# FROBENIUS–SCHUR FUNCTIONS

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(with an appendix by Vladimir Ivanov)

Abstract. We introduce and study a new basis in the algebra of symmetric functions. The elements of this basis are called the Frobenius–Schur functions ($FS$-functions, for short).

Our main motivation for studying the $FS$-functions is the fact that they enter a formula expressing the combinatorial dimension of a skew Young diagram in terms of the Frobenius coordinates. This formula plays a key role in the asymptotic character theory of the symmetric groups. The $FS$-functions are inhomogeneous, and their top homogeneous components coincide with the conventional Schur functions ($S$-functions, for short). The $FS$-functions are best described in the super realization of the algebra of symmetric functions. As supersymmetric functions, the $FS$-functions can be characterized as a solution to an interpolation problem.

We find the transition coefficients between the $FS$- and $S$-functions. We also establish the $FS$ analogs for a number of basic facts concerning the $S$-functions: Jacobi–Trudi formula together with its dual form; combinatorial formula (expression in terms of tableaux); Giambelli formula; Sergeev–Pragacz formula.

All these results hold for a large family of bases interpolating between the $FS$-functions and the ordinary $S$-functions.

Contents



## §0. Introduction

Let $\Lambda$ denote the algebra of symmetric functions. Recall that $\Lambda$ is a graded algebra, isomorphic to the algebra of polynomials in the power sums $\boldsymbol{p}_1, \boldsymbol{p}_2, \dots$. A natural homogeneous basis in $\Lambda$ is formed by the *Schur functions* (or $S$-functions, for short). An $S$-function is denoted as $s_\mu$, where the index $\mu$ is a Young diagram.

Let $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ be two infinite collections of indeterminates. We will mainly deal with the *super realization* of $\Lambda$, which is defined by the following specialization of the generators $\boldsymbol{p}_1, \boldsymbol{p}_2, \dots$:

$$\boldsymbol{p}_k \to \boldsymbol{p}_k(x;y) = \sum_{i=1}^{\infty} x_i^k + (-1)^{k-1} \sum_{i=1}^{\infty} y_i^k .$$

In this realization, each element $F \in \Lambda$ turns into a supersymmetric function in $x, y$, which will be denoted as $F(x;y)$. In particular, the supersymmetric Schur function will be denoted as $s_\mu(x;y)$.

Let $\mu$ and $\nu$ be Young diagrams. We are interested in the combinatorial function

$$\nu \mapsto \frac{\dim(\mu,\nu)}{\dim \nu},$$

where $\mu$ is fixed while $\nu$ ranges over the set of Young diagrams; $\dim \nu$ is the number of standard tableaux of shape $\nu$; $\dim(\mu,\nu)$ is the number of standard tableaux of skew shape $\nu/\mu$ provided that $\mu \subseteq \nu$, and 0 otherwise.

Write $\nu$ in the *Frobenius notation*:

$$\nu = (p_1, \dots, p_d \,|\, q_1, \dots, q_d),$$

where $d = d(\nu)$ (the *depth* of $\nu$) is the number of diagonal squares in $\nu$, $p_i = \nu_i - i$, $q_i = \nu_i' - i$, and $\nu'$ stands for the transposed diagram. Let $m = |\mu|$, $n = |\nu|$, where $|\cdot|$ denotes the number of squares of a Young diagram. Note that the function $s_\mu$ is homogeneous of degree $|\mu|$.

Our starting point is the following result:

$$\frac{\dim(\mu,\nu)}{\dim \nu} = \frac{s_\mu(x(\nu); y(\nu)) + \dots}{n(n-1)\dots(n-m+1)}, \tag{0.1}$$

where $(x(\nu); y(\nu))$ are the so–called *modified Frobenius coordinates* of $\nu$:

$$(x(\nu); y(\nu)) = (p_1 + \tfrac{1}{2}, \dots, p_d + \tfrac{1}{2}, 0, 0, \dots \,;\, q_1 + \tfrac{1}{2}, \dots, q_d + \tfrac{1}{2}, 0, 0, \dots), \tag{0.2}$$

and the dots in the numerator denote the remainder term, which is a (super)symmetric function of degree strictly less than $|\mu|$, evaluated at $(x(\nu); y(\nu))$.

An equivalent form of (0.1) is as follows ([VK], [W, III.6]):

$$\frac{\chi^\nu_{\rho \cup 1^{n-m}}}{\dim \nu} = \frac{\boldsymbol{p}_\rho(x(\nu); y(\nu)) + \dots}{n(n-1)\dots(n-m+1)}, \tag{0.1$'$}$$

where $\rho = (\rho_1, \rho_2, \dots)$ is a partition of $m$, $\chi^\nu$ is the irreducible character of the symmetric group $S(n)$ which corresponds to $\nu$, $\chi^\nu_{\rho\cup 1^{n-m}}$ is the character value on the conjugacy class in $S(n)$ indexed by the partition $\rho \cup 1^{n-m}$ of $n$, and $\boldsymbol{p}_\rho = \boldsymbol{p}_{\rho_1} \boldsymbol{p}_{\rho_2} \dots$ .

Formula (0.1$'$) was one of the basic ingredients in the asymptotic approach to Thoma's description of characters of the infinite symmetric group, see [VK], [KV2], [W]. Note that $\dim \nu$ is a well–known combinatorial function for which several nice formulas are known, so that the complexity of (0.1) is mainly caused by the *relative dimension function* $\dim(\mu, \nu)$.

The aim of the present paper is to study in detail the function which appears in the numerator of the right–hand side of (0.1). By the very definition, this is an inhomogeneous supersymmetric function, and by our convention, we can view it as an element of $\Lambda$. We call it the *Frobenius–Schur function* with index $\mu$ (or the $FS$-function, for short) and we denote it as $FS_\mu$. In this notation, (0.1) takes the form

$$\frac{\dim(\mu,\nu)}{\dim\nu} = \frac{FS_\mu(x(\nu);y(\nu))}{n(n-1)\dots(n-m+1)}\,. \tag{0.3}$$

Our main results about the $FS$-functions, and their key properties are as follows:

• The $FS$-functions can be characterized in terms of a *multivariate interpolation problem.*

Specifically, given a diagram $\mu$, $FS_\mu$ is the only (up to a scalar multiple) supersymmetric function in $(x;y)$ that vanishes at $(x;y)=(x(\nu);y(\nu))$ when $|\nu|\le|\mu|$, $\nu\ne\mu$, and does not vanish when $\nu=\mu$.[1]

This characterization follows at once from a correspondence between the $FS$-functions and the so–called shifted Schur functions (see below), and the interpolation properties of the latter functions established in [Ok1], [OO].

• As $FS_\mu$ differs from $s_\mu$ in lower terms only, the $FS$-functions form a basis in $\Lambda$. We find explicitly the *transition coefficients* between the both bases, $\{s_\mu\}$ and $\{FS_\mu\}$; they are given by simple determinantal expressions. Specifically, we have

$$FS_\mu = \sum_\nu \det[c_{p_i,p'_j}]\,\det[c_{q_i,q'_j}]\,s_\nu, \tag{0.4}$$

where $p_1,p_2,\dots,$ $q_1,q_2,\dots$ denote the Frobenius coordinates of $\mu$, and $p'_1,p'_2,\dots,$ $q'_1,q'_2,\dots$ denote the Frobenius coordinates of $\nu$. The summation in (0.4) is taken over diagrams $\nu$ with the length of the diagonal equal to that of $\mu$ (say, $d$). So, the determinants are of order $d$. Finally, the coefficients $c_{pp}$ are as follows:

$$c_{pp'} = \begin{cases} (-1)^{p-p'}e_{p-p'}(\frac12,\frac32,\dots,\frac{2p-1}{2}), & p'\le p,\\ 0, & p'>p,\end{cases} \tag{0.5}$$

for any $p,p'=0,1,2,\dots,$ where $e_1,e_2,\dots$ stand for the elementary symmetric functions. This implies, in particular, that only diagrams $\nu$ contained in $\mu$ may contribute. See Theorem 2.6 and a more general Theorem 7.3.

• Consider the *duality map* $\omega:\Lambda\to\Lambda$ that sends $\boldsymbol{p}_k$ to $(-1)^{k-1}\boldsymbol{p}_k$ (in the super realization, $(\omega(F))(x;y)=F(y;x)$ for $F\in\Lambda$). Then (Proposition 2.2)

$$\omega(FS_\mu)=FS_{\mu'}\,,$$

which is parallel to the well–known property $\omega(s_\mu)=s_{\mu'}$ of the Schur functions.

• For natural $FS$ analogs of the complete homogeneous symmetric functions, elementary symmetric functions, and the hook functions,

$$Fh_k = FS_{(k)},\quad Fe_k = FS_{(1^k)},\quad FS_{(k\,|\,l)} = FS_{(k+1,1^l)},$$

[1] Actually, vanishing holds for all $\nu$ which do not contain $\mu$.

we get counterparts of the well–known generating series. See formulas (2.3), (2.4), and Theorem 2.3.

• There exists a wider family of 'multiparameter Schur functions' which depend on a doubly infinite string $(a_i)_{i\in\mathbb{Z}}$ of complex parameters and interpolate between the conventional Schur functions (case $a_i \equiv 0$) and the $FS$-functions (case $a_i = i-\frac{1}{2}$). See §3.

• Recall that the supersymmetric Schur functions in finitely many variables can be obtained via the *Sergeev–Pragacz formula,* which is an analog of the basic formula for the Schur polynomials,

$$s_\mu(x_1,\dots,x_n)=\frac{\det[x_i^{\mu_j+n-j}]}{\det[x_i^{n-j}]}\,. \tag{0.6}$$

In Theorem 6.1 we get a version of the Sergeev–Pragacz formula for the multiparameter Schur functions (in particular, for the $FS$-functions).

• Recall the classical combinatorial formula:

$$s_\mu(x_1,x_2,\dots)=\sum_T\prod_{(ij)\in\mu}x_{T(i,j)}\,,$$

summed over the semi–standard tableaux $T$ of the shape $\mu$, and note that it also has a super analog. We get a version of this formula for the $FS$-functions (as well as for multiparameter Schur functions), see Theorem 4.6 and the Appendix.

• The classical Jacobi–Trudi formula and its dual version,

$$s_\mu=\det[h_{\mu_i-i+j}],\qquad s_\mu=\det[e_{\mu'_i-i+j}],$$

also have counterparts for the $FS$-functions and multiparameter Schur functions, see formulas just before Theorem 2.3, formula (3.4), and Corollary 4.9.

• Exactly as in the classical Giambelli formula, we have

$$FS_\mu=\det[FS_{(p_i\,|\,q_j)}],$$

where $p_1,\dots,p_d,q_1,\dots,q_d$ are the Frobenius coordinates for $\mu$, $d=d(\mu)$. Moreover, the same formula holds for the multiparameter Schur functions.

Now we shall briefly describe the history of the formula (0.1) and relationships between our work and [VK], [W], [KO], [OO], [Mo], [L1], [L2].

The formula (0.1) (or rather (0.1′)) first appeared in [VK, §5] in connection with the asymptotic approach to Thoma's classification of the characters of the infinite symmetric group. However, the original proof of (0.1′) was not included in [VK] and remained unpublished. Then a proof was given in [W, III.6, Theorem 6] but this work also remained unpublished.

A first published proof of (0.1′)) was given in [KO, Theorem 5]. This paper used the concept of *shifted symmetric functions* (see §1 for the definition). In [KO], it was shown that there exists a natural isomorphism between the algebra $\Lambda^*$ of shifted symmetric functions and the algebra of supersymmetric functions (hence, the algebra $\Lambda$). The both algebras are viewed as algebras of functions on the set of Young diagrams. Under this isomorphism, a shifted symmetric function in the row coordinates $(\nu_1, \nu_2, \dots)$ of a Young diagram $\nu$ turns into a supersymmetric function in the modified Frobenius coordinates (0.2).

The algebra $\Lambda^*$ of shifted symmetric functions was studied in detail in [OO]. In particular, it was proved that

$$\frac{\dim(\mu,\nu)}{\dim\nu} = \frac{s^*_\mu(\nu_1,\nu_2,\dots)}{n(n-1)\dots(n-m+1)}\,, \tag{0.7}$$

where $s^*_\mu \in \Lambda^*$ are the so–called *shifted Schur functions.* By (0.3), this means that under the isomorphism $\Lambda^* \to \Lambda$ mentioned above, $FS_\mu$ is the image of $s^*_\mu$. For $s^*_\mu$, a number of explicit expressions is available, see [OO]. This provides us with explicit formulas for $\dim(\mu,\nu)$ expressed in terms of the row coordinates. The initial purpose of our work was to get similar results in terms of the Frobenius coordinates.

Another source of inspiration for our work was Molev's paper [Mo]. Molev deals with supersymmetric polynomials with finitely many indeterminates $(x_1,\dots,x_m; y_1,\dots,y_n)$ and studies multiparameter Schur polynomials with an arbitrary sequence of parameters $a = (a_i)_{i\in\mathbb{Z}}$. He obtains, among other things, generating series for the one–row and one–column polynomials, a combinatorial formula, and a Sergeev–Pragacz–type formula. However, Molev's approach is not quite consistent with our purposes, because his polynomials are not stable as $m,n\to\infty$ and, consequently, do not define supersymmetric functions in infinitely many indeterminates. Also, his formulas are not symmetric with respect to the duality map $\omega$ that interchanges the $x$'s and the $y$'s.

We show how to make Molev's construction stable and $\omega$-symmetric. The recipe is simple: it suffices to impose the restriction $m = n$ and to introduce in the initial definition of the polynomials a certain 'shift' that depends on $n$. In this way we come to a generalization of the $FS$-functions depending on $a = (a_i)$. In our approach, we start with Molev's generating series, which needs only a slight adaptation. However, our versions of the combinatorial formula and the Sergeev–Pragacz–type formula differ from Molev's versions in a more substantial way.

As was pointed out by Lascoux [L1], [L2], [OO, §15.6], the shifted Schur functions can be introduced by means of the Schubert polynomials. Possibly, the $FS$ functions can also be handled using the techniques of the Schubert polynomials.

As is well known, there exists a deep analogy between the conventional Schur functions (also called Schur's $S$-functions) and Schur's $Q$-functions. The latter are related to projective representations of symmetric groups just in the same way as the former are related to ordinary representations. Schur's $Q$-functions span a subalgebra in $\Lambda$, whose elements can be characterized by a supersymmetry property of another kind, see, e.g., [Ma1, §III.8], [P]. The results of the present paper have counteparts for Schur's $Q$-functions: these are due to Ivanov [I1], [I2].

