *product gate*. The *size* of a formula is the number of edges in it. The *depth* of a formula is the length of the longest directed path in it¹. The *fanin* of a gate is its in-degree.

Every node of an arithmetic formula computes a polynomial in the ring $\mathbb{F}[x_1, \dots, x_n]$ as follows. A leaf just computes the input variable or field element that labels it. A sum gate computes the sum of the polynomials computed by its children. A product gate computes the product of the polynomials computed by its children. The *output* of the formula is the polynomial computed by the root. The root of the formula is also called the *output node*.

An *arithmetic circuit* is defined in the same way as arithmetic formula, except that the underlying graph is a general directed acyclic graph (rather than a directed tree). For simplicity, we assume that every circuit has exactly one output node.

Proving super-polynomial lower bounds for the size of arithmetic circuits and formulas (for explicit polynomials) is one of the most interesting and most challenging open problems in computational complexity. Such lower bounds are only known for restricted cases. For example, super-polynomial lower bounds were proved for non-commutative formulas [Nis91], for depth 3 formulas over finite fields [GK98, GR98], and for multilinear formulas [R04a, R04b].

1.2 Homogenous Formulas

A polynomial f in the ring $\mathbb{F}[x_1, \dots, x_n]$ is *homogenous* if all the monomials that occur in f are of the same degree. An arithmetic formula (or circuit) is *homogenous* if the polynomial computed by each of its nodes is homogenous.

A standard (and straightforward) homogenization technique by Strassen [Str73] shows that for any homogenous polynomial $f \in \mathbb{F}[x_1, \dots, x_n]$ of degree r , if there exists a formula of size s for f then there exists a homogenous formula of size $\text{poly}(s^{\log r})$ for f . It was conjectured (see for example [NW95]) that this technique is optimal for every degree r . We show that this is not the case. In particular, we show that for any homogenous polynomial $f \in \mathbb{F}[x_1, \dots, x_n]$ of degree $r \leq O(\log n)$, if there exists a polynomial size formula for f then there exists a polynomial size homogenous formula for f . Thus, super-polynomial lower bounds for homogenous formulas of degree up to $O(\log n)$ imply super-polynomial lower bounds for general arithmetic formulas.

¹When counting the depth of a formula, it is customary not to count *scalar-products*, that is, product gates of fanin 2 that at least one of their children is a leaf labeled by a field element.

1.3 Set-Multilinear Formulas

Let X_1, \dots, X_r be sets of variables. A polynomial f over the set of variables $X_1 \cup \dots \cup X_r$ is *set-multilinear* in the sets X_1, \dots, X_r if every monomial that occurs in f is multilinear and contains exactly one variable from each set X_i . An arithmetic formula (or circuit) is *set-multilinear* in the sets X_1, \dots, X_r if the polynomial computed by its output node is set-multilinear in the sets X_1, \dots, X_r and the polynomial computed by each of its other nodes is set-multilinear in a subset of $\{X_1, \dots, X_r\}$.

A standard (and straightforward) multilinearization technique shows that for any set-multilinear polynomial f over the sets X_1, \dots, X_r , if there exists a formula of size s for f then there exists a set-multilinear formula of size $\text{poly}(s^r)$ for f . We show that this technique is not optimal. In particular, we show that for any set-multilinear polynomial f of degree $r \leq O(\log n / \log \log n)$, if there exists a polynomial size formula for f then there exists a polynomial size set-multilinear formula for f . Thus, super-polynomial lower bounds for set-multilinear formulas of degree up to $O(\log n / \log \log n)$ imply super-polynomial lower bounds for general arithmetic formulas.

Set-multilinear formulas were first studied in [NW95]. Super-polynomial lower bounds for multilinear formulas (that are more general than set-multilinear formulas) were proved in [R04a, R04b] (see also [Aar04, RSY07, RY08a, RY08b]). These techniques however do not give super-polynomial lower bounds for polynomials of very small degree.

1.4 Tensor-Rank

A tensor $A : [n]^r \rightarrow \mathbb{F}$ is of rank 1 if there exist r vectors $a_1, \dots, a_r : [n] \rightarrow \mathbb{F}$ such that $A = a_1 \otimes \dots \otimes a_r$ (where \otimes denotes tensor product). More generally, the *tensor-rank* of A is the minimal k such that there exist k tensors $A_1, \dots, A_k : [n]^r \rightarrow \mathbb{F}$ of rank 1 such that $A = \sum_{i=1}^k A_i$. This is a natural generalization of matrix-rank.

Given a tensor $A : [n]^r \rightarrow \mathbb{F}$ and r sets of variables X_1, \dots, X_r , where $X_i = \{x_{i,1}, \dots, x_{i,n}\}$, one can define the set-multilinear polynomial f_A as follows

$$f_A(x_{1,1}, \dots, x_{r,n}) = \sum_{i_1, \dots, i_r \in [n]} A(i_1, \dots, i_r) \cdot \prod_{j=1}^r x_{j,i_j}$$

A beautiful well known and straightforward insight, going back to [Str73] (see [Gat88] for a survey), shows that the tensor-rank of tensors of order $r = 3$ is very related to *bilinear complexity* and hence also to general arithmetic circuit complexity. In particular (using also [BS83]), if the tensor-rank of $A : [n]^3 \rightarrow \mathbb{F}$ is k then the smallest circuit for f_A is of size $\Omega(k)$. Thus, one can prove lower bounds for arithmetic circuit size by proving lower bounds for tensor-rank. Note, however, that the tensor-rank of $A : [n]^3 \rightarrow \mathbb{F}$ is bounded by $O(n^2)$. Hence, this approach can only give lower bounds of up to $\Omega(n^2)$ for arithmetic circuit size. We note also that to date no lower bound better than $\Omega(n)$ is known for the tensor-rank of any explicit tensor $A : [n]^3 \rightarrow \mathbb{F}$.