For an application of the $FS$ functions, see [BO, §6].

Finally, we would like to notice a remarkable stability of the Giambelli formula: in various generalizations of the Schur functions, examined in [Ma1], [OO], [Mo], and the present paper, it remains intact while the Jacobi–Trudi identity must be suitably modified. According to a general theorem due to Macdonald [Ma2], this effect holds under rather wide assumptions. The Giambelli formula is substantially exploited in our approach.

The paper is organized as follows. In §1, we collected definitions and results from [KO], [OO] which are employed in our work. In §2, we introduce the $FS$-functions, then we write down the generating series and calculate the transition coefficients between the $S$- and $FS$-functions. In §3, we introduce a wider family of 'multiparameter Schur functions' depending on sequence of parameters $a = (a_i)$ and we explain their connection with generalized factorial supersymmetric Schur polynomials studied by Molev [Mo]. In §4, we state the combinatorial formula, which expresses the multiparameter Schur functions in terms of tableaux. Its proof is postponed to the Appendix, written by Vladimir Ivanov. §5 is devoted to an analog of the Sergeev–Pragacz formula. In §6 we calculate the transition coefficients between multiparameter Schur functions corresponding to different sequences of parameters.

A short exposition of the present paper is given in [ORV].

*Acknowledgment.* We are grateful to Alain Lascoux for discussions and fruitful criticism. The authors were supported by ???

## 1. Preliminaries on supersymmetric and shifted symmetric functions

In this section, our main references are Macdonald's book [Ma1] (symmetric functions in general), [BR] and [Ma1] (supersymmetric functions), and [OO] (shifted symmetric functions). We take $\mathbb{C}$ as the base field.

The algebra $\Lambda$ of symmetric functions can be initially defined as the algebra of polynomials $\mathbb{C}[\boldsymbol{p}_1, \boldsymbol{p}_2, \dots]$. Then it can be realized, in different ways, as an algebra of functions, depending on a specialization of the generators $\boldsymbol{p}_k$. In the conventional realization, the generators $\boldsymbol{p}_k$ are specialized to the Newton power sums,

$$\boldsymbol{p}_k \mapsto \boldsymbol{p}_k(x) = \sum_{i=1}^{\infty} x_i^k \,,$$

where $x = (x_1, x_2, \dots)$ is an infinite collection of indeterminates. Then elements $f \in \Lambda$ turn into symmetric functions in $x$. The term 'functions' makes sense if one assumes, e.g., that only finitely many of $x_i$'s are different from 0.

Let $\Lambda_n$ be the algebra of symmetric polynomials in $x_1, \dots, x_n$.[2] In the conventional realization, $\Lambda$ is identified with $\varprojlim \Lambda_n$, the projective limit taken in the category of graded algebras, where the projection $\Lambda_n \to \Lambda_{n-1}$ is defined as the specialization $x_n = 0$.

We will mainly deal with another realization of $\Lambda$, which may be called the *super realization.* In this realization, the generators $\boldsymbol{p}_k$ are viewed as super power sums,

$$\boldsymbol{p}_k \mapsto \boldsymbol{p}_k(x; y) = \sum_{i=1}^{\infty} x_i^k + (-1)^{k-1} \sum_{i=1}^{\infty} y_i^k = \boldsymbol{p}_k(x) - \boldsymbol{p}_k(-y) \,,$$

where $y = (y_1, y_2, \dots)$ is another collection of indeterminates. Note that our definition of the super power sums agrees with that of [BR] but slightly differs from that of [Ma1].

[2] To emphasize the difference between the case of infinitely many indeterminates and that of finitely many indeterminates, we shall employ the terms 'functions' and 'polynomials', respectively.

Denote by $\Lambda_{m,n}$ the algebra of polynomials in $x_1,\dots,x_m$, $y_1,\dots,y_n$, which are separately symmetric in $x$'s and $y$'s and which satisfy the following *cancellation property*: for any $i$ and $j$, the result of the specialization $x_i=-y_j=t$ does not depend on $t$. Such polynomials are called *supersymmetric,* and the (double) projective limit algebra $\varprojlim \Lambda_{m,n}$ is called the *algebra of supersymmetric functions* in $x$ and $y$. Here the limit is again taken in the category of graded algebras, and the projections $\Lambda_{m,n}\to\Lambda_{m-1,n}$ and $\Lambda_{m,n}\to\Lambda_{m,n-1}$ are defined by the specializations $x_m=0$ and $y_n=0$, respectively.

By a well–known theorem (see [St], [P]), the super realization of $\Lambda$ establishes an isomorphism between the algebra $\Lambda$ and the algebra $\varprojlim \Lambda_{m,n}$ of supersymmetric functions, so that we may identify these two algebras. For this reason, we will not introduce a separate notation for the algebra of supersymmetric functions.

Given $f\in\Lambda$, we write $f(x;y)$ for the corresponding supersymmetric function in $x$ and $y$, and we write $f(x_1,\dots,x_m;y_1,\dots,y_n)$ for the supersymmetric polynomial that is the image of $f(x;y)$ in $\Lambda_{m,n}$. In §§4–6 it will be convenient to assume $m=n$ and to think about $\Lambda$ as of the projective limit algebra $\varprojlim \Lambda_{n,n}$.

By the cancellation property, the algebra $\Lambda_{n,n}$ of supersymmetric polynomials is invariant under any shift of variables of the form

$$(x_1,\dots,x_n;y_1,\dots,y_n)\ \mapsto\ (x_1+r,\dots,x_n+r;y_1-r,\dots,y_n-r),\qquad r\in\mathbb{C}.$$

Let $T_r$ denote the corresponding automorphism of $\Lambda_{n,n}$:

$$(T_rf)(x_1,\dots,x_n;y_1,\dots,y_n)=f(x_1+r,\dots,x_n+r;y_1-r,\dots,y_n-r).$$

Applying again the cancellation property we see that the automorphisms $T_r:\Lambda_{n,n}\to\Lambda_{n,n}$ and $T_r:\Lambda_{n+1,n+1}\to\Lambda_{n+1,n+1}$ are compatible with the canonical projection $\Lambda_{n+1,n+1}\to\Lambda_{n,n}$. Thus, for any $r\in\mathbb{C}$, we get an automorphism $T_r$ of the algebra $\Lambda=\varprojlim \Lambda_{n,n}$ which informally can be written as follows:

$$(T_rf)(x_1,x_2,\dots;y_1,y_2,\dots)=f(x_1+r,x_2+r,\dots;y_1-r,y_2-r,\dots)\qquad \forall f\in\Lambda. \tag{1.1}$$

Equivalently, $T_r$ can be defined by

$$T_r:\boldsymbol{p}_k\ \mapsto\ \sum_{j=1}^{k}\binom{k}{j}r^{k-j}\boldsymbol{p}_j\,.$$

Recall that the elements $h_k$ and $e_k$ (complete homogeneous symmetric functions and elementary symmetric functions) can be introduced through the generating series:

$$1+\sum_{k=1}^{\infty}h_kt^k=\exp\left(\sum_{k=1}^{\infty}\boldsymbol{p}_k\frac{t^k}{k}\right)=\left(1+\sum_{k=1}^{\infty}e_kt^k\right)^{-1},$$

where $t$ is a formal indeterminate. In the super realization,

$$1+\sum_{k=1}^{\infty}h_kt^k\ \mapsto\ \prod_{i=1}^{\infty}\frac{1+y_it}{1-x_it},\qquad 1+\sum_{k=1}^{\infty}e_kt^k\ \mapsto\ \prod_{i=1}^{\infty}\frac{1+x_it}{1-y_it}.$$

It will be convenient for us to take $t = \frac{1}{u}$ and to redefine the generating series for $\{h_k\}$ and $\{e_k\}$ as formal series in $\frac{1}{u}$, i.e., as elements of $\Lambda[[\frac{1}{u}]]$:

$$H(u) = 1 + \sum_{k=1}^{\infty} \frac{h_k}{u^k}, \qquad E(u) = 1 + \sum_{k=1}^{\infty} \frac{e_k}{u^k}. \tag{1.2}$$

Then their super specialization takes the form

$$H(u)(x;y) = \prod_{i=1}^{\infty} \frac{1 + y_i/u}{1 - x_i/u}, \qquad E(u)(x;y) = \prod_{i=1}^{\infty} \frac{1 + x_i/u}{1 - y_i/u}. \tag{1.2$'$}$$

Obviously,

$$H(u)E(-u) = 1. \tag{1.3}$$

**Proposition 1.1.** *In terms of the generating series* (1.2), *the automorphisms* $T_r$ *defined in* (1.1) *act on the elements* $h_k$ *and* $e_k$ *as follows*:

$$T_r(H(u)) = H(u - r), \qquad T_r(E(u)) = E(u + r).$$

(It should be noted that for any formal series in $\frac{1}{u}$ (in contrast to a series in $u$), the change of a variable $u \mapsto u + \text{const}$ makes sense. )

*Proof.* This follows from the equalities

$$\frac{1 + (y_i - r)/u}{1 - (x_i + r)/u} = \frac{1 + y_i/(u - r)}{1 - x_i/(u - r)}, \qquad \frac{1 + (x_i + r)/u}{1 - (y_i - r)/u} = \frac{1 + x_i/(u + r)}{1 - y_i/(u + r)}.$$

□

Recall that the *duality map* is defined as an algebra isomorphism $\omega : \Lambda \to \Lambda$ such that $\omega(\boldsymbol{p}_k) = (-1)^{k-1}\boldsymbol{p}_k$. In the super realization, $\omega$ reduces to interchanging $x$ and $y$. We have $\omega(h_k) = e_k$, $\omega(e_k) = h_k$.

The *Schur function* $s_\mu$ indexed by a Young diagram $\mu$ can be introduced through the *Jacobi–Trudi formula*:

$$s_\mu = \det[h_{\mu_i - i + j}],$$

where, by convention, $h_0 = 1$, $h_{-1} = h_{-2} = \cdots = 0$, and the order of the determinant is any number greater or equal to $\ell(\mu)$, the number of nonzero row lengths of $\mu$. The Schur functions form a homogeneous basis in $\Lambda$ (it is convenient to agree that $s_\varnothing = 1$).

Given $m$ and $n$, the supersymmetric Schur polynomial $s_\mu(x_1, \ldots, x_m; y_1, \ldots, y_n)$ in $m + n$ indeterminates does not vanish identically if and only if the diagram $\mu$ does not contain the square $(m+1, n+1)$. Such polynomials form a basis in $\Lambda_{m,n}$.

We have

$$\omega(s_\mu) = s_{\mu'},$$

so that (*dual version* of Jacobi–Trudi, or *Nägelsbach–Kostka formula*)

$$s_\mu = \det[e_{\mu'_i - i + j}].$$

For $p,q=0,1,\dots$, let $(p\,|\,q)$ denote the 'hook' Young diagram $(p+1,1^q)$, and let $s_{(p\,|\,q)}$ denote the corresponding 'hook Schur function'.[3] The hook Schur functions form a convenient system of generators of $\Lambda$. However, they are not algebraically independent. The relations between them are as follows:

$$s_{(p+1\,|\,q)}+s_{(p\,|\,q+1)}=s_{(p\,|\,0)}s_{(0\,|\,q)}\,, \qquad p,q=0,1,2,\dots \tag{1.4}$$

(see, e.g., [Ma1, Ex. I.3.9]). This is equivalent to

$$1+(u+v)\sum_{p,q=0}^{\infty}\frac{s_{(p\,|\,q)}}{u^{p+1}v^{q+1}}=\left(1+\sum_{p=1}^{\infty}\frac{h_p}{u^p}\right)\left(1+\sum_{q=1}^{\infty}\frac{e_q}{v^q}\right)=H(u)E(v). \tag{1.5}$$

In the *Frobenius notation,* a diagram is written as $\mu=(p_1,\dots,p_d\,|\,q_1,\dots,q_d)$ or, in more detail,

$$\mu=(p_1(\mu),\dots,p_d(\mu)\,|\,q_1(\mu),\dots,q_d(\mu)),$$

where $d=d(\mu)$, the *depth* of $\mu$, is the number of diagonal squares, and $p_i=\mu_i-i$, $q_i=\mu_i'-i$.

A very useful expression of the Schur functions is given by the *Giambelli formula*:

$$s_\mu=\det[s_{(p_i\,|\,q_j)}]_{i,j=1}^{d}\,.$$

We proceed to the algebra $\Lambda^*$ of *shifted symmetric functions.* This is a filtered algebra such that the associated graded algebra is canonically isomorphic to $\Lambda$. Its definition is parallel to that of the algebra $\Lambda$ in its conventional realization. First, let $\Lambda_n^*$ be the subalgebra in $\mathbb{C}[x_1,\dots,x_n]$ formed by the polynomials which are symmetric in 'shifted' variables $x_j'=x_j-j$, $j=1,\dots,n$. Define the projection map $\Lambda_n^*\to\Lambda_{n-1}^*$ as the specialization $x_n=0$ and note that this projection preserves the filtration defined by ordinary degree of polynomials. Now set $\Lambda^*=\varprojlim\Lambda_n^*$, where the limit is understood in the category of filtered algebras.

The definition of the *shifted Schur functions* $s_\mu^*\in\Lambda^*$ is parallel to the definition of the conventional Schur functions via formula (0.6); the difference is that the ordinary powers $x^m$ are replaced by the *falling factorial powers*

$$x^{\downarrow m}=\begin{cases} x(x-1)\dots(x-m+1), & m\ge 1,\\ 1, & m=0,\end{cases}$$

and a shift of variables is introduced:

$$s_\mu^*(x_1,\dots,x_n)=\frac{\det[(x_i+n-i)^{\downarrow\,\mu_j+n-j}]}{\det[(x_i+n-i)^{\downarrow\,n-j}]}\,. \tag{1.6}$$

Clearly, this is a shifted symmetric polynomial of degree $|\mu|=\sum_i\mu_i$. Moreover, the formula is stable as $n\to\infty$ and, thus, indeed determines an element of $\Lambda^*$. We agree that $s_\varnothing^*=1$.

[3]Note that in [BR], the term 'hook Schur functions' has another meaning.