Here we consider tensors $A : [n]^r \rightarrow \mathbb{F}$ where $r = r(n)$ is super-constant and satisfies $r \leq O(\log n / \log \log n)$. We show that for any such A , if there exists an arithmetic formula of size n^c for f_A then the tensor-rank of A is at most $n^{r \cdot (1 - 2^{-O(c)})}$. Thus, a lower bound of $n^{r \cdot (1 - o(1))}$ for the tensor-rank of A implies a super-polynomial lower bound for the size of any arithmetic formula for² f_A .

Since the tensor-rank of A corresponds to computations of f_A by depth-3 (set-multilinear) formulas, our result shows that strong enough lower bounds for the size of arithmetic formulas of depth 3 imply super-polynomial lower bounds for the size of general arithmetic formulas. Previously, it was well known that strong enough lower bounds for the size of arithmetic circuits of depth 4 imply exponential lower bounds for the size of general arithmetic circuits (see for example [RY08a, R08]). Moreover, a striking recent result of Agrawal and Vinay (based on [VSB83]) shows that *any* exponential lower bound for the size of arithmetic circuits of depth 4 implies an exponential lower bound for the size of general arithmetic circuits [AV08].

We note that it is very easy to give lower bounds of $n^{\lfloor r/2 \rfloor}$ for the tensor-rank of tensors $A : [n]^r \rightarrow \mathbb{F}$ (by taking a full-rank matrix of size $n^{\lfloor r/2 \rfloor} \times n^{\lfloor r/2 \rfloor}$). We are not aware of any better lower bound. We note also that it was proved by Håstad that computing the tensor-rank is an NP-complete problem [H89].

1.5 Preliminaries

We say that an arithmetic formula (or circuit) is of *fanin 2* if the fanin of every gate in it is 2. We say that an arithmetic formula (or circuit) is of *product-fanin-2* if the fanin of every product gate in it is 2. The *product-depth* of a product-fanin-2 formula (or circuit) is the maximal number of product gates along a directed path in it.

For a formula (or circuit) Φ and a node u in it, we denote by Φ_u the sub-formula of Φ rooted at u and by $\hat{\Phi}_u$ the polynomial computed by the formula Φ_u .

It is well known that for any fanin-2 formula Φ of size s , one can assume without loss of generality that the depth of Φ is $O(\log s)$. That is, there exists a formula of size $\text{poly}(s)$ and depth $O(\log s)$ that computes the same polynomial computed by Φ .

1.6 Discussion

A well known approach, first suggested by Strassen [Str73], is to consider the tensor-product of tensors of high tensor-rank as a candidate for a larger tensor with high tensor-rank.

Let n, m, r be such that $m^r = n$, and for simplicity assume that \mathbb{F} is a finite field. Let $A_1, \dots, A_r : [m]^r \rightarrow \mathbb{F}$ be random tensors, and let $A = A_1 \otimes \dots \otimes A_r : [n]^r \rightarrow \mathbb{F}$ be their

²Moreover, it was noted to us by Amir Yehudayoff that by the completeness of the permanent, if the tensor A is “explicit” the super-polynomial lower bound holds for the permanent.

tensor-product, defined by

$$A((i_{1,1}, \dots, i_{1,r}), \dots, (i_{r,1}, \dots, i_{r,r})) = A_1(i_{1,1}, \dots, i_{r,1}) \cdots A_r(i_{1,r}, \dots, i_{r,r}).$$

Since A_1, \dots, A_r are random, with high probability their tensor-rank is high. If one can prove that with probability larger than 0 the tensor-rank of A is at least $n^{r \cdot (1-o(1))}$, one obtains super-polynomial lower bounds for arithmetic formulas for the polynomial f_A , where the entries of the tensors A_1, \dots, A_r are viewed as additional input variables (note that there are only $r \cdot n$ such entries).

2 Homogenization

For a polynomial $f \in \mathbb{F}[x_1, \dots, x_n]$, denote by f_i the homogenous part of f of degree i . That is, $f = \sum_i f_i$ where each f_i is a homogenous polynomial of degree i . In the same way, for a formula (or circuit) Φ and a node u in it denote by $\hat{\Phi}_{u,i}$ the homogenous part of degree i of the polynomial $\hat{\Phi}_u$.

Obviously, if u is a fanin-2 sum gate with children v, w then for every i

$$\hat{\Phi}_{u,i} = \hat{\Phi}_{v,i} + \hat{\Phi}_{w,i} \tag{1}$$

and if u is a fanin-2 product gate with children v, w then for every i

$$\hat{\Phi}_{u,i} = \sum_{j=0}^i \hat{\Phi}_{v,j} \cdot \hat{\Phi}_{w,i-j} \tag{2}$$

Let Φ be a fanin-2 circuit of size s and depth d that computes a homogenous polynomial of degree r . There is a standard and straightforward technique to turn Φ into a fanin-2 homogenous circuit of size $s \cdot \text{poly}(r)$ and depth $O(d \cdot \log r)$ that computes the same polynomial [Str73]. The main idea of the homogenization technique is to split every node u in the circuit into $r + 1$ nodes u_0, \dots, u_r , where each node u_i in the new circuit computes the homogenous part of degree i of the polynomial computed by u ; that is, u_i computes the polynomial $\hat{\Phi}_{u,i}$. The computation of each $\hat{\Phi}_{u,i}$ is done recursively using Equation 1 and Equation 2. If u is the output node of the original circuit, the output node of the new circuit is the node u_r that computes the polynomial $\hat{\Phi}_{u,r} = \hat{\Phi}_u$ (since we assumed that the original circuit computes a homogenous polynomial of degree r). The size of the new circuit is $s \cdot \text{poly}(r)$. When we turn the new circuit into a fanin-2 circuit its depth may increase to $O(d \cdot \log r)$ since the sum in Equation 2 is on up to $r + 1$ elements.

Let Φ be a formula of size s that computes a homogenous polynomial of degree r . The standard homogenization technique turns Φ into a homogenous formula of size $\text{poly}(s^{\log r})$ that computes the same polynomial, as follows. First, assume for simplicity and without loss of generality that Φ is a fanin-2 formula, and assume without loss of generality that the depth of Φ is $O(\log s)$ (see Section 1.5) - this step may increase the size of Φ polynomially. Next, use

the homogenization technique described above to turn Φ into a fanin-2 homogenous circuit of depth $O(\log s \cdot \log r)$ for the same polynomial. Finally, turn that circuit into a formula of size $2^{O(\log s \cdot \log r)} = \text{poly}(s^{\log r})$.

It was conjectured in [NW95] that the increase in the formula-size by a power of $O(\log r)$ in the exponent is necessary. We note also that for the special case of multilinear formulas, it was recently proved in [HY09] that for $r = \text{poly}(n)$ this is indeed the case.

2.1 A New Homogenization Theorem

Our approach can be viewed as a tighter analysis of the standard technique described above. For simplicity and without loss of generality we state and prove our theorem for fanin-2 formulas.

Theorem 1. *Let Φ be a fanin-2 formula of size s and product-depth d that computes a homogenous polynomial of degree r . Then there exists a fanin-2 homogenous formula Ψ of size $O\left(\binom{d+r+1}{r} \cdot s\right)$ and product-depth d that computes the same polynomial³.*

Proof. For every node u in the formula Φ , denote by $\text{path}(u)$ the set of all nodes on the directed path from u to the root (including the node u). For every node u denote by N_u the set of all functions $D : \text{path}(u) \rightarrow \{0, \dots, r\}$ such that:

1. For every $v, w \in \text{path}(u)$ such that v is a sum gate and w is a child of v , $D(v) = D(w)$.
2. For every $v, w \in \text{path}(u)$ such that v is a product gate and w is a child of v , $D(v) \geq D(w)$.

Intuitively, a function $D \in N_u$ describes a possible progress of the degree of a monomial along the path from u to the root.

Construction of Ψ :

The formula Ψ is constructed as follows. For every node u in Φ and $D \in N_u$ we will have a node (u, D) . Every node (u, D) will compute in Ψ the polynomial

$$\hat{\Psi}_{(u,D)} = \hat{\Phi}_{u,D(u)}$$

that is, the homogenous part of degree $D(u)$ of the polynomial $\hat{\Phi}_u$ computed by u in Φ . To compute $\hat{\Phi}_{u,D(u)}$, the node (u, D) will only use the outputs of nodes (v, D') such that v is a child of u and D' extends D . This is done inductively in a straightforward manner (using Equation 1 and Equation 2) as follows:

³In the same way, if the polynomial computed by Φ is not homogenous and is of an arbitrary degree, and r is an integer, there exists Ψ as above that computes the homogenous part of degree r of the polynomial computed by Φ .

1. **u is a leaf:** If u is a leaf in Φ then for every $D \in N_u$ the node (u, D) will be a leaf. If u is labeled by a field element, (u, D) is labeled by the same field element if $D(u) = 0$ and by 0 if $D(u) \neq 0$. If u is labeled by an input variable, (u, D) is labeled by the same input variable if $D(u) = 1$ and by 0 if $D(u) \neq 1$. By the definitions, for every $D \in N_u$,

$$\hat{\Psi}_{(u,D)} = \hat{\Phi}_{u,D(u)}$$

2. **u is a sum gate:** Assume that u is a sum gate in Φ with children v, w . For every $D \in N_u$ denote by $D_v \in N_v$ the function that agrees with D on $\text{path}(u)$ and satisfies $D_v(v) = D(u)$, and in the same way denote by $D_w \in N_w$ the function that agrees with D on $\text{path}(u)$ and satisfies $D_w(w) = D(u)$. The node (u, D) will sum the outputs of the nodes (v, D_v) and (w, D_w) . By the induction hypothesis and Equation 1 we have

$$\hat{\Psi}_{(u,D)} = \hat{\Psi}_{(v,D_v)} + \hat{\Psi}_{(w,D_w)} = \hat{\Phi}_{v,D_v(v)} + \hat{\Phi}_{w,D_w(w)} = \hat{\Phi}_{v,D(u)} + \hat{\Phi}_{w,D(u)} = \hat{\Phi}_{u,D(u)}$$

3. **u is a product gate:** Assume that u is a product gate in Φ with children v, w . For every $D \in N_u$ and $i \in \{0, \dots, D(u)\}$ denote by $D_{v,i} \in N_v$ the function that agrees with D on $\text{path}(u)$ and satisfies $D_{v,i}(v) = i$, and in the same way denote by $D_{w,i} \in N_w$ the function that agrees with D on $\text{path}(u)$ and satisfies $D_{w,i}(w) = i$. The node (u, D) will compute $\hat{\Psi}_{(u,D)}$ from the outputs of the nodes $\{(v, D_{v,i}) : i \in \{0, \dots, D(u)\}\}$ and $\{(w, D_{w,i}) : i \in \{0, \dots, D(u)\}\}$ by the formula

$$\hat{\Psi}_{(u,D)} = \sum_{i=0}^{D(u)} \hat{\Psi}_{(v,D_{v,i})} \cdot \hat{\Psi}_{(w,D_{w,D(u)-i})}$$

By the induction hypothesis and Equation 2 we have

$$\begin{aligned} \hat{\Psi}_{(u,D)} &= \sum_{i=0}^{D(u)} \hat{\Psi}_{(v,D_{v,i})} \cdot \hat{\Psi}_{(w,D_{w,D(u)-i})} = \\ &= \sum_{i=0}^{D(u)} \hat{\Phi}_{v,D_{v,i}(v)} \cdot \hat{\Phi}_{w,D_{w,D(u)-i}(w)} = \sum_{i=0}^{D(u)} \hat{\Phi}_{v,i} \cdot \hat{\Phi}_{w,D(u)-i} = \hat{\Phi}_{u,D(u)} \end{aligned}$$

We fix the output node of Ψ to be the node (u, D) such that u is the output node of Φ and $D \in N_u$ is the function that satisfies $D(u) = r$.