We have

$$s^*_\mu(x_1,\dots,x_n) = s_\mu(x_1,\dots,x_n) \, + \text{ lower terms}.$$

It follows that, under the canonical isomorphism $\operatorname{gr}\Lambda^* = \Lambda$, the highest term of $s^*_\mu$ coincides with $s_\mu$. This implies, in particular, that the shifted Schur functions form a basis in $\Lambda^*$.

Recall the definition of the *relative dimension function* $\dim(\mu,\nu)$, where $\mu$ and $\nu$ are arbitrary Young diagrams:

If $\mu \subseteq \nu$ then $\dim(\mu,\nu)$ is the number of standard tableaux of skew shape $\nu/\mu$, i.e., the number of chains of diagrams of the form

$$\mu = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^k = \nu, \qquad k = |\nu| - |\mu|.$$

In particular, $\dim(\mu,\mu) = 1$. Next, if $\mu$ is not contained in $\nu$ then $\dim(\mu,\nu) = 0$. Finally, we also set $\dim\mu = \dim(\varnothing,\mu)$, which is equal to the number of standard tableaux of shape $\mu$.

The relative dimension can be expressed in terms of the shifted Schur functions:

**Proposition 1.2.** *For arbitrary Young diagrams $\mu,\nu$,*

$$\frac{\dim(\mu,\nu)}{\dim\nu} = \frac{s^*_\mu(\nu_1,\nu_2,\dots)}{n^{\downarrow m}}, \qquad m = |\mu|, n = |\nu|.$$

Several proofs of this result are given in [OO, Theorem 8.1]. The argument presented below is a modification of one of them.

*Outline of proof.* The starting point is the well–known formula

$$s_\mu \cdot \boldsymbol{p}_1 = \sum_{\nu:\, \nu\supset\mu,\, |\nu|=|\mu+1} s_\nu\,.$$

It follows that

$$\boldsymbol{p}_1^n = \sum_{\nu:\,|\nu|=n} \dim\nu\, s_\nu$$

and, more generally,

$$s_\mu \cdot \boldsymbol{p}_1^{n-m} = \sum_{\nu:\,|\nu|=n} \dim(\mu,\nu)\, s_\nu\,, \qquad |\mu| = m, \quad n \ge m.$$

Arguing as in [Ma1, Ex. I.1.7] we get from these formulas the following ones:

$$\dim\nu = n!\,\det\left[\frac{1}{(\nu_i - i + j)!}\right]_{i,j=1}^{l} = \frac{n!\,\prod_{1\le i<j\le l}(\nu_i - \nu_j - i + j)}{\prod_{i=1}^{l}(\nu_i + l - i)!},$$

$$\dim(\mu,\nu) = (n-m)!\,\det\left[\frac{1}{(\nu_i - \mu_j - i + j)!}\right]_{i,j=1}^{l},$$

where $l$ is any sufficiently large natural number ($l \ge n$ is enough) and $\frac{1}{k!} = \frac{1}{\Gamma(k+1)}$ equals 0 when $k = -1, -2, \dots$ .

This implies

$$\frac{\dim(\mu,\nu)}{\dim\nu} = \frac{1}{n^{\downarrow m}} \cdot \det\left[\frac{1}{(\nu_i-\mu_j-i+j)!}\right]_{i,j=1}^{l} \cdot \frac{\prod_{i=1}^{l}(\nu_i+l-i)!}{\prod_{1\le i<j\le l}(\nu_i-\nu_j-i+j)}\,.$$

After simple transformations we get

$$\frac{\dim(\mu,\nu)}{\dim\nu} = \frac{1}{n^{\downarrow m}}\,\frac{\det\left[(\nu_i+l-i)^{\downarrow\,\mu_j+l-j}\right]_{i,j=1}^{l}}{\prod_{1\le i<j\le l}(\nu_i-\nu_j-i+j)}\,,$$

as required. □

**Proposition 1.3.** *For arbitrary Young diagrams $\mu,\nu$,*

$$s^*_\mu(\nu_1,\nu_2,\dots) = 0 \qquad \textit{unless } \mu\subseteq\nu, \tag{1.7}$$

$$s^*_\mu(\mu_1,\mu_2,\dots) = \frac{|\mu|!}{\dim\mu} \neq 0. \tag{1.8}$$

*Proof.* See [Ok1] and [OO, Theorem 3.1]. □

In the same way as for the conventional Schur functions, we set

$$h^*_k = s^*_{(k)}, \qquad e^*_k = s^*_{(1^k)}, \qquad s^*_{(p\,|\,q)} = s^*_{(p+1,1^q)}.$$

The generating series for $\{h^*_k\}$ and $\{e^*_k\}$ are defined by analogy with (1.2) but the ordinary powers of $u$ are replaced by the falling factorial powers:

$$H^*(u) = 1+\sum_{k=1}^{\infty}\frac{h^*_k}{u^{\downarrow k}}\,, \qquad E^*(u) = 1+\sum_{k=1}^{\infty}\frac{e^*_k}{u^{\downarrow k}}\,.$$

**Proposition 1.4.** *The specialization of these series at $x=(x_1,x_2,\dots)$ takes the form*

$$H^*(u)(x) = \prod_{i=1}^{\infty}\frac{1+i/u}{1+(i-x_i)/u}\,, \qquad E^*(u)(x) = \prod_{i=1}^{\infty}\frac{1+(x_i-i+1)/u}{1+(-i+1)/u}\,,$$

*and the following relation holds (cf. (1.3)):*

$$H^*(u)E^*(-u-1) = 1. \tag{1.9}$$

*Proof.* See [OO, §12]. □

Since $\Lambda^*$ is isomorphic to the algebra of polynomials in $h^*_1, h^*_2,\dots$, we may define a 1-parameter family $\{T_r\}$, $r\in\mathbb{C}$, of automorphisms of the algebra $\Lambda^*$ by

$$T^*_r(H^*(u)) = 1+\sum_{k=1}^{\infty}\frac{T^*_r(h^*_k)}{u^{\downarrow k}} = H^*(u-r).$$

Together with (1.9) this implies

$$T^*_r(E^*(u)) = 1+\sum_{k=1}^{\infty}\frac{T^*_r(e^*_k)}{u^{\downarrow k}} = E^*(u+r).$$

It readily follows that

$$T^*_r(h^*_k) = h^*_k \;+\; \{\text{a linear combination of } h^*_{k-1},\dots,h^*_1,1\},$$
$$T^*_r(e^*_k) = e^*_k \;+\; \{\text{a linear combination of } e^*_{k-1},\dots,e^*_1,1\}.$$

**Proposition 1.5.** *The following analogs of the Jacobi–Trudi formula and its dual version hold:*

$$s^*_\mu = \det[T^*_{j-1}(h^*_{\mu_i-i+j})] = \det[T^*_{-j+1}(e^*_{\mu'_i-i+j})].$$

*Proof.* See [OO, Theorem 13.1]. □

The following general result is due to Macdonald:

**Proposition 1.6.** *Let $A$ be a commutative algebra and $\{h_{k,r}\}$ be a double sequence of elements of $A$. Here $k = 1, 2, \dots$, $r = 0, 1, \dots$. We also agree that $h_{0,r} \equiv 1$ and $h_{k,r} \equiv 0$ for all $k < 0$. For an arbitrary Young diagram $\mu$, set*

$$S_\mu = \det[h_{\mu_i-i+j,j-1}],$$

*where the order of the determinant is any number $\ge \ell(\mu)$. Finally, set*

$$S_{(p\,|\,q)} = S_{(p+1,1^q)}.$$

*Then the Giambelli formula holds:*

$$S_\mu = \det[S_{(p_i\,|\,q_j)}]_{i,j=1}^d, \qquad \text{where } p_i = p_i(\mu),\ q_i = q_i(\mu),\ d = d(\mu).$$

*Proof.* See [Ma1, Ex. I.3.21] or [Ma2]. □

As a corollary we get the following result, which was pointed out in [OO, Remark 13.2]:

**Proposition 1.7.** *For shifted Schur functions, the Giambelli formula remains intact:*

$$s^*_\mu = \det[s^*_{(p_i(\mu)\,|\,q_j(\mu))}].$$

*Proof.* We employ Proposition 1.5 and apply Proposition 1.6 to the two–parameter family $h_{k,r} = T^*_r h^*_k$. □

Following [KO], we define an algebra isomorphism $\varphi : \Lambda^* \to \Lambda$ by

$$\varphi(H^*(u)) = H(u + \tfrac{1}{2}).$$

Clearly, $\varphi$ intertwines the automorphisms $T_r : \Lambda \to \Lambda$ and $T^*_r : \Lambda^* \to \Lambda^*$. The next result yields a characterization of $\varphi$. The key idea is to realize both $\Lambda^*$ and $\Lambda$ as algebras of functions on Young diagrams; here the realization of $\Lambda^*$ is defined in terms of the row coordinates while the realization of $\Lambda$ requires Frobenius coordinates.

**Proposition 1.8.** *Let $f \in \Lambda^*$ be arbitrary. For any Young diagram $\lambda$,*

$$f(\lambda_1, \lambda_2, \dots) = \varphi(f)(p_1 + \tfrac{1}{2}, \dots, p_d + \tfrac{1}{2}\,;\ q_1 + \tfrac{1}{2}, \dots, q_d + \tfrac{1}{2}),$$

*where $\lambda_i$ are the row lengths of $\lambda$ and $(p_1,\dots,p_d\,|\,q_1,\dots,q_d)$ is its Frobenius notation.*

*Proof.* Our argument is a slight simplification of the proof given in [KO]. Let $\square$ range over the squares of $\lambda$ and let $c(\square)$ denote the content of $\square$, i.e., $c(\square)=j-i$ if $\square=(i,j)$. Using Proposition 1.4, we have

$$\begin{aligned}
H^*(u)(\lambda_1,\dots,\lambda_l) &= \prod_{i=1}^{l}\frac{u+i}{u+i-\lambda_i}\\
&= \prod_{i=1}^{l}\frac{u+i}{u+i-1}\,\frac{u+i-1}{u+i-2}\cdots\frac{u+i-\lambda_i+1}{u+i-\lambda_i}\\
&= \prod_{\square\in\lambda}\frac{u-c(\square)+1}{u-c(\square)}.
\end{aligned}$$

In the latter product, we may first fix a diagonal hook and then let $\square$ range along this hook. In the $i$th diagonal hook, the content ranges from $-q_i$ to $p_i$. From this we conclude that

$$\begin{aligned}
H^*(u)(\lambda_1,\dots,\lambda_l) &= \prod_{i=1}^{d}\frac{u+q_i+1}{u-p_i}=\prod_{i=1}^{d}\frac{u+\frac12+(q_i+\frac12)}{u+\frac12-(p_i+\frac12)}\\
&= H(u+\tfrac12)(p_1+\tfrac12,\dots,p_d+\tfrac12\,;q_1+\tfrac12,\dots,q_d+\tfrac12)
\end{aligned}$$

(here we have used (1.2′)). $\square$

## §2. The Frobenius–Schur functions

We keep to the notation of §1.

Define the *Frobenius–Schur function* indexed by a Young diagram $\mu$ as the following element of $\Lambda$:

$$FS_\mu=\varphi(s^*_\mu). \tag{2.1}$$

Here $\varphi$ is the isomorphism $\Lambda^*\to\Lambda$ introduced just before Proposition 1.8.

Then the following characterization theorem holds:

**Proposition 2.1.** *$FS_\mu(x;y)$ is the only supersymmetric function such that for any Young diagram $\nu=(p_1,\dots,p_d\,|\,q_1,\dots,q_d)$,*

$$\frac{\dim(\mu,\nu)}{\dim\nu}=\frac{FS_\mu(p_1+\frac12,\dots,p_d+\frac12,\;;\;q_1+\frac12,\dots,q_d+\frac12)}{n(n-1)\dots(n-m+1)}, \tag{2.2}$$

*where $m=|\mu|$, $n=|\nu|$.*

*Proof.* Formula (2.2) holds by virtue of Proposition 1.2 and Proposition 1.8. The uniqueness claim follows from the fact that a supersymmetric function is uniquely defined by its values on the set of modified Frobenius coordinates of Young diagrams. $\square$

**Proposition 2.2.** *We have*

$$\omega(FS_\mu) = FS_{\mu'}.$$

*Proof.* This follows from the fact that the left–hand side of (2.2) is invariant under transposition of the diagrams, which interchanges Frobenius coordinates. $\square$

Set

$$Fh_k = FS_{(k)} = \varphi(h_k^*), \qquad Fe_k = FS_{(1^k)} = \varphi(e_k^*).$$

The formulas of §1 together with the definition (2.1) lead to the following formulas for the $FS$ functions.

*Generating series*:

$$1 + \sum_{k=1}^{\infty} \frac{Fh_k}{(u-\frac{1}{2})\dots(u-\frac{2k-1}{2})} = H(u) = 1 + \sum_{k=1}^{\infty} \frac{h_k}{u^k} \tag{2.3}$$

$$1 + \sum_{k=1}^{\infty} \frac{Fe_k}{(u-\frac{1}{2})\dots(u-\frac{2k-1}{2})} = E(u) = 1 + \sum_{k=1}^{\infty} \frac{e_k}{u^k} \tag{2.4}$$

Or, in terms of the super realization of $\Lambda$,

$$\begin{aligned} 1 + \sum_{k=1}^{\infty} \frac{Fh_k(x;y)}{(u-\frac{1}{2})\dots(u-\frac{2k-1}{2})} &= \prod_{i=1}^{\infty} \frac{1+y_i/u}{1-x_i/u} \\ 1 + \sum_{k=1}^{\infty} \frac{Fe_k(x;y)}{(u-\frac{1}{2})\dots(u-\frac{2k-1}{2})} &= \prod_{i=1}^{\infty} \frac{1+x_i/u}{1-y_i/u}. \end{aligned}$$

*Jacobi–Trudi and Nägelsbach–Kostka*:

$$\begin{aligned} FS_\mu &= \det[T_{j-1}(Fh_{\mu_i-i+j})] \\ &= \det[T_{1-j}(Fe_{\mu_i'-i+j})]. \end{aligned}$$

Or, in terms of the super realization,

$$\begin{aligned} FS_\mu(x;y) &= \det[Fh_{\mu_i-i+j}(x+j-1; y-j+1)] \\ &= \det[Fe_{\mu_i'-i+j}(x-j+1; y+j-1)], \end{aligned}$$

where we abbreviate

$$x + \text{const} = (x_1 + \text{const}, x_2 + \text{const}, \dots).$$

*Giambelli*:

$$FS_\mu = \det[FS_{(p_i \,|\, q_j)}].$$

The next result is a generating series for the elements $FS_{(p\,|\,q)}$. It will be used in Theorem 2.4 and is of independent interest.

**Theorem 2.3.** *We have*

$$1+(u+v)\sum_{p,q=0}^{\infty}\frac{FS_{(p\,|\,q)}}{(u-\frac12)\dots(u-\frac{2p+1}{2})(v-\frac12)\dots(v-\frac{2q+1}{2})}$$
$$=H(u)E(v), \tag{2.5}$$

*cf.* (1.5). *Equivalently,*

$$FS_{(p+1\,|\,q)}+FS_{(p\,|\,q+1)}+(p+q+1)FS_{(p\,|\,q)}=FS_{(p\,|\,0)}FS_{(0\,|\,q)}\,,\qquad p,q=0,1,2,\dots, \tag{2.6}$$

*cf.* (1.4).

*Proof.* Since $FS_{(p\,|\,q)}=\varphi(s^*_{(p\,|\,q)})$ and

$$H(u)=\varphi(H^*(u-\tfrac12)),\qquad E(v)=\varphi(E^*(v-\tfrac12)),$$

the relation (2.5) is equivalent to

$$1+(u+v)\sum_{p,q=0}^{\infty}\frac{s^*_{(p\,|\,q)}}{(u-\frac12)\dots(u-\frac{2p+1}{2})(v-\frac12)\dots(v-\frac{2q+1}{2})}$$
$$=H^*(u-\tfrac12)E^*(v-\tfrac12)$$
$$=\left(1+\sum_{p=0}^{\infty}\frac{s^*_{(p\,|\,0)}}{(u-\frac12)\dots(u-\frac{2p+1}{2})}\right)\left(1+\sum_{q=0}^{\infty}\frac{s^*_{(0\,|\,q)}}{(v-\frac12)\dots(v-\frac{2q+1}{2})}\right).$$

Writing

$$u+v=(u-\tfrac{2p+1}{2})+(v-\tfrac{2q+1}{2})+(p+q+1),$$

we reduce the above formula to the system of relations

$$s^*_{(p+1\,|\,q)}+s^*_{(p\,|\,q+1)}+(p+q+1)s^*_{(p\,|\,q)}=s^*_{(p\,|\,0)}s^*_{(0\,|\,q)},\qquad p,q=0,1,2,\dots. \tag{2.7}$$

Note that this system is equivalent to (2.6).

Thus, it suffices to check (2.7). This is a particular case of a Littlewood–Richardson–type rule for the $s^*$-functions, found by Molev and Sagan [MS]. Here is an elementary derivation of (2.7).

Let us expand the product $s^*_{(p\,|\,0)}s^*_{(0\,|\,q)}$ into a linear combination of the shifted Schur functions; such an expansion exists, because the shifted Schur functions form a basis in $\Lambda^*$. By virtue of (1.4) we know the highest terms of this expansion, so we get

$$s^*_{(p\,|\,0)}s^*_{(0\,|\,q)}=s^*_{(p+1\,|\,q)}+s^*_{(p\,|\,q+1)}+\sum_{\nu:\,|\nu|\le p+q+1}c(\nu)s^*_\nu\,,$$

where $c(\nu)$ are certain number coefficients. Let $X$ be the set of those $\nu$'s that enter this sum with nonzero coefficients $c(\nu)$. We claim that $X$ contains only the diagram $(p\,|\,q)$; here we will use Proposition 1.3 and the fact that $|\nu|\le p+q+1$.

Indeed, let $\lambda$ be a minimal (with respect to inclusion) diagram in $X$ and evaluate both sides at the point $\lambda=(\lambda_1,\lambda_2,\dots)$. On the right, the result is nonzero, because $s^*_\lambda(\lambda)\ne 0$ while all other terms on the right have zero contributions, because neither a diagram $\nu\ne\lambda$ from $X$ nor $(p+1\,|\,q)$ and $(p\,|\,q+1)$ are contained in $\lambda$. So, the result of the evaluation on the left is nonzero, too. This implies that $\lambda$ contains both $(p\,|\,0)$ and $(0\,|\,q)$, which is only possible for $\lambda=(p\,|\,q)$.

Thus, our expansion takes the form

$$s^*_{(p\,|\,0)}s^*_{(0\,|\,q)}=s^*_{(p+1\,|\,q)}s^*_{(p\,|\,q+1)}+\text{const}\,s^*_{(p\,|\,q)}\,,$$

and it remains to find the constant. Evaluating both sides at $\lambda=(p\,|\,q)$ we get

$$\text{const}=\frac{s^*_{(p\,|\,0)}(\lambda)s^*_{(0\,|\,q)}(\lambda)}{s^*_{(p\,|\,q)}(\lambda)}\,.$$

Applying Proposition 1.2 we get

$$s^*_{(p\,|\,0)}(\lambda)=\frac{\dim((p\,|\,0),(p\,|\,q))}{\dim(p\,|\,q)}\,(p+q+1)^{\downarrow p+1}=(p+q+1)p!$$

Similarly,

$$s^*_{(0\,|\,q)}(\lambda)=(p+q+1)q!$$

Finally, by (1.8),

$$s^*_{(p\,|\,q)}(\lambda)=\frac{(p+q+1)!}{\dim(p\,|\,q)}=(p+q+1)p!q!$$

This implies const $=p+q+1$, as was required. □

Our aim is to expand the $FS$ functions in the $S$ functions.

**Proposition 2.4.** *We have*

$$FS_{(p\,|\,q)}=\sum_{p'=0}^{p}\sum_{q'=0}^{q}c_{pp'}c_{qq'}s_{(p'\,|\,q')}\,, \tag{2.8}$$

*where*

$$c_{pp'}=\begin{cases}(-1)^{p-p'}e_{p-p'}(\frac12,\frac32,\dots,\frac{2p-1}{2}), & p'\le p,\\ 0, & p'>p,\end{cases} \tag{2.9}$$

*for any* $p,p'=0,1,2,\dots$.

(Note that $c_{pp}=1$.)

*Proof.* By virtue of (2.5) and (1.5),

$$(u+v)\sum_{p,q=0}^{\infty}\frac{FS_{(p\,|\,q)}}{(u-\frac12)\dots(u-\frac{2p+1}{2})(v-\frac12)\dots(v-\frac{2q+1}{2})}$$
$$=H(u)E(v)-1=(u+v)\sum_{p,q=0}^{\infty}\frac{s_{(p\,|\,q)}}{u^{p+1}v^{q+1}}$$

Note that $u+v$ can be written as $(\frac{1}{u}+\frac{1}{v})(\frac{1}{u}\cdot\frac{1}{v})^{-1}$ and that the algebra of formal series in the indeterminates $\frac{1}{u}$, $\frac{1}{v}$ has no zero divisors. Consequently, we may divide both sides by $u+v$, which gives

$$\sum_{p,q=0}^{\infty} \frac{FS_{(p\,|\,q)}}{(u-\frac{1}{2})\dots(u-\frac{2p+1}{2})(v-\frac{1}{2})\dots(v-\frac{2q+1}{2})} = \sum_{p,q=0}^{\infty} \frac{s_{(p\,|\,q)}}{u^{p+1}v^{q+1}}. \tag{2.10}$$

We need the following Lemma.

**Lemma 2.5.** *If $a_1, a_2, \dots$ and $b_1, b_2, \dots$ are two number sequences, then for any $p' = 0, 1, \dots,$*

$$\frac{1}{(u-b_1)\dots(u-b_{p'+1})} = \sum_{p=p'}^{\infty} \frac{h_{p-p'}(b_1,\dots,b_{p'+1}\,;\,-a_1,\dots,-a_p)}{(u-a_1)\dots(u-a_{p+1})},$$

*where $h_0 = 1, h_2, \dots$ denote the conventional complete homogeneous functions in the super realization of the algebra $\Lambda$.*

*Proof of Lemma.* Indeed, we have

$$\frac{1}{(u-b_1)\dots(u-b_{p'+1})} = \sum_{p=p'}^{\infty} \frac{d(p,p')}{(u-a_1)\dots(u-a_{p+1})}$$

with certain coefficients $d(p,p')$. It is readily seen that $d(p,p')$ is equal to the coefficient of $u^{-1}$ in the expansion of

$$\frac{(u-a_1)\dots(u-a_p)}{(u-b_1)\dots(u-b_{p'+1})} \in \mathbb{C}((\tfrac{1}{u})),$$

or, equivalently, to the coefficient of $u^{p'-p}$ in the expansion of

$$\frac{(u-a_1)\dots(u-a_p)}{(u-b_1)\dots(u-b_{p'+1})u^{p-p'-1}} \in \mathbb{C}[[\tfrac{1}{u}]].$$

By $(1.2')$, the latter coefficient is exactly $h_{p-p'}(b_1,\dots,b_{p'+1}\,;\,-a_1,\dots,-a_p)$. $\square$

Returning to the proof of the proposition, let us apply Lemma 2.5 to

$$a_1 = \tfrac{1}{2},\ a_2 = \tfrac{3}{2},\ a_3 = \tfrac{5}{2},\dots, \quad b_1 = b_2 = \dots = 0.$$

Then, using the relation

$$h_{p-p'}(0,\dots,0\,;\,-\tfrac{1}{2},-\tfrac{3}{2},\dots,-\tfrac{2p-1}{2}) = (-1)^{p-p'}e_{p-p'}(\tfrac{1}{2},\tfrac{3}{2},\dots,\tfrac{2p-1}{2}) = c_{pp'},$$

we get

$$\frac{1}{u^{p'+1}} = \sum_{p=p'}^{\infty} \frac{c_{pp'}}{(u-\frac{1}{2})\dots(u-\frac{2p+1}{2})}.$$

Likewise, we have

$$\frac{1}{u^{q'+1}} = \sum_{q=q'}^{\infty} \frac{c_{qq'}}{(u-\frac{1}{2})\dots(u-\frac{2q+1}{2})},$$

whence

$$\frac{1}{u^{p'+1}v^{q'+1}} = \sum_{p=p'}^{\infty}\sum_{q=q'}^{\infty} \frac{c_{pp'}c_{qq'}}{(u-\frac{1}{2})\dots(u-\frac{2p+1}{2})(u-\frac{1}{2})\dots(u-\frac{2q+1}{2})}.$$

Substituting this into the right–hand side of (2.10) we get

$$FS_{(p\,|\,q)}) = \sum_{p'=0}^{p}\sum_{q'=0}^{q} c_{pp'}\, c_{qq'}\, s_{(p'\,|\,q')},$$

which concludes the proof of Proposition 2.4. □

**Theorem 2.6.** (i) *We have*

$$FS_\mu = \sum_\nu c_{\mu\nu} s_\nu,$$

*summed over diagrams $\nu$ which are contained in $\mu$ and have the same number of diagonal squares as $\mu$.*

(ii) *Write $\mu$ and $\nu$ in Frobenius notation*

$$\begin{aligned} \mu &= (p_1,\dots,p_d\,|\,q_1,\dots,q_d), \qquad p_1>\dots>p_d\ge 0, \quad q_1>\dots>q_d\ge 0,\\ \nu &= (p'_1,\dots,p'_d\,|\,q'_1,\dots,q'_d), \qquad p'_1>\dots>p'_d\ge 0, \quad q'_1>\dots>q'_d\ge 0. \end{aligned}$$

*Then we have*

$$c_{\mu\nu} = \det[c_{p_i,p'_j}]\,\det[c_{q_i,q'_j}], \tag{2.11}$$

*where the determinants are of order $d$ and the coefficients $c_{pp'}$ are defined by* (2.9).

*Proof.* Both $FS_\mu$ and $s_\nu$ can be expressed via the Giambelli formula,

$$FS_\mu = \det[FS_{(p_i\,|\,q_j)}], \qquad s_\nu = \det[s_{(p'_i\,|\,q'_j)}].$$

Consequently, it suffices to prove the claim of the theorem in the simplest case when $\mu$ is a hook diagram, $\mu = (p\,|\,q)$, which was done in Proposition 2.4. □

## §3. Multiparameter Schur functions

Let $a = (a_i)_{i\in\mathbb{Z}}$ be an arbitrary sequence of complex numbers. All the symmetric functions introduced in this section will depend on $a$.

First, we define the multiparameter Schur functions $h_{k;a}$ which are indexed by one row diagrams $(k)$, i.e., certain analogs of the complete homogeneous functions. To do this we shall employ generating functions which are formal series in $u^{-1}$, cf. (2.3):

$$1+\sum_{k=1}^{\infty}\frac{h_{k;a}}{(u-a_1)\dots(u-a_k)} = H(u) = 1+\sum_{k=1}^{\infty}\frac{h_k}{u^k}. \tag{3.1}$$

Clearly, we have

$$h_{k;a} = h_k + \text{lower terms}. \tag{3.2}$$

This implies, in particular, that $\{h_{k;a}\}_{k=1,2,\dots}$ is a system of algebraically independent generators of $\Lambda$.

We agree that

$$h_{0;a} = 1, \qquad h_{-1;a} = h_{-2;a} = \dots = 0.$$

We also need the following notation: for $r \in \mathbb{Z}$, let $\tau^r a$ be the result of shifting $a$ by $r$ digits to the left,

$$(\tau^r a)_i = a_{i+r}\,. \tag{3.3}$$

Now we are in a position to define the *multiparameter Schur function* indexed by an arbitrary Young diagram $\mu$:

$$s_{\mu;a} = \det[h_{\mu_i - i + j;\tau^{1-j}a}] \tag{3.4}$$

where the order of the determinant is any number greater or equal to $\ell(\mu)$, the number of rows in $\mu$. Clearly, $h_{k;a} = s_{(k);a}$.

Note that our definition (3.4) is a particular case of a very general concept of *multi–Schur functions* due to Lascoux [L1] (see also [L2] and [Ma3]).

**Proposition 3.1.** *Multiparameter Schur functions as defined above satisfy the Giambelli formula*:

$$s_{\mu;a} = \det[s_{(p_i\,|\,q_j);a}],$$

*where the determinant has order $d = d(\mu)$ and $p_1, \dots, p_d; q_1, \dots, q_d$ denote the Frobenius coordinates of $\mu$.*

*Proof.* This is immediate from Macdonald's result stated above as Proposition 1.6. □

Note that if $a \equiv 0$ then $s_{\mu;a} = s_\mu$. For arbitrary $a$, it follows from (3.2) and (3.4) that

$$s_{\mu;a} = s_\mu + \text{lower terms},$$

which implies that the elements $s_{\mu;a}$ form a basis in $\Lambda$.