Comment: Note that in the construction above there may be nodes (u, D) that are not connected by a path to the output node of Ψ . These nodes do not contribute to the functionality of Ψ and should be removed so that the final Ψ is a tree rather than a union of trees.

Functionality of Ψ :

We proved by induction that every node (u, D) in Ψ computes the polynomial $\hat{\Psi}_{(u,D)} = \hat{\Phi}_{u,D(u)}$. In particular, the output node (u, D) computes the polynomial $\hat{\Psi}_{(u,D)} = \hat{\Phi}_{u,D(u)} = \hat{\Phi}_{u,r} = \hat{\Phi}_u$, which is the polynomial computed by Φ .

Properties of Ψ :

To see that Ψ is a formula note that the output of a node (v, D') is only used by a node (u, D) such that u is a parent of v and D agrees with D' on $\text{path}(u)$, and there is at most one such node (u, D) . Thus, the out-degree of every node is at most 1.

Ψ is homogenous since each of its nodes computes a homogenous polynomial.

The product-depth of Ψ is the same as the product-depth of Φ since the “product-depth” of Equation 1 is 0 and the “product-depth” of Equation 2 is 1.

Finally, since the size of every N_u is bounded by $\binom{d+r+1}{r}$, the size of Ψ is at most $O\left(\binom{d+r+1}{r} \cdot s\right)$. \square

Corollary 2. *Let $f \in \mathbb{F}[x_1, \dots, x_n]$ be a homogenous polynomial of degree $\leq O(\log n)$. If there exists a polynomial size formula for f then there exists a polynomial size homogenous formula for f .*

Proof. Let Φ be a polynomial size formula for f . Without loss of generality Φ is a fanin-2 formula of depth $O(\log n)$. The proof hence follows from Theorem 1. \square

3 Multilinearization

For vectors $a, b \in \{0, 1\}^r$, we say that $b \leq a$ if for every $i \in \{1, \dots, r\}$, $b(i) \leq a(i)$.

Let X_1, \dots, X_r be sets of variables. For a vector $a \in \{0, 1\}^r$ and a monomial q in the set of variables $X_1 \cup \dots \cup X_r$, we say that q is set-multilinear of type a if it is multilinear and contains exactly one variable from each set X_i such that $a(i) = 1$ and no variables from sets X_i such that $a(i) = 0$. For a polynomial f over the set of variables $X_1 \cup \dots \cup X_r$, and a vector $a \in \{0, 1\}^r$, denote by f_a the set-multilinear part of f of type a . That is, f_a is the sum of all the set-multilinear monomials of type a that occur in f with the same coefficient as their occurrence in f .

In the same way, for a formula (or circuit) Φ and a node u in it denote by $\hat{\Phi}_{u,a}$ the set-multilinear part of type a of the polynomial $\hat{\Phi}_u$.

Obviously, if u is a fanin-2 sum gate with children v, w then for every a

$$\hat{\Phi}_{u,a} = \hat{\Phi}_{v,a} + \hat{\Phi}_{w,a} \quad (3)$$

and if u is a fanin-2 product gate with children v, w then for every a

$$\hat{\Phi}_{u,a} = \sum_{b \in \{0,1\}^r \text{ s.t. } b \leq a} \hat{\Phi}_{v,b} \cdot \hat{\Phi}_{w,a-b} \quad (4)$$

Let Φ be a fanin-2 circuit of size s and depth d that computes a set-multilinear polynomial in the sets X_1, \dots, X_r . There is a standard and straightforward technique to turn Φ into a fanin-2 set-multilinear circuit of size $s \cdot \text{poly}(2^r)$ and depth $O(d \cdot r)$ that computes the same

polynomial. The main idea of the multilinearization technique is to split every node u in the circuit into 2^r nodes $\{u_a\}_{a \in \{0,1\}^r}$, where each node u_a in the new circuit computes the set-multilinear part of type a of the polynomial computed by u ; that is, u_a computes the polynomial $\hat{\Phi}_{u,a}$. The computation of each $\hat{\Phi}_{u,a}$ is done recursively using Equation 3 and Equation 4. If u is the output node of the original circuit, the output node of the new circuit is the node $u_{\vec{1}}$ (where $\vec{1} \in \{0,1\}^r$ denotes the vector $(1, \dots, 1)$) that computes the polynomial $\hat{\Phi}_{u,\vec{1}} = \hat{\Phi}_u$ (since we assumed that the original circuit computes a set-multilinear polynomial in the sets X_1, \dots, X_r). The size of the new circuit is $s \cdot \text{poly}(2^r)$. When we turn the new circuit into a fanin-2 circuit its depth may increase to $O(d \cdot r)$ since the sum in Equation 4 is on up to 2^r elements.

Let Φ be a formula of size s that computes a set-multilinear polynomial in the sets X_1, \dots, X_r . The standard multilinearization technique turns Φ into a set-multilinear formula of size $\text{poly}(s^r)$ that computes the same polynomial, as follows. First, assume for simplicity and without loss of generality that Φ is a fanin-2 formula, and assume without loss of generality that the depth of Φ is $O(\log s)$ (see Section 1.5) - this step may increase the size of Φ polynomially. Next, use the multilinearization technique described above to turn Φ into a fanin-2 set-multilinear circuit of depth $O(\log s \cdot r)$ for the same polynomial. Finally, turn that circuit into a formula of size $2^{O(\log s \cdot r)} = \text{poly}(s^r)$.