**Proposition 3.2.** *If $a_i = i - \frac{1}{2}$ then $s_{\mu;a} = FS_\mu$.*

*Proof.* Both functions can be given by a Jacobi–Trudi–type formula, see §2 and (3.4). Consequently, it suffices to prove that for $a_i = i - \frac{1}{2}$

$$h_{k;a} = Fh_k, \qquad h_{k;\tau^{-r}a} = T_r(Fh_k).$$

The first equality is immediate from the comparison of (3.1) and (2.3). Let us prove the more general second equality. By (3.1) and (3.3),

$$1 + \sum_{k=1}^{\infty} \frac{h_{k;\tau^{-r}a}}{(u - a_{1-r}) \dots (u - a_{k-r})} = H(u).$$

By our assumption on $a$, we have $a_{i-r} = a_i - r$. Substitute this in the latter expression and then replace $u+r$ by $u$. Then we get

$$1+\sum_{k=1}^{\infty}\frac{h_{k;\tau^{-r}a}}{(u-a_1)\dots(u-a_k)} = H(u-r).$$

The left–hand side is equal to

$$1+\sum_{k=1}^{\infty}\frac{h_{k;\tau^{-r}a}}{(u-\frac{1}{2})\dots(u-\frac{2k-1}{2})}$$

while the right–hand side, by virtue of Proposition 1.1 and (2.3), is equal to

$$1+\sum_{k=1}^{\infty}\frac{T_r(Fh_k)}{(u-\frac{1}{2})\dots(u-\frac{2k-1}{2})}.$$

This proves the second equality. □

Thus, the multiparameter Schur functions interpolate between the conventional $S$-functions and the $FS$-functions.

**Remark 3.3.** Following the general philosophy of symmetric functions, one can define multiparameter Schur functions indexed by *skew* diagrams $\lambda/\mu$ by making use of the canonical comultiplication $\Delta : \Lambda \to \Lambda\otimes\Lambda$. We recall that $\Delta$ is specified by setting $\Delta(\boldsymbol{p}_k) = \boldsymbol{p}_k\otimes 1 + 1\otimes \boldsymbol{p}_k$, or, which is the same, it corresponds to splitting the collection of the variables into two disjoint parts: $x = x'\sqcup x''$ (and, in the super case, $y = y'\sqcup y''$). Then $s_{\lambda/\mu;a}$ is defined by

$$\Delta(s_{\lambda;a}) = \sum_{\mu} s_{\mu;a}\otimes s_{\lambda/\mu;a}.$$

The result of the Appendix shows that $s_{\lambda/\mu;a}$ vanishes unless $\mu\subseteq\lambda$, and an analogue of (3.4) holds:

$$s_{\lambda/\mu;a} = \det[h_{\lambda_i-\mu_j-i+j;\tau^{\mu_j-j+1}a}].$$

□

As was emphasized in §0, our definition of the multiparameter Schur functions was suggested by Molev's work [Mo]. In the rest of this section we discuss the connection with [Mo].

Let $\Lambda_{m,n}$ denote the algebra of supersymmetric polynomials in $m+n$ variables [BR],[4] and let $\mathbb{Y}_{m,n}$ denote the set of Young diagrams not containing the square $(m+1, n+1)$. It is well known that the conventional Schur polynomials $s_\mu(x_1,\dots,x_m;y_1,\dots,y_n)$, where $\mu$ ranges over $\mathbb{Y}_{m,n}$, form a homogeneous basis in

[4] Here we say 'supersymmetric polynomials' in place of 'supersymmetric functions' in order to emphasize that one deals with finitely many variables.

$\Lambda_{m,n}$. The algebra of supersymmetric functions can be identified with the projective limit of the graded algebras $\Lambda_{m,n}$ as both $m, n$ go to infinity.

In [Mo], Molev introduced a family of multiparameter supersymmetric Schur polynomials, which depend on $a$ and are denoted as

$$s_\mu(x_1, \dots, x_m/y_1, \dots, y_n|a). \tag{3.5}$$

These polynomials are inhomogeneous, and their top degree homogeneous components coincide with the conventional supersymmetric Schur polynomials. This implies that the polynomials (3.5) form a basis in $\Lambda_{m,n}$. When $a \equiv 0$, they reduce to the conventional supersymmetric Schur polynomials.

Molev's initial definition is given in terms of a combinatorial formula. Next, he writes down generating series for the $h$- and $e$-functions and establishes a Jacobi–Trudi–type formula and its dual analog. He then gets a version of the Sergeev–Pragacz formula. His paper also contains a number of other results which will not be discussed here.

Recall that a fundamental property of the conventional supersymmetric Schur polynomials is their stability: specializing $x_m = 0$ gives the supersymmetric Schur polynomial in $(m-1)+n$ variables with the same index, and similarly for $y_n = 0$. Due to the stability property, one can define the supersymmetric Schur functions in $\infty+\infty$ variables. It is pointed out in [Mo] that the polynomials (3.5) lose the stability property. Our observation is that stability is recovered if we restrict ourselves to the case $m = n$ and slightly modify Molev's definition. The exact correspondence between Molev's polynomials and our multiparameter Schur functions is as follows:

**Proposition 3.4.** *For any $\mu$ and any $n$,*

$$s_{\mu;a}(x_1, \dots, x_n; y_1, \dots, y_n) = s_\mu(x_1, \dots, x_n/y_1, \dots, y_n|\tau^{-n}a)$$

*Proof.* Assume first that $\mu = (k)$, where $k = 1, 2, \dots$, and write $h_k(\dots)$ instead of $s_{(k)}(\dots)$. By [Mo, (2.6)] we have

$$1 + \sum_{k=1}^{\infty} \frac{h_k(x_1, \dots, x_n/y_1, \dots, y_n|a)}{(u-a_{n+1})\dots(u-a_{n+k})} = \frac{(u+y_1)\dots(u+y_n)}{(u-x_1)\dots(u-x_n)}.$$

Replacing $a$ by $\tau^{-n}a$ we get, by (1.2′) and (3.1),

$$1 + \sum_{k=1}^{\infty} \frac{h_k(x_1, \dots, x_n/y_1, \dots, y_n|\tau^{-n}a)}{(u-a_1)\dots(u-a_k)} = \frac{(u+y_1)\dots(u+y_n)}{(u-x_1)\dots(u-x_n)}$$

$$= H(u)(x_1, \dots, x_n; y_1, \dots, y_n) = 1 + \sum_{k=1}^{\infty} \frac{h_{k;a}(x_1, \dots, x_n; y_1, \dots, y_n)}{(u-a_1)\dots(u-a_k)}.$$

This implies that

$$h_{k;a}(x_1, \dots, x_n; y_1, \dots, y_n) = h_k(x_1, \dots, x_n/y_1, \dots, y_n|\tau^{-n}a), \tag{3.6}$$

which is our claim for $\mu = (k)$.

For a general $\mu$ we employ Molev's Jacobi–Trudi formula [Mo,(3.1)], which gives

$$s_\mu(x_1, \ldots, x_n/y_1, \ldots, y_n|a) = \det[h_{\mu_i - i + j}(x_1, \ldots, x_n/y_1, \ldots, y_n|\tau^{-j+1}a)],$$

whence

$$\begin{aligned} s_\mu(x_1, \ldots, x_n/y_1, \ldots, y_n|\tau^{-n}a) &= \det[h_{\mu_i - i + j}(x_1, \ldots, x_n/y_1, \ldots, y_n|\tau^{-n-j+1}a)] \\ &= \det[h_{\mu_i - i + j; \tau^{1-j}a}(x_1, \ldots, x_n; y_1, \ldots, y_n)], \quad \text{by (3.6)} \\ &= s_{\mu;a}(x_1, \ldots, x_n; y_1, \ldots, y_n), \quad \text{by (3.4)}. \end{aligned}$$

This completes the proof. $\square$

It should be emphasized that this correspondence fails when $m \neq n$.

## §4. Combinatorial formula

We attach to $a$ the 'dual' sequence $\widehat{a}$ such that

$$\widehat{a}_i = -a_{-i+1}\,. \tag{4.1}$$

Note that in the $FS$ case, we have $\widehat{a} = a$.

Let $\mathbb{Z}' = \{\ldots, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots\}$ stand for the set of proper half–integers. Certain formulas will look more symmetric if we agree to label the terms of the sequence $a$ by the half–integers. For this reason we introduce the alternative notation

$$a' = (a'_\varepsilon)_{\varepsilon \in \mathbb{Z}'}, \qquad a'_\varepsilon = a_{\varepsilon + 1/2}\,. \tag{4.2}$$

In this notation, (4.1) takes symmetric form:

$$(\widehat{a})'_\varepsilon = -a'_{-\varepsilon}. \tag{4.3}$$

Note that in the $FS$ case, $a'_\varepsilon = \varepsilon$.

Recall that a *horizontal strip* is a skew Young diagram containing at most one square in each column. Dually, a *vertical strip* contains at most one square in each row. More generally, we shall deal with skew diagrams $\nu$ of the following kind: there exists a skew subdiagram $\nu_1 \subseteq \nu$ such that $\nu_1$ is a horizontal strip while $\nu/\nu_1$ is a vertical strip (equivalently, there exists $\nu_2 \subseteq \nu$ which is a vertical strip while $\nu/\nu_2$ is a horizontal strip. These are exactly skew diagrams $\nu$ containing no $2 \times 2$ block of squares (equivalently, the contents of the squares $\square \in \nu$ are pairwise distinct). Such a diagram is called a *skew hook*[5] if, in addition, it is connected. Thus, a skew diagram with no $2 \times 2$ block of squares is a disjoint union of skew hooks.

To each skew diagram $\nu$ containing no $2 \times 2$ block of squares we attach a polynomial $f_{\nu;a}(u, v)$ in two variables $u, v$, of degree $|\nu|$, as follows.

First, assume $\nu$ is a skew hook. Consider the interior sides $s$ of the shape $\nu$: an interior side is adjacent to two squares of $\nu$; the total number of the interior

[5] Other terms: border strip, ribbon, see [Ma1].

sides is equal to $|\nu|-1$. To each interior side $s$ we attach the coordinates $(\varepsilon,\delta)$ of its midpoint,[6] and we write $s=(\varepsilon,\delta)$. Note that one of the coordinates is always half–integral while another coordinate is integral. Specifically, if $s$ is a vertical side then $\varepsilon\in\mathbb{Z}'$, $\delta\in\mathbb{Z}$, and the ends of $s$ are the points $(\varepsilon-1/2,\delta)$ and $(\varepsilon+1/2,\delta)$; if $s$ is a horizontal side then $\varepsilon\in\mathbb{Z}$, $\delta\in\mathbb{Z}'$, and the ends of $s$ are the points $(\varepsilon,\delta-1/2)$, $(\varepsilon,\delta+1/2)$.

For both vertical and horizontal sides, $\delta-\varepsilon\in\mathbb{Z}'$. Using the notation (4.2), we set

$$f_{\nu;a}(u,v)=(u+v)\prod_{\substack{\text{vertical interior}\\ \text{sides } s=(\varepsilon,\delta) \text{ of } \nu}}(u-a'_{\delta-\varepsilon})\prod_{\substack{\text{horizontal interior}\\ \text{sides } s=(\varepsilon,\delta) \text{ of } \nu}}(v+a'_{\delta-\varepsilon}). \tag{4.4}$$

For instance, if $\nu=(4,2,2)/(1,1)$ then there are 6 squares and 5 interior sides with midpoints

$$(\tfrac{5}{2},1),\quad (2,\tfrac{3}{2}),\quad (1,\tfrac{3}{2}),\quad (\tfrac{1}{2},2),\quad (\tfrac{1}{2},3),$$

and we have

$$\begin{aligned} f_{\nu;a}(u,v)&=(u+v)(u-a'_{-3/2})(u-a'_{3/2})(u-a'_{5/2})(v+a'_{-1/2})(v+a'_{1/2})\\ &=(u+v)(u-a_{-1})(u-a_2)(u-a_3)(v+a_0)(v+a_1)\end{aligned}$$

When $\nu$ is an arbitrary skew diagram with no $2\times2$ block of squares, we define $f_{\nu;a}(u,v)$ as the product of the polynomials attached to its connected components.

**Proposition 4.1.** *Let $\nu$ be a skew Young diagram containing no $2\times2$ block of squares. Then*

$$f_{\nu;a}(v,u)=f_{\nu';\widehat{a}}(u,v).$$

*Proof.* This is immediate from (4.3) and (4.4). $\square$

Let $\mu$ be a Young diagram. Recall that a *semistandard* (or *column–strict*) tableau of shape $\mu$ is a function $\mathcal{T}(\square)$ from the squares of $\mu$ to $\{1,2,\dots\}$ such that the numbers $\mathcal{T}(\square)$ weakly increase from left to right along the rows and strictly increase down the columns. For such a tableau $\mathcal{T}$, the pull–back $\mathcal{T}^{-1}(i)\subset\mu$ is a horizontal strip for any $i=1,2,\dots$, see [Ma1, I.5]. Dually, for a *row–strict* tableau, each subset of the form $\mathcal{T}^{-1}(i)$ is a vertical strip. Now, we introduce the following definition:

A *diagonal–strict* tableau of shape $\mu$ is a function $\mathcal{T}(\square)$ from the squares of $\mu$ to $\{1,2,\dots\}$ such that the numbers $\mathcal{T}(\square)$ weakly increase both along the rows (from left to right) and down the columns, and strictly increase along the diagonals $j-i=$ const. We will also consider diagonal–strict tableaux with entries in $\{1,\dots,n\}$.

Clearly, each subset of form $\mathcal{T}^{-1}(i)$ is a skew diagram with no $2\times2$ block of squares, i.e., a disjoint union of skew hooks. Thus, a diagonal–strict tableau $\mathcal{T}$ with entries in $\{1,\dots,n\}$ may be viewed as a chain of Young diagrams,

$$\varnothing=\mu^{(0)}\subseteq\mu^{(1)}\subseteq\cdots\subseteq\mu^{(n)}=\mu,$$

[6] According to the ‘English’ manner of drawing Young diagrams, we assume that the first coordinate axis is directed downwards and the second coordinate axis is directed to the right.

such that $\mu^{(i)}/\mu^{(i-1)}$ has no $2\times 2$ block of squares for each $i = 1,\dots,n$.