3.1 A New Multilinearization Theorem

Our approach can be viewed as a tighter analysis of the standard technique described above. For simplicity and without loss of generality we state and prove our theorem for fanin-2 formulas. The technique that we use is very similar to the technique that we used in Section 2.1.

Theorem 3. *Let Φ be a fanin-2 formula of size s and product-depth d that computes a set-multilinear polynomial over the sets X_1, \dots, X_r . Then there exists a fanin-2 set-multilinear formula Ψ of size $O((d+2)^r \cdot s)$ and product-depth d that computes the same polynomial.*

Proof. For every node u in the formula Φ , denote by $\text{path}(u)$ the set of all nodes on the directed path from u to the root (including the node u). For every node u denote by N_u the set of all functions $D : \text{path}(u) \rightarrow \{0,1\}^r$ such that:

1. For every $v, w \in \text{path}(u)$ such that v is a sum gate and w is a child of v , $D(v) = D(w)$.
2. For every $v, w \in \text{path}(u)$ such that v is a product gate and w is a child of v , $D(v) \geq D(w)$.

Intuitively, a function $D \in N_u$ describes a possible progress of the type of a set-multilinear monomial along the path from u to the root.

Construction of Ψ :

The formula Ψ is constructed as follows. For every node u in Φ and $D \in N_u$ we will have a node (u, D) . Every node (u, D) will compute in Ψ the polynomial

$$\hat{\Psi}_{(u,D)} = \hat{\Phi}_{u,D(u)}$$

that is, the set-multilinear part of type $D(u)$ of the polynomial $\hat{\Phi}_u$ computed by u in Φ . To compute $\hat{\Phi}_{u,D(u)}$, the node (u, D) will only use the outputs of nodes (v, D') such that v is a child of u and D' extends D . This is done inductively in a straightforward manner (using Equation 3 and Equation 4) as follows:

1. **u is a leaf:** If u is a leaf in Φ then for every $D \in N_u$ the node (u, D) will be a leaf. If u is labeled by a field element, (u, D) is labeled by the same field element if $D(u) = \vec{0}$ and by 0 if $D(u) \neq \vec{0}$ (where $\vec{0}$ denotes the 0 vector in $\{0, 1\}^r$). If u is labeled by an input variable from the set X_i , (u, D) is labeled by the same input variable if $D(u) = \vec{e}_i$ and by 0 if $D(u) \neq \vec{e}_i$ (where $\vec{e}_i \in \{0, 1\}^r$ is defined by $\vec{e}_i(j) = 1$ iff $j = i$). By the definitions, for every $D \in N_u$,

$$\hat{\Psi}_{(u,D)} = \hat{\Phi}_{u,D(u)}$$

2. **u is a sum gate:** Assume that u is a sum gate in Φ with children v, w . For every $D \in N_u$ denote by $D_v \in N_v$ the function that agrees with D on $\text{path}(u)$ and satisfies $D_v(v) = D(u)$, and in the same way denote by $D_w \in N_w$ the function that agrees with D on $\text{path}(u)$ and satisfies $D_w(w) = D(u)$. The node (u, D) will sum the outputs of the nodes (v, D_v) and (w, D_w) . By the induction hypothesis and Equation 3 we have

$$\hat{\Psi}_{(u,D)} = \hat{\Psi}_{(v,D_v)} + \hat{\Psi}_{(w,D_w)} = \hat{\Phi}_{v,D_v(v)} + \hat{\Phi}_{w,D_w(w)} = \hat{\Phi}_{v,D(u)} + \hat{\Phi}_{w,D(u)} = \hat{\Phi}_{u,D(u)}$$

3. **u is a product gate:** Assume that u is a product gate in Φ with children v, w . For every $D \in N_u$ and $a \in \{0, 1\}^r$ such that $a \leq D(u)$ denote by $D_{v,a} \in N_v$ the function that agrees with D on $\text{path}(u)$ and satisfies $D_{v,a}(v) = a$, and in the same way denote by $D_{w,a} \in N_w$ the function that agrees with D on $\text{path}(u)$ and satisfies $D_{w,a}(w) = a$. The node (u, D) will compute $\hat{\Psi}_{(u,D)}$ from the outputs of the nodes $\{(v, D_{v,a}) : a \leq D(u)\}$ and $\{(w, D_{w,a}) : a \leq D(u)\}$ by the formula

$$\hat{\Psi}_{(u,D)} = \sum_{a \leq D(u)} \hat{\Psi}_{(v,D_{v,a})} \cdot \hat{\Psi}_{(w,D_{w,D(u)-a})}$$

By the induction hypothesis and Equation 4 we have

$$\begin{aligned} \hat{\Psi}_{(u,D)} &= \sum_{a \leq D(u)} \hat{\Psi}_{(v,D_{v,a})} \cdot \hat{\Psi}_{(w,D_{w,D(u)-a})} = \\ &= \sum_{a \leq D(u)} \hat{\Phi}_{v,D_{v,a}(v)} \cdot \hat{\Phi}_{w,D_{w,D(u)-a}(w)} = \sum_{a \leq D(u)} \hat{\Phi}_{v,a} \cdot \hat{\Phi}_{w,D(u)-a} = \hat{\Phi}_{u,D(u)} \end{aligned}$$

We fix the output node of Ψ to be the node (u, D) such that u is the output node of Φ and $D \in N_u$ is the function that satisfies $D(u) = \vec{1}$.

Comment: Note that in the construction above there may be nodes (u, D) that are not connected by a path to the output node of Ψ . These nodes do not contribute to the functionality of Ψ and should be removed so that the final Ψ is a tree rather than a union of trees.

Functionality of Ψ :

We proved by induction that every node (u, D) in Ψ computes the polynomial $\hat{\Psi}_{(u,D)} = \hat{\Phi}_{u,D(u)}$. In particular, the output node (u, D) computes the polynomial $\hat{\Psi}_{(u,D)} = \hat{\Phi}_{u,D(u)} = \hat{\Phi}_{u,\vec{1}} = \hat{\Phi}_u$, which is the polynomial computed by Φ .

Properties of Ψ :

To see that Ψ is a formula note that the output of a node (v, D') is only used by a node (u, D) such that u is a parent of v and D agrees with D' on $\text{path}(u)$, and there is at most one such node (u, D) . Thus, the out-degree of every node is at most 1.

Ψ is set-multilinear since each of its nodes computes a set-multilinear polynomial.

The product-depth of Ψ is the same as the product-depth of Φ since the “product-depth” of Equation 3 is 0 and the “product-depth” of Equation 4 is 1.

Finally, since the size of every N_u is bounded by $(d+2)^r$, the size of Ψ is at most $O((d+2)^r \cdot s)$. \square

Corollary 4. *Let f be a set-multilinear polynomial over sets X_1, \dots, X_r of size n each, where $r \leq O(\log n / \log \log n)$. If there exists a polynomial size formula for f then there exists a polynomial size set-multilinear formula for f .*

Proof. Let Φ be a polynomial size formula for f . Without loss of generality Φ is a fanin-2 formula of depth $O(\log n)$. The proof hence follows from Theorem 3. \square

4 Tensor-Rank and Formula Size

Recall that given a tensor $A : [n]^r \rightarrow \mathbb{F}$ and r sets of variables X_1, \dots, X_r , where $X_i = \{x_{i,1}, \dots, x_{i,n}\}$, we defined the set-multilinear polynomial f_A by

$$f_A(x_{1,1}, \dots, x_{r,n}) = \sum_{i_1, \dots, i_r \in [n]} A(i_1, \dots, i_r) \cdot \prod_{j=1}^r x_{j,i_j}$$

In this section, we show that for any tensor $A : [n]^r \rightarrow \mathbb{F}$ with $r \leq O(\log n / \log \log n)$, if there exists a polynomial size formula for the polynomial f_A then the tensor-rank of A is not too high.

Theorem 5. Let $A : [n]^r \rightarrow \mathbb{F}$ be a tensor such that $r \leq O(\log n / \log \log n)$. If there exists a formula of size n^c for the polynomial f_A then the tensor-rank of A is at most $n^{r \cdot (1 - 2^{-O(c)})}$.

Proof. Let Φ be a formula of size n^c for f_A . Without loss of generality Φ is a fanin-2 formula of depth $O(\log(n^c))$. Hence, by Theorem 3 we can assume without loss of generality that Φ is a set-multilinear formula (in the sets X_1, \dots, X_r) of size $n^{O(c)}$.

Formulas in Normal Form:

It will be convenient in this proof to allow a leaf of a formula to be labeled by a product of a field element and an input variable (rather than by only one of them). The polynomial computed by such a leaf is the product that labels it.

We will say that a set-multilinear formula is in a *normal form* if it satisfies the following properties (that can be assumed without loss of generality):

1. The fanin of every product gate in the formula is 2.
2. The sum gates in the formula are collapsed so that a child of a sum gate is never a sum gate.
3. No node of the formula computes the 0 polynomial.
4. Every leaf of the formula is labeled by a product of a field element (different than 0) and an input variable (and hence the polynomial computed by any node in the formula is of degree larger than 0).

Without loss of generality we assume that the formula Φ is in normal form.

Tensor-Rank:

We will use the following 3 (straightforward) properties of tensor-rank:

1. For any $A' : [n]^{r'} \rightarrow \mathbb{F}$, (where $r' > 0$),

$$\text{rank}(A') \leq n^{r'-1} \tag{5}$$

2. For any $A_1, \dots, A_k : [n]^{r'} \rightarrow \mathbb{F}$,

$$\text{rank} \left(\sum_{i=1}^k A_i \right) \leq \sum_{i=1}^k \text{rank}(A_i) \tag{6}$$

3. For any $A_1 : [n]^{r_1} \rightarrow \mathbb{F}$ and $A_2 : [n]^{r_2} \rightarrow \mathbb{F}$

$$\text{rank}(A_1 \otimes A_2) \leq \text{rank}(A_1) \cdot \text{rank}(A_2) \tag{7}$$

(where $A_1 \otimes A_2 : [n]^{r_1+r_2} \rightarrow \mathbb{F}$ is the tensor-product of A_1 and A_2 , defined by $A_1 \otimes A_2(i_1, \dots, i_{r_1+r_2}) = A_1(i_1, \dots, i_{r_1}) \cdot A_2(i_{r_1+1}, \dots, i_{r_1+r_2})$).

Syntactic-Rank:

For a set-multilinear formula Ψ in normal form in the sets of variables X_1, \dots, X_r , we define the *syntactic-rank* of Ψ inductively as follows:

1. If u is a leaf,

$$\text{syn-rank}(\Psi_u) = 1$$

2. If u is a sum gate with children u_1, \dots, u_k ,

$$\text{syn-rank}(\Psi_u) = \min \left(n^{r'-1} ; \sum_{i=1}^k \text{syn-rank}(\Psi_{u_i}) \right)$$

where r' is the degree of $\hat{\Psi}_u$ (and note that since Ψ is in normal form, $r' > 0$).