Note that the definition of a diagonal–strict tableau also makes sense for a skew Young diagram.

Given an ordinary or skew Young diagram $\mu$ and indeterminates $x = (x_i)$, $y = (y_i)$, consider the combinatorial sum

$$\Sigma_{\mu;a}(x;y) = \sum_{\mathcal{T}} \prod_{i\geq 1} f_{\mathcal{T}^{-1}(i);a}(x_i,y_i) \tag{4.5}$$

summed over all diagonal–strict tableaux of shape $\mu$.

By Proposition 4.1, we have

$$\Sigma_{\mu;a}(y;x) = \Sigma_{\mu';\widehat{a}}(x;y). \tag{4.6}$$

**Proposition 4.2.** *Assume $x_i = y_i = 0$ for $i > n$. Then only tableaux $\mathcal{T}$ with entries in $\{1,\dots,n\}$ have nonzero contribution to the sum* (4.5).

*Proof.* Indeed, assume $\mathcal{T}$ takes a certain value $i > n$. Then for this $i$, the shape $\mathcal{T}^{-1}(i) \subset \mu$ is nonempty. By the definition of the polynomials $f_{\nu;a}$ (see (4.4)), $f_{\mathcal{T}^{-1}(i);a}(x_i,y_i)$ contains the factor $x_i + y_i$, which is zero by the assumption. Consequently, the contribution of $\mathcal{T}$ is zero. $\square$

Thus, under the above assumption, the sum (4.5) is actually finite. Note that the same holds under the weaker assumption that $x_i = -y_i$ for all $i > n$.

**Proposition 4.3.** *The sum* (4.5) *can also be defined by recurrence as follows: for any $k < n$, if $\lambda$ is a skew diagram, then*

$$\begin{aligned}&\Sigma_{\lambda;a}(x_1,\dots,x_k,x_{k+1},\dots,x_n;y_1,\dots,y_k,y_{k+1},\dots,y_n)\\ &\qquad= \sum_{\mu\subseteq\lambda} \Sigma_{\mu;a}(x_1,\dots,x_k;y_1,\dots,y_k)\Sigma_{\lambda/\mu;a}(x_{k+1},\dots,x_n;y_{k+1},\dots,y_n),\end{aligned}$$

*summed over skew diagrams $\mu$ contained in $\lambda$, and*

$$\Sigma_{\mu;a}(x_1;y_1) = \begin{cases} f_{\mu;a}(x_1;y_1), & \textit{if } \mu \textit{ contains no } 2\times 2 \textit{ block of squares,}\\ 0, & \textit{otherwise.}\end{cases}$$

*Proof.* This is evident. $\square$

We proceed to an alternative description of the sum (4.5). Consider the ordered alphabet

$$\mathbb{A} = \{1' < 1 < 2' < 2 < \dots\}$$

and call an $\mathbb{A}$-*tableau* of shape $\mu$ any map $T(\cdot)$ from the set of squares of $\mu$ to the alphabet $\mathbb{A}$ such that:

(*) The symbols $T(\square)$ weakly increase from left to right along each row and down each column.

(**) For each $i = 1,2,\dots,$ there is at most one symbol $i'$ in each row and at most one symbol $i$ in each column.

This definition (as well as that of diagonal–strict tableaux) is suggested by the branching rules for the supersymmetric Schur polynomials, see [BR, §2], especially Theorem 2.15 in [BR]. Note also that the $\mathbb{A}$-tableaux can be obtained via an appropriate 'super' version of the Robinson–Schensted–Knuth correspondence, see [BR, §2] and [RS].[7]

On the other hand, $\mathbb{A}$-tableaux were employed for *shifted* Young diagrams, in the combinatorial formula for the Schur $Q$-functions and their factorial analogs, see [Ma1, III.8, (8.16′)], [I].

**Proposition 4.4.** *The combinatorial sum* (4.5) *can be written as follows*

$$\Sigma_{\mu;a}(x;y) = \sum_T \left( \prod_{\substack{\square\in\mu \\ T(\square)=1,2,\dots}} (x_{T(\square)} - a_{c(\square)}) \prod_{\substack{\square\in\mu \\ T(\square)=1',2',\dots}} (y_{|T(\square)|} + a_{c(\square)}) \right), \quad (4.7)$$

*summed over all $\mathbb{A}$-tableaux of shape $\mu$, where we use the notation $|i'| = i$ for $i = 1, 2, \dots$ and $c(\square)$ denotes the content of $\square$, i.e., if $\square = (i, j)$ then $c(\square) = j - i$.*

*Proof.* Let $\nu$ be a skew hook and $i \in \{1, 2, \dots\}$ be fixed. Clearly, there exists exactly one diagonal–strict tableau $\mathcal{T}$ of shape $\nu$, with entries in $\{i\}$, and we claim that there exist exactly two $\mathbb{A}$-tableaux $T$ of the same shape, with entries in $\{i, i'\}$. Indeed, let $\square_1, \dots, \square_k$ be the squares of $\nu$ written down in the order of increment of contents. Then $T(\square_1)$ may be chosen arbitrarily, while for any $r = 2, \dots, k$, the value of $T(\square_r)$ is defined uniquely, according to whether the squares $\square_{r-1}$, $\square_r$ lie in the same row or in the same column: in the former case, $T(\square_r) = i$, and in the latter case, $T(\square_r) = i'$.

Next, if $\nu$ is a skew diagram with no $2 \times 2$ block of squares then the same is true, with the only exception that the number of the $T$'s is equal to $2^l$, where $l$ stands for the number of connected components of $\nu$.

Now, let $\mu$ be an arbitrary (skew) diagram. To any $\mathbb{A}$-tableau $T$ of shape $\mu$ we assign a diagonal–strict tableau $\mathcal{T}$ by replacing each primed index $i'$ by the corresponding nonprimed index $i$. Conversely, any diagonal–strict tableau $\mathcal{T}$ of shape $\mu$ can be obtained in this way from a certain (nonunique) $\mathbb{A}$-tableau $T$. To get all such $T$'s, we have to choose, for any nonempty diagram $\nu = \mathcal{T}^{-1}(i)$, an arbitrary $\mathbb{A}$-tableau of shape $\nu$, with entries in $\{i, i'\}$, as described above.

This means that the right–hand side in (4.7) can be written as a double sum, where the exterior sum is taken over the diagonal–strict tableaux $\mathcal{T}$ of shape $\mu$, and each interior sum is taken over all $\mathbb{A}$-tableaux $T$ 'over' a fixed $\mathcal{T}$. It follows that the claim of the proposition can be reduced to the following one: let $\nu$ by a skew

[7] Strictly speaking, the version of the RSK correspondence given in [BR,§2] is related to another ordering of the alphabet $\mathbb{A}$. However, the construction can be readily rephrased to handle our ordering. Actually, there are many different 'super' versions of RSK, related to different shuffles of the primed and nonprimed indices. This fact was briefly pointed out at the bottom of page 125 of [BR]. For a detailed analysis, see [RS]. Note that the RSK correspondence implies various formulas for the enumeration of the $\mathbb{A}$-tableaux. Finally, note that from another point of view, a 'super' version of the RSK correspondence was also discussed in [KV1].

Young diagram with no $2 \times 2$ block of squares; then

$$f_{\nu;a}(u;v) = \sum_{T:\nu \to \{1',1\}} \left( \prod_{\substack{\square \in \nu \\ T(\square)=1}} (u - a_{c(\square)}) \prod_{\substack{\square \in \nu \\ T(\square)=1'}} (v + a_{c(\square)}) \right),$$

summed over all $\mathbb{A}$-tableaux of shape $\nu$, with entries in $\{1, 1'\}$.

Finally, without loss of generality, we may assume that $\nu$ is a skew hook. Then, as was shown above, there are exactly two $T$'s, so that the sum in the right–hand side of the last formula consists of two summands. On the other hand, the left–hand side is given by (4.4). Writing in that expression the factor $u + v$ as the sum of $u - a_c$ and $v + a_c$, where $c = c(\square_1)$ is the smallest content, we split the left–hand side into two summands, too. Then the desired equality is readily verified. $\square$

The following claim is a slight refinement of Proposition 4.4:

**Proposition 4.5.** *Assume $x_i = y_i = 0$ (or, more generally, $x_i = -y_i$) for all $i > n$. Then, in the right–hand side of* (4.7)*, one can take only tableaux $T$ with entries in $\mathbb{A}_n = \{1' < 1 < \cdots < n' < n\}$.*

*Proof.* Indeed, this follows from Proposition 4.3 and the proof of Proposition 4.4. $\square$

Note that the $\mathbb{A}$-tableaux $T$, in contrast to the diagonal–strict tableaux $\mathcal{T}$, are not consistent with transposition. In particular, the symmetry (4.6) is not evident from (4.7). However, in certain circumstances, it is more convenient to use formula (4.7) than formulas (4.4) and (4.5).

**Theorem 4.6** (Combinatorial formula)**.** *We have*

$$s_{\mu;a}(x;y) = \Sigma_{\mu;a}(x;y)$$

*where the right–hand side is given by* (4.5) *or* (4.7)*.*

*Proof.* See the Appendix. $\square$

Note that for the first time, an 'inhomogeneous' combinatorial formula probably appeared in [BL1], [BL2], see also [CL]. Other examples can be found in [GG], [Ma2], [Mo], [Ok1], [OO], [Ok2]. See also further references in [Ok2] to works by Knop, Okounkov, and Sahi about combinatorial formulas for interpolation Jack and Macdonald polynomials.

**Corollary 4.7** (Duality)**.** *We have*

$$\omega(s_{\mu;a}) = s_{\mu';\widehat{a}}.$$

*Proof.* Indeed, this follows from Theorem 4.6 and (4.6). $\square$

Set

$$e_{k;a} = s_{(1^k);a}\,.$$

By Corollary 4.7,

$$\omega(h_{k;a}) = e_{k;\widehat{a}}. \tag{4.8}$$

**Corollary 4.8** (Generating series for $e$-functions)**.** *We have*

$$1+\sum_{k=1}^{\infty}\frac{e_{k;a}}{(u-\widehat{a}_1)\dots(u-\widehat{a}_k)}=E(u)=1+\sum_{k=1}^{\infty}\frac{e_k}{u^k}. \tag{4.9}$$

*Proof.* Applying $\omega$ to both sides of (3.1) we get

$$1+\sum_{k=1}^{\infty}\frac{e_{k;\widehat{a}}}{(u-a_1)\dots(u-a_k)}=\omega(H(u))=E(u).$$

Next, replacing $a$ by $\widehat{a}$, we get (4.9). □

**Corollary 4.9** (Nägelsbach–Kostka formula)**.** *We have*

$$s_{\mu;a}=\det[e_{\mu'_i-i+j;\tau^{j-1}a}]$$

*with the understanding that*

$$e_{0;a}=1, \qquad e_{-1;a}=e_{-2;a}=\dots=0.$$

*Proof.* This follows from (3.4), Corollary 4.7, (4.8), and the fact that $(\tau^r a)\widehat{\ } = \tau^{-r}\widehat{a}$. □

**Remark 4.10.** Let us specialize $a_i=i-\frac{1}{2}$. That is, take $a'_{\delta-\varepsilon}=\delta-\varepsilon$ in (4.4), and $a_{c(\square)}=c(\square)-\frac{1}{2}$ in (4.7). Then Theorem 4.6 turns into a combinatorial formula for the $FS$- functions.

## §5. Vanishing property

We fix a Young diagram $\mu$ and write it in the Frobenius notation,

$$\mu=(p_1,\dots,p_d\,|\,q_1,\dots,q_d).$$

Let $\lambda$ be an arbitrary diagram,

$$\lambda=(P_1,\dots,P_D\,|\,Q_1,\dots,Q_D).$$

Then $\mu\subseteq\lambda$ means that $d\le D$, $p_i\le P_i$, $q_i\le Q_i$ for $i=1,\dots,d$.

We define a collection of variables $(x(\lambda);y(\lambda))$ as follows

$$\begin{gathered} x(\lambda)_i=a_{P_i+1}, \quad y(\lambda)_i=\widehat{a}_{Q_i+1}, \quad 1\le i\le D,\\ x(\lambda)_i=y(\lambda)_i=0, \quad i>D. \end{gathered}$$

For instance, if $a_i=i-\frac{1}{2}$ then $(x(\lambda),y(\lambda))$ is exactly the collection of the modified Frobenius coordinates of $\lambda$.

**Theorem 5.1** (Vanishing Theorem). *If $\mu \not\subseteq \lambda$ then $s_{\mu;a}(x(\lambda); y(\lambda)) = 0$.*

*Proof.* We employ, in a slightly modified form, an argument due to Ok1ounkov, cf. [Ok1, proof of Prop. 3.8] and [OO, second proof of Theorem 11.1].

Assume that $s_{\mu;a}(x(\lambda); y(\lambda)) \neq 0$ and let us prove that $\lambda \supseteq \mu$.

*Step 1.* Let us prove that $D \geq d$. Employ for $s_{\mu;a}(x(\lambda); y(\lambda))$ the expression given by Theorem 4.6 and formulas (4.4) and (4.5). By Proposition 4.2, we can take in (4.5) only tableaux $\mathcal{T}$ with entries in $\{1, \dots, D\}$. On the other hand, the main diagonal in $\mu$ has length $d$ and, by the definition of a diagonal–strict tableau, it is filled by strictly increasing numbers. Consequently, $D \geq d$.

*Step 2.* By Corollary 4.7, the quantity $s_{\mu;a}(x(\lambda); y(\lambda))$ does not change under $\mu \mapsto \mu'$, $\lambda \mapsto \lambda'$, $a \mapsto \widehat{a}$. Consequently, to conclude that $\lambda \supseteq \mu$, it suffices to prove that

$$P_1 \geq p_1, \dots, P_d \geq p_d \,. \tag{5.1}$$

Write the coordinates $x_i(\lambda)$, $y_i(\lambda)$, $i = 1, \dots, D$, in the reverse order,

$$\begin{aligned} \bar{x} &= (\bar{x}_1, \dots, \bar{x}_D) = (a_{P_D+1}, \dots, a_{P_1+1}), \\ \bar{y} &= (\bar{y}_1, \dots, \bar{y}_D) = (\widehat{a}_{Q_D+1}, \dots, \widehat{a}_{Q_1+1}). \end{aligned}$$

Since $s_{\mu;a}(x; y)$ is symmetric in $x$ and in $y$, we get

$$s_{\mu;a}(\bar{x}; \bar{y}) = s_{\mu;a}(x(\lambda); y(\lambda)) \neq 0.$$

By Theorem 4.6, $s_{\mu;a}(\bar{x}; \bar{y})$ is given by formula (4.7). Let us fix a tableau $T$ which has nonzero contribution to the sum (4.7). For this $T$, we get, in particular,

$$\prod_{\substack{\square \in \mu \\ T(\square) = 1, 2, \dots}} (\bar{x}_{T(\square)} - a_{c(\square)}) \neq 0. \tag{5.2}$$

We aim to prove that (5.2) implies (5.1). Note that, by Proposition 4.5, $|T(\square)|$ takes values in $\{1, \dots, D\}$.