3. If u is a product gate with children u_1, u_2 ,

$$\text{syn-rank}(\Psi_u) = \text{syn-rank}(\Psi_{u_1}) \cdot \text{syn-rank}(\Psi_{u_2})$$

By Equation 5, Equation 6, and Equation 7, it is straightforward to verify by induction on the formula that the syntactic-rank bounds the tensor-rank in the following sense: if A_Ψ is the tensor such that the formula Ψ computes the polynomial f_{A_Ψ} then $\text{rank}(A_\Psi) \leq \text{syn-rank}(\Psi)$. Hence, in order to bound the tensor-rank of A it's sufficient to bound the syntactic-rank of Φ .

Extended-Formulas:

It will be convenient to extend the definition of a set-multilinear formula in normal form (in the sets of variables X_1, \dots, X_r), so that each node in the formula can be labeled (in addition to its label in the formula) by an additional real number larger or equal to 1, referred to as the *weight* of the node (if the node is not labeled we think of the weight as 1). We refer to such formulas as *extended-formulas* (in the sets of variables X_1, \dots, X_r). (Note that *extended-formula* already means that the formula is set-multilinear in normal form).

Intuitively, one should think of a weight w of a node as indicating that the sub-formula rooted at that node is duplicated w times and all the w copies are summed. Note however that w is a real number and not necessarily integer. An extended-formula is not viewed as a computational device and we will not care about its functionality; we define it in order to explore the relations between size and syntactic-rank of a formula.

Size of an Extended-Formula:

We define the *size* of an extended-formula to be the sum over all the leaves of the formula of the product of all weights along the path from that leaf to the root of the formula. Note that

if all weights are 1, the size of an extended-formula is the number of leaves in the formula. This is different than our definition of the size as the number of edges in a formula, but the two notions differ by a factor of at most 2 and hence the difference will not be important.

Note that if in an extended-formula Ψ , u is a node with children u_1, \dots, u_k , then

$$\text{size}(\Psi_u) = \text{weight}(u) \cdot \sum_{i=1}^k \text{size}(\Psi_{u_i}) \quad (8)$$

(where $\text{size}(\Psi_u)$ and $\text{size}(\Psi_{u_i})$ denote the sizes of the extended-formulas Ψ_u and Ψ_{u_i}).

Syntactic-Degree of an Extended Formula:

We define the *syntactic-degree* of an extended-formula Ψ inductively as follows:

1. If u is a leaf, $\text{syn-deg}(\Psi_u) = 1$.
2. If u is a sum gate with children u_1, \dots, u_k , $\text{syn-deg}(\Psi_u) = \text{syn-deg}(\Psi_{u_1})$, (note that for every i, j , $\text{syn-deg}(\Psi_{u_i}) = \text{syn-deg}(\Psi_{u_j})$).
3. If u is a product gate with children u_1, u_2 , $\text{syn-deg}(\Psi_u) = \text{syn-deg}(\Psi_{u_1}) + \text{syn-deg}(\Psi_{u_2})$.

Syntactic-Rank of an Extended Formula:

We extend the definition of syntactic-rank to extended-formulas. We define the *syntactic-rank* of an extended-formula Ψ (in the sets of variables X_1, \dots, X_r) inductively as follows:

1. If u is a leaf,

$$\text{syn-rank}(\Psi_u) = 1 \quad (9)$$

2. If u is a sum gate with children u_1, \dots, u_k ,

$$\text{syn-rank}(\Psi_u) = \min \left(n^{r'-1} ; \text{weight}(u) \cdot \sum_{i=1}^k \text{syn-rank}(\Psi_{u_i}) \right) \quad (10)$$

where $r' = \text{syn-deg}(\Psi_u)$ (and note that $r' > 0$).

3. If u is a product gate with children u_1, u_2 ,

$$\text{syn-rank}(\Psi_u) = \min \left(n^{r'-1} ; \text{weight}(u) \cdot \text{syn-rank}(\Psi_{u_1}) \cdot \text{syn-rank}(\Psi_{u_2}) \right) \quad (11)$$

where $r' = \text{syn-deg}(\Psi_u)$ (and note that $r' > 0$).

Note that if all weights are 1 the definition coincides with the original definition of syntactic-rank.

By the definition, for any node u in Ψ , $\text{syn-rank}(\Psi_u) \leq n^{r'-1}$, where $r' = \text{syn-deg}(\Psi_u)$. We say that a node u in Ψ is of *full-rank* if $\text{syn-rank}(\Psi_u) = n^{r'-1}$.

Size versus Syntactic-Rank of an Extended-Formula:

For a fixed $s = n^{c'}$, where $1 \leq c' \leq \log r - 3$, consider all the extended-formulas Ψ' (in the sets of variables X_1, \dots, X_r) such that $\text{syn-deg}(\Psi') = r$ and $\text{size}(\Psi') \leq s$; and let Ψ be an extended-formula with the largest syntactic-rank among all these formulas (if there are many extended-formulas with the same syntactic-rank, take Ψ to be an extended-formula with the smallest size among all these formulas). We will show that without loss of generality the extended-formula Ψ is of a very specific form.

First we claim that without loss of generality we can assume that every sum gate in Ψ is of fanin 1. Let u be a sum gate in Ψ , with children u_1, \dots, u_k (where $k > 1$). Note that for every i , $\text{syn-deg}(\Psi_u) = \text{syn-deg}(\Psi_{u_i})$. Let $i \in \{1, \dots, k\}$ be such that $\text{syn-rank}(\Psi_{u_i})/\text{size}(\Psi_{u_i})$ is maximal, and let $q = \sum_{j=1}^k \text{size}(\Psi_{u_j})$. We will multiply the weight of u_i by $q/\text{size}(\Psi_{u_i})$ and remove all children of u (together with the sub-trees below them) except u_i . By Equation 8, $\text{size}(\Psi_u)$ and hence also $\text{size}(\Psi)$ didn't change, while by Equation 10 (using also Equation 9 and Equation 11), $\text{syn-rank}(\Psi_u)$ and hence also $\text{syn-rank}(\Psi)$ may only increase. Thus, we have turned u into a fanin 1 gate without increasing the size of the extended-formula and without decreasing its syntactic-rank. By repeating this process we can assume that every sum gate in Ψ is of fanin 1.

Next we claim that without loss of generality we can assume that there are no sum gates in Ψ . This is obvious because any sum gate u of fanin 1 can be removed by connecting its child directly to its parent (or just removing u if it has no parent) and passing its weight to its child (that is, multiplying the weight of the child of u by $\text{weight}(u)$). We can hence assume that Ψ has no sum gates. Thus, Ψ is a Binary tree and every non-leaf node in it is a product gate. Since $\text{syn-deg}(\Psi) = r$, the number of leaves in Ψ is r .

We will now analyze the possible weights of the nodes of Ψ .

First we claim that for every leaf u , $\text{weight}(u) = 1$. This is because for every leaf u , $\text{syn-rank}(\Psi_u) = 1$ (regardless of its weight). Therefore, $\text{weight}(u) = 1$, since a higher weight increases the size of the extended-formula without increasing its syntactic-rank.

Let u be a non-leaf node with children u_1, u_2 . Denote $r_1 = \text{syn-deg}(\Psi_{u_1})$ and $r_2 = \text{syn-deg}(\Psi_{u_2})$. Thus, $\text{syn-deg}(\Psi_u) = r_1 + r_2$. We know that $\text{syn-rank}(\Psi_{u_1}) \leq n^{r_1-1}$ and $\text{syn-rank}(\Psi_{u_2}) \leq n^{r_2-1}$. We claim the following:

1. If $\text{weight}(u) > 1$ then u_1, u_2 are of full-rank. *Proof:* Assume for a contradiction that $\text{weight}(u) > 1$, and u_1 is not of full-rank. (In particular, u_1 is not a leaf). Then by multiplying $\text{weight}(u_1)$ by $1 + \epsilon$ and dividing $\text{weight}(u)$ by $1 + \epsilon$ (for a small enough ϵ) we preserve the syntactic-rank of Ψ_u (by Equation 11) while decreasing its size (by Equation 8). Thus, we preserve the syntactic-rank of Ψ while decreasing its size, in contradiction to the definition of Ψ .
2. If u is of full-rank then $\text{weight}(u) = n$. *Proof:* Since u is of full-rank, $\text{syn-rank}(\Psi_u) = n^{r_1+r_2-1}$. Hence by Equation 11, $\text{weight}(u) \geq n$. Thus, by 1, u_1, u_2 are of full-rank. Thus, $\text{syn-rank}(\Psi_{u_1}) = n^{r_1-1}$ and $\text{syn-rank}(\Psi_{u_2}) = n^{r_2-1}$. Hence, by Equation 11,

weight(u) = n (otherwise, by reducing weight(u) to n we preserve the syntactic-rank of Ψ while decreasing its size, in contradiction to the definition of Ψ).

3. By 1, 2, if u is of full-rank then both u_1, u_2 are of full-rank.

Consider the nodes along a path from a leaf to the root in Ψ . The leaf is always of full-rank and, as we saw, the weight of the leaf is 1. By 1,2,3, following the leaf we have a certain number of full-rank nodes with weight n . After that we have at most one node that is not of full-rank and its weight is larger than 1, and after that we have nodes that are not of full-rank and are with weight 1. Since the size of Ψ is $s = n^{c'}$, the number of full-rank nodes with weight n along a path from a leaf to the root in Ψ is at most $\log_n(s) = c'$.

Denote by V the set of non-leaf nodes in Ψ that are not of full-rank and their weight is 1. Thus, from any leaf of Ψ we can reach a node in the set V by a path of length at most $c' + 2$. Hence, since the number of leaves in Ψ is r and since Ψ is a Binary tree, $|V| \geq \frac{r}{2^{c'+2}}$. Also, by 1,2,3, V is a Binary tree (of product gates) such that if $v \in V$ and v is a child of u then $u \in V$.

Denote by U the set of nodes that are not in V but are connected to V by an edge. Let $l = |U|$ and let u_1, \dots, u_l be the nodes in U and let $r_i = \text{syn-deg}(\Psi_{u_i})$. Note that $l > |V|$ and that for every i , $\text{syn-rank}(\Psi_{u_i}) \leq n^{r_i-1}$. Since V is a Binary tree of product gates $\sum_{i=1}^l r_i = \text{syn-deg}(\Psi) = r$, and since the weight of every node in V is 1, by Equation 11 (applied $|V|$ times)

$$\text{syn-rank}(\Psi) \leq \prod_{i=1}^l \text{syn-rank}(\Psi_{u_i}) \leq n^{\sum_{i=1}^l r_i - l} \leq n^{r - r/2^{c'+2}}$$

Since Ψ was chosen to be the extended-formula with the highest syntactic-rank among all the extended-formulas of size $n^{c'}$, we have the same bound for every extended-formula (as a function of its size). In particular, since Φ is a formula of size $n^{O(c)}$,

$$\text{rank}(A) \leq \text{syn-rank}(\Phi) \leq n^{r - r/2^{O(c)}}$$

□

Corollary 6. *Let $A : [n]^r \rightarrow \mathbb{F}$ be a tensor such that $r \leq O(\log n / \log \log n)$. If the tensor-rank of A is $\geq n^{r \cdot (1 - o(1))}$ then there is no polynomial size formula for the polynomial f_A .*

(Note that the corollary is interesting only when $r = r(n)$ is super-constant. For $r = O(1)$, the corollary holds trivially since the tensor-rank of any $A : [n]^r \rightarrow \mathbb{F}$ is at most n^{r-1}).

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