Introduce the notation

$$(k(1), \dots, k(D)) = (P_D + 1, \dots, P_1 + 1),$$

so that

$$1 \leq k(1) < \dots < k(D). \tag{5.3}$$

In this notation,

$$\bar{x}_r = a_{k(r)}, \qquad r = 1, \dots, D.$$

Our argument will employ the following evident fact:

(*) If, for a certain square $\square \in \mu$, $T(\square)$ is nonprimed then $k(T(\square)) \neq c(\square)$. (Indeed, otherwise we would get

$$\bar{x}_{T(\square)} - a_{c(\square)} = a_{k(T(\square))} - a_{c(\square)} = 0,$$

in contradiction with (5.2).)

*Step 3.* Let $\mathcal{T}$ be the diagonal–strict tableau corresponding to $T$. By its definition, $\mathcal{T}(\square) = |T(\square)|$ for any square $\square \in \mu$. Consider the squares $\square = (1, j)$ of the first row in $\mu$. For these squares,

$$\mathcal{T}(1,1) \le \mathcal{T}(1,2) \le \cdots \le \mathcal{T}(1,\mu_1).$$

We claim that

$$k(\mathcal{T}(1,j)) \ge j, \qquad j = 1, \ldots, \mu_1. \tag{5.4}$$

Indeed, (5.4) is trivial for $j = 1$. Assuming that (5.4) is true for $j \le j_0$, let us check it for $j = j_0 + 1$.

The numbers $k(\mathcal{T}(1,j))$ weakly increase. Therefore, if $k(\mathcal{T}(1, j_0+1)) \le j_0$ then, by the assumption,

$$k(\mathcal{T}(1,j_0)) = k(\mathcal{T}(1,j_0+1)) = j_0.$$

It follows that $T(1, j_0 + 1)$ is nonprimed (indeed, if $T(1, j_0 + 1)$ were primed then $T(1, j_0)$ would be the same primed index, which contradicts to the definition of $\mathbb{A}$-tableaux). Then for the square $\square = (1, j_0 + 1)$ we get: $T(\square)$ is nonprimed and $k(T(\square)) = j_0 = c(\square)$, which is in contradiction with (*). Thus, we have proved (5.4) by induction.

*Step 4.* Recall that the numbers $\mathcal{T}(\square) = |T(\square)|$ take values in $\{1, \ldots, D\}$. On the other hand, these numbers strictly increase as $\square$ ranges over any diagonal in $\mu$ from top to bottom. Together with the inequalities (5.3) this implies that the numbers $k(\mathcal{T}(\square))$ also strictly increase along diagonals. Consequently, for any square $\square = (i, j) \in \mu$ with $i \le j$,

$$k(\mathcal{T}(i,j)) > k(\mathcal{T}(i-1,j-1)) > \cdots > k(\mathcal{T}(1,j-i+1)) \ge j-i+1, \tag{5.5}$$

where, on the last step, we have used (5.4).

Let us fix $i = 1, \ldots, d$ and set $j = \mu_i$, which corresponds to the last square in the $i$th row of $\mu$ (since $i \le d$, the assumption $j \ge i$ is satisfied). Then $j - i + 1 = p_i + 1$. On the other hand, by the definition, each number $k(\,\cdot\,)$ is equal to a certain $P_r + 1$, and recall that $P_1 > \cdots > P_D$, so that $P_1 + 1 > \cdots > P_D + 1$. Thus, (5.5) means that, for certain $r_1 < \cdots < r_i$,

$$P_{r_1} + 1 > \cdots > P_{r_i} + 1 \ge p_i + 1.$$

Since $r_i \ge i$, it follows that $P_i \ge p_i$, which concludes the proof. $\square$

**Theorem 5.2.** *Let $\mu$ be a Young diagram and let the symbols $x(\cdot)$, $y(\cdot)$ be as defined in the beginning of the section. We have*

$$s_{\mu;a}(x(\mu); y(\mu)) = \prod_{(i,j)\in\mu} (a_{\mu_i-i+1} - a_{j-\mu'_j}). \tag{5.6}$$

Note that

$$a_{\mu_i-i+1} - a_{j-\mu'_j} = a'_{\mu_i-i+1/2} - a'_{-\mu'_j+j-1/2} = a_{\mu_i-i+1} + \widehat{a}_{\mu'_j-j+1}\,,$$

where $a' = (a'_\varepsilon)$ was defined in (4.2). Hence, (5.6) can be rewritten as follows:

$$s_{\mu;a}(x(\mu); y(\mu)) \\ = \prod_{(i,j)\in\mu} (a'_{\mu_i-i+1/2} - a'_{-\mu'_j+j-1/2}) = \prod_{(i,j)\in\mu} (a_{\mu_i-i+1} + \widehat{a}_{\mu'_j-j+1}). \quad (5.7)$$

These two expressions are symmetric with respect to $\mu \leftrightarrow \mu'$. One more expression is given below in Proposition 5.4.

*Proof.* As before, write $\mu$ in the Frobenius notation, $\mu = (p_1, \dots, p_d \,|\, q_1, \dots, q_d)$. Let $\mu^* \subset \mu$ denote the shape which is obtained from $\mu$ by removing the border skew hook — the set of the squares $(i,j) \in \mu$ such that $(i+1, j+1) \notin \mu$. In Frobenius notation, $\mu^*$ is obtained by removing the Frobenius coordinates $p_1$ and $q_1$,

$$\mu^* = (p_2, \dots, p_d \,|\, q_2, \dots, q_d).$$

Note that $\mu^*$ is the smallest subdiagram in $\mu$ such that $\mu/\mu^*$ has no $2 \times 2$ block of squares.

Let $\bar{x}(\mu)$ and $\bar{y}(\mu)$ be obtained from $x(\mu)$ and $y(\mu)$ by writing the coordinates in the reversed order,

$$\bar{x}(\mu) = (\bar{x}_1, \dots, \bar{x}_d) = (a_{p_d+1}, \dots, a_{p_1+1}),$$
$$\bar{y}(\mu) = (\bar{y}_1, \dots, \bar{y}_d) = (\widehat{a}_{q_d+1}, \dots, \widehat{a}_{q_1+1}).$$

Then

$$(\bar{x}_1, \dots, \bar{x}_{d-1}; \bar{y}_1, \dots, \bar{y}_{d-1}) = (\bar{x}(\mu^*); \bar{y}(\mu^*)).$$

We use the fact that

$$s_{\mu;a}(x(\mu); y(\mu)) = s_{\mu;a}(\bar{x}(\mu); \bar{y}(\mu)) = s_{\mu;a}(\bar{x}_1, \dots, \bar{x}_d; \bar{y}_1, \dots, \bar{y}_d)$$

and we employ the combinatorial formula (4.5) and Theorem 4.6 to compute the latter expression,

$$s_{\mu;a}(\bar{x}_1, \dots, \bar{x}_d; \bar{y}_1, \dots, \bar{y}_d) = \sum_{\mathcal{T}} \prod_{i=1}^{d} f_{\mathcal{T}^{-1}(i);a}(\bar{x}_i, \bar{y}_i), \quad (5.8)$$

summed over all diagonal–strict tableaux of shape $\mu$. By Proposition 4.3 and Theorem 4.6, this can also be written as follows

$$s_{\mu;a}(\bar{x}(\mu); \bar{y}(\mu)) = \sum_{\nu \subset \mu} s_{\nu;a}(\bar{x}(\mu^*); \bar{y}(\mu^*)) f_{\mu/\nu;a}(\bar{x}_d, \bar{y}_d),$$

summed over all subdiagrams $\nu$ such that $\mu/\nu$ contains no $2 \times 2$ block of squares.

By Theorem 5.1, $s_{\nu;a}(\bar{x}(\mu^*); \bar{y}(\mu^*)) = 0$ unless $\nu$ is contained in $\mu^*$. By the minimality property of $\mu^*$ mentioned above, this means that $\nu = \mu^*$. Repeating this argument we conclude that in the sum (5.8), there is only one tableau $\mathcal{T} = \mathcal{T}_0$

with (possibly) nonzero contribution: the chain of diagrams corresponding to $\mathcal{T}_0$ looks as follows:

$$\cdots \subset (\mu^*)^* \subset \mu^* \subset \mu.$$

Finally, let us check that the contribution of $\mathcal{T}_0$ is indeed given by formula (5.6). Let $\mu[1] \subset \mu$ stand for the first diagonal hook in $\mu$,

$$\mu[1] = \{(1,j) \in \mu\} \cup \{(i,1) \in \mu\}.$$

We shall prove that

$$f_{\mu/\mu^*;a}(\bar{x}_d, \bar{y}_d) = \prod_{(i,j)\in\mu[1]} (a'_{\mu_i - i + 1/2} - a'_{-\mu'_j + j - 1/2}). \tag{5.9}$$

Moreover, the same argument will prove that

$$f_{\mu^*/(\mu^*)^*;a}(\bar{x}_{d-1}, \bar{y}_{d-1}) = \prod_{(i,j)\in\mu[2]} (a'_{\mu_i - i + 1/2} - a'_{-\mu'_j + j - 1/2}),$$

where $\mu[2]$ stands for the second diagonal hook, and so on. This will imply (5.6).

By the definition (4.4),

$$f_{\mu/\mu^*;a}(\bar{x}_d, \bar{y}_d) = (u+v) \prod_{\substack{k,\\ s_k \text{ is vertical}}} (u - a'_{\delta_k - \varepsilon_k}) \prod_{\substack{k,\\ s_k \text{ is horizontal}}} (v + a'_{\delta_k - \varepsilon_k}), \tag{5.10}$$

where $(\varepsilon_k, \delta_k)$ are the midpoints of the interior sides $s_k$ of the shape $\mu/\mu^*$ and

$$u = \bar{x}_d = a_{p_1+1} = a'_{p_1+1/2}, \quad v = \bar{y}_d = \widehat{a}_{q_1+1} = -a'_{-(q_1+1/2)}.$$

We establish a bijective correspondence $s \leftrightarrow (i,j)$ between the sides $s = s_k$ and the squares $(i,j) \in \mu[1]$, except the diagonal square $(1,1)$, as follows:

$$s = (\varepsilon, \delta) \leftrightarrow (i,j) = \begin{cases} (1, \delta+1), & \text{if } s \text{ is vertical,} \\ (\varepsilon+1, 1), & \text{if } s \text{ is horizontal.} \end{cases}$$

Note that if $(i,j) = (1, \delta+1)$ then $\varepsilon + 1/2 = \mu'_j$, and if $(i,j) = (\varepsilon+1, 1)$ then $\delta + 1/2 = \mu_i$. It follows that, under the above correspondence, the contribution of $s$ to (5.10) coincides with the contribution of $(i,j)$ to (5.9). As for the factor $(u+v)$ in (5.10), it coincides with the contribution of the square $(1,1)$. This proves (5.9) and concludes the proof of the theorem. $\square$

**Corollary 5.3.** *If the numbers $a_i$ are pairwise distinct then $s_{\mu;a}(x(\mu); y(\mu)) \neq 0$.*

*Proof.* Recall a well–known claim: the sets $\{\mu_i - i + 1\}_{i=1}^{\infty}$ and $\{j - \mu'_j\}_{j=1}^{\infty}$ do not intersect (and, moreover, their union is the whole $\mathbb{Z}$), see, e.g., [Ma1, I, (1.7)]. It follows that, under the assumption on $a$, all the factors in the product (5.6) do not vanish. $\square$

In the special case $a_i = i - \frac{1}{2}$, Theorem 5.2 means that for any Young diagram $\mu = (p_1, \dots, p_d \,|\, q_1, \dots, q_d)$,

$$FS_\mu(p_1 + \tfrac{1}{2}, \dots, p_d + \tfrac{1}{2}; q_1 + \tfrac{1}{2}, \dots, q_d + \tfrac{1}{2}) = \prod_{(i,j)\in\mu} (\mu_i - i + \mu'_j - j + 1), \quad (5.11)$$

which is equal to

$$\prod_{(i,j)\in\mu} h(i,j) = \frac{|\mu|!}{\dim\mu},$$

the product of the hook lengths. There exist other useful expressions for this product, in particular,

$$\prod_{(i,j)\in\mu} h(i,j) = \frac{\prod\limits_{i=1}^{d} p_i! q_i! \prod\limits_{i,j=1}^{d} (p_i + q_j + 1)}{\prod\limits_{1\le i<k\le d} (p_i - p_k)(q_i - q_k)}$$

(see, e.g., [BR, 7.14.1], where one must take $k = l = d$, or [Ol]). The next proposition provides a generalization of the last identity.

**Proposition 5.4.** *For any Young diagram $\mu = (p_1, \dots, p_d \,|\, q_1, \dots, q_d)$, the product*

$$\prod_{(i,j)\in\mu} (a_{\mu_i - i + 1} - a_{j - \mu'_j})$$

$$= \prod_{(i,j)\in\mu} (a'_{\mu_i - i + 1/2} - a'_{-\mu'_j + j - 1/2}) = \prod_{(i,j)\in\mu} (a_{\mu_i - i + 1} + \widehat{a}_{\mu'_j - j + 1}), \quad (5.12)$$

*which gives the value of $s_{\mu;a}(x(\mu); y(\mu))$, is equal to*

$$\frac{\prod\limits_{i=1}^{d} (a_{p_i+1} - a_1)\dots(a_{p_i+1} - a_{p_i})(\widehat{a}_{q_i+1} - \widehat{a}_1)\dots(\widehat{a}_{q_i+1} - \widehat{a}_{q_i}) \prod\limits_{i,j=1}^{d} (a_{p_i+1} + \widehat{a}_{q_j+1})}{\prod\limits_{1\le i<k\le d} (a_{p_i+1} - a_{p_k+1})(\widehat{a}_{q_i+1} - \widehat{a}_{q_k+1})}. \quad (5.12')$$

Note that in the special case $a_i = i - \frac{1}{2}$, this coincides with the right–hand side of (5.11).

*Proof.* The part of the product (5.12) corresponding to the squares $(i,j) \in \mu$ with $i, j \le d$ exactly coincides with the second product in the numerator of (5.12'). It follows that the equality of (5.12) and (5.12') can be reduced to the following identity

$$\prod_{\substack{(i,j)\in\mu \\ i\le d, j>d}} (a_{\mu_i - i + 1} - a_{j - \mu'_j}) \cdot \prod_{\substack{(i,j)\in\mu \\ j\le d, i>d}} (\widehat{a}_{\mu'_j - j + 1} - \widehat{a}_{i - \mu_i})$$

$$= \frac{\prod\limits_{i=1}^{d} (a_{p_i+1} - a_1)\dots(a_{p_i+1} - a_{p_i})}{\prod\limits_{1\le i<k\le d} (a_{p_i+1} - a_{p_k+1})} \cdot \frac{\prod\limits_{i=1}^{d} (\widehat{a}_{q_i+1} - \widehat{a}_1)\dots(\widehat{a}_{q_i+1} - \widehat{a}_{q_i})}{\prod\limits_{1\le i<k\le d} (\widehat{a}_{q_i+1} - \widehat{a}_{q_k+1})}.$$

By symmetry, it suffices to prove the identity

$$\prod_{\substack{(i,j)\in\mu\\ i\le d,j>d}} (a_{\mu_i-i+1}-a_{j-\mu'_j}) = \frac{\prod_{i=1}^{d}(a_{p_i+1}-a_1)\dots(a_{p_i+1}-a_{p_i})}{\prod_{1\le i<k\le d}(a_{p_i+1}-a_{p_k+1})}, \tag{5.13}$$

which reduces to the following claim: for any diagram $\mu$ and any $i=1,\dots,d$,

$$\{1,\dots,\mu_i-i\} = \{\mu_j-j+1\}_{j=i+1}^{d} \sqcup \{j-\mu'_j\}_{j=d+1}^{\mu_i}, \tag{5.14}$$

disjoint union of two sets.

By virtue of the combinatorial fact mentioned in the proof of Corollary 5.2, the two sets in the right-hand side of (5.14) are indeed disjoint.

Next, each of the two sets in the right–hand side of (5.14) is contained in the set from the left–hand side. Indeed, the minimal element of the first set is $\mu_d-d+1\ge 1$ and the maximal element is $\mu_{i+1}-i\le\mu_i-i$. Similarly, in the second set, the minimal element is $d+1-\mu'_{d+1}\ge d+1-d=1$ and the maximal element is $\mu_i-\mu'_{\mu_i}\le \mu_i-i$. Finally, the total number of elements in both sets is $(d-i)+(\mu_i-d)=\mu_i-i$, which is equal to the number of elements in the left–hand side. This proves (5.14) and concludes the proof of the proposition. □

The next two results are similar to the characterization theorems for shifted Schur functions, cf. [Ok1], [OO, §3].

**Theorem 5.5** (Characterization theorem I). *Let $\mu$ be an arbitrary Young diagram and*

$$D(\mu)=\{\lambda\in\mathbb{Y}\mid |\lambda|\le|\mu|,\lambda\ne\mu\}.$$

*Assume that the numbers $a_i$ are pairwise distinct. Then, as an element of $\Lambda$, $s_{\mu;a}$ is uniquely determined by the following two properties: first, its top degree homogeneous component is the Schur function $s_\mu$; second, it vanishes at $(x(\lambda),y(\lambda))$ for all $\lambda\in D(\mu)$.*

*Proof.* Assume $F$ is a symmetric function with the same two properties. Then there exists an expansion of the form

$$F=s_{\mu;a}+\sum_{\nu\in D(\mu)} c(\nu)s_{\nu;a}$$

with certain number coefficients $c(\nu)$. We must prove that these coefficients are actually equal to zero. Let $X$ stand for the set of $\nu$'s with $c(\nu)\ne 0$. Assume that $X$ is nonempty and choose a minimal diagram $\lambda\in X$ with respect to the partial ordering by inclusion. We get $F(x(\lambda);y(\lambda))=s_{\mu;a}(x(\lambda);y(\lambda))=0$, because $\lambda\in D(\mu)$. On the other hand, since $\lambda$ is minimal, we also have $s_{\nu;a}(x(\lambda);y(\lambda))=0$ for any $\nu\in X\setminus\{\lambda\}$, because $\nu$ is not contained in $\lambda$. But $s_{\lambda;a}(x(\lambda);y(\lambda))\ne 0$, because of Corollary 5.3. This leads to a contradiction. □

**Theorem 5.6** (Characterization theorem II). *Let $\mu$ be an arbitrary Young diagram and $D(\mu)$ be as above. Assume again that the numbers $a_i$ are pairwise distinct. Then, as an element of $\Lambda$, $s_{\mu;a}$ is uniquely determined by the following three properties: first, its degree is less or equal to $|\mu|$; second, its value at $(x(\mu); y(\mu))$ is given by formula* (5.6); *third, it vanishes at $(x(\lambda), y(\lambda))$ for all $\lambda \in D(\mu)$.*

*Proof.* Let $F \in \Lambda$ possess the same three properties. Then

$$F = \sum_{\nu \in D(\mu) \cup \{\mu\}} c(\nu) s_{\nu;a}$$

with certain number coefficients $c(\nu)$. The same argument as above proves that $c(\nu) = 0$ for all $\nu \in D(\mu)$, so that $F$ is proportional to $s_{\mu;a}$. Then the second property implies that $F$ is exactly equal to $s_{\mu;a}$. $\square$

In the particular case of the Frobenius–Schur functions, the results of this section are equivalent to similar claims for the shifted Schur functions contained in [Ok1], [OO, §3]. Interpolation of arbitrary polynomials in terms of Schubert polynomials is discussed in [L2]. A very general scheme of Newton interpolation for symmetric polynomials is developed in [Ok2]. In the latter paper one can also find the references to earlier works by Knop, Sahi, and Okounkov.

## §6. Sergeev–Pragacz formula

Recall first the conventional Sergeev–Pragacz formula. Let $m, n$ be arbitrary nonnegative integers and let $\mu$ be an arbitrary diagram not containing the square $(m+1, n+1)$. Then

$$s_\mu(x_1, \dots, x_m; y_1, \dots, y_n) = \frac{\sum_{w \in \mathfrak{S}_m \times \mathfrak{S}_n} \varepsilon(w)\, w[f_\mu(x_1, \dots, x_m; y_1, \dots, y_n)]}{V(x_1, \dots, x_m) V(y_1, \dots, y_n)}, \quad (6.1)$$

where

$$f_\mu(x_1, \dots, x_m; y_1, \dots, y_n) = \prod_{i=1}^{m} x_i^{(\mu_i - i)_+ + m - i} \cdot \prod_{j=1}^{n} y_j^{(\mu'_j - j)_+ + n - j} \cdot \prod_{\substack{i \le m,\, j \le n \\ (i,j) \in \mu}} (x_i + y_j). \quad (6.2)$$

Here $\mathfrak{S}_m \times \mathfrak{S}_n$ is the product of two symmetric groups acting on polynomials in $x_1, \dots, x_m$ and $y_1, \dots, y_n$ by permuting separately $x$'s and $y$'s; for $w = (w_1, w_2) \in \mathfrak{S}_m \times \mathfrak{S}_n$, the symbol $\varepsilon(w)$ means $\operatorname{sgn}(w_1) \operatorname{sgn}(w_2)$;

$$V(x_1, \dots, x_m) = \prod_{1 \le i < k \le m} (x_i - x_k), \qquad V(y_1, \dots, y_n) = \prod_{1 \le j < l \le n} (y_j - y_l),$$

and

$$(k)_+ = \max(k, 0).$$

Note that for $n = 0$ the formula (6.1) reduces to the classical formula (0.6) for the Schur polynomial $s_\mu(x_1, \dots, x_m)$. Proofs of (6.1)–(6.2) can be found in [PT], [Ma1, I.3, Ex. 23].

Our aim in this section is to establish an analog of the formulas (6.1)–(6.2) for the multiparameter supersymmetric Schur polynomials depending on a sequence of parameters $a = (a_i)$. According to our basic principle, we shall deal with equal number of the $x$'s and $y$'s.

**Theorem 6.1.** *Let $n = 1, 2, \dots$ and let $\mu$ be an arbitrary Young diagram such that $d = d(\mu) \le n$. Then, in the above notation,*

$$s_{\mu;a}(x_1,\dots,x_n;y_1,\dots,y_n) = \frac{\sum_{w\in\mathfrak{S}_n\times\mathfrak{S}_n} \varepsilon(w)\, w[f_{\mu;a}(x_1,\dots,x_n;y_1,\dots,y_n)]}{V(x_1,\dots,x_n)V(y_1,\dots,y_n)}, \tag{6.3}$$

*where*

$$\begin{aligned} f_{\mu;a}(x_1,\dots,x_n;y_1,\dots,y_n) &= \prod_{i=1}^{d} (x_i\,|\,a)^{\mu_i-i} x_i^{(n-\mu_i)_+} (y_i\,|\,\widehat{a})^{\mu'_i-i} y_i^{(n-\mu'_i)_+} \\ &\times \prod_{i=d+1}^{n} x_i^{n-i} y_i^{n-i} \cdot \prod_{\substack{i,j\le n\\(i,j)\in\mu}} (x_i+y_j) \end{aligned} \tag{6.4}$$

*and*

$$(x\,|\,a)^m = \begin{cases} (x-a_1)\dots(x-a_m), & m\ge 1,\\ 0, & m=0. \end{cases}$$

Note that for the sequence $a \equiv 0$ formulas (6.3)–(6.4) reduce to formulas (6.1)–(6.2) with $m = n$. Indeed, it suffices to check that $f_{\mu;a}$ reduces to $f_\mu$. To do this, let us compare (6.2) and (6.4). The last product in the both formulas is the same. Then we remark that

$$(\mu_i - n)_+ + n - i = \begin{cases} \mu_i - i + (n-\mu_i)_+, & i = 1,\dots,d,\\ n-i, & i = d+1,\dots,n, \end{cases}$$

and likewise for $\mu'$, which implies the equality $f_\mu = f_{\mu;a}\big|_{a\equiv 0}$.

*Proof.* The proof of Theorem 6.1 is divided into three lemmas.

**Lemma 6.2.** *The expression* (6.3) *is a supersymmetric polynomial.*

*Proof.* One may argue exactly as in the proof of Proposition 2.3 in [PT] (see also [Ma1, I.3, Ex. 24]). For the reader's convenience we present the argument.

Obviously, (6.3) is a polynomial, which is separately symmetric in the $x$'s and $y$'s. Let us verify the cancellation property: assume that $x_i = t = -y_j$ for certain indices $i, j$, and let us prove that (6.3) does not depend on $t$. Consider the expression (6.3) as a function in $t$. This is a rational function, which is actually a polynomial. The degree of the denominator is exactly $2n-2$, so that it suffices to prove that the numerator has degree less or equal to $2n-2$. Next, it suffices to prove the latter claim for the polynomial (6.4).

Without loss of generality, one may assume $i \le j$. Consider three cases.

First case: $i \le j \le d$. Then $(i,j) \in \mu$, so that (6.4) contains the factor $(x_i + y_j)$. This factor vanishes when $x_i = -y_j$, consequently, (6.4) identically vanishes.

Second case: $i \le d < j \le n$. The degree of the expression (6.4) with respect to $t$ is less or equal to the sum of three terms:

$$(\mu_i - i + (n-\mu_i)_+) + (n-j) + (\mu_i + \mu'_j).$$

Again, if $(i,j) \in \mu$ then (6.4) is identically equal to 0, so that one may assume $(i,j) \notin \mu$. This means that $\mu_i \le j-1$ and $\mu'_j \le i-1$; in particular, $\mu_i < n$. It follows that the above sum reduces to

$$(n-i)+(n-j)+(\mu_i+\mu'_j),$$

which is $\le 2n-2$, because $\mu_i+\mu'_j \le i+j-2$, as was mentioned above.

Third case: $d+1 \le i \le j \le n$. Then the degree in question is

$$(n-i)+(n-j)+(\mu_i+\mu'_j) \le 2n-i-j+(j-1)+(i-1) = 2n-2.$$

This concludes the proof of the lemma. $\square$

Let us temporarily denote the right–hand side of (6.3) by $s'_{\mu;a}(x_1,\dots,x_n;y_1,\dots,y_n)$.

**Lemma 6.3.** *We have*

$$s'_{\mu;a}(x_1,\dots,x_n;y_1,\dots,y_n)\,\Big|_{x_n=y_n=0} = \begin{cases} 0, & d=n, \\ s'_{\mu;a}(x_1,\dots,x_{n-1};y_1,\dots,y_{n-1}), & d \le n-1. \end{cases}$$

*Proof.* a) If $d=n$ then the polynomial $f_{\mu;a}(x_1,\dots,x_n;y_1,\dots,y_n)$ is divisible by

$$\prod_{i,j=1}^{n}(x_i+y_j),$$

and the same holds for its transformations by elements $w$. Since the above product vanishes when $x_n=y_n=0$, the whole expression $s'_{\mu;a}(x_1,\dots,x_n;y_1,\dots,y_n)$ vanishes, too.

b) Next, let us assume that $d \le n-1$ and let us prove the following claim:

$$f_{\mu;a}(x_1,\dots,x_n;y_1,\dots,y_n)\,\Big|_{x_i=y_j=0} = 0 \qquad \text{unless } i=j=n.$$

Indeed, vanishing holds if $(i,j)\in\mu$, because then the polynomial in question is divisible by $x_i+y_j$. Assume $i \le d$. If $\mu_i \ge n$ then $(i,j)\in\mu$, which implies vanishing. If $\mu_i < n$ then $(n-\mu_i)_+ > 0$, so that vanishing holds thanks to the factor $x_i^{(n-\mu_i)_+}$ in the expression (6.4). Similar argument also holds when $j \le d$.

Thus, we may assume that both $i > d$ and $j > d$. Then the expression (6.4) contains the factor $x_i^{n-i}y_j^{n-j}$, so that the only possibility of nonvanishing may occur for $i=j=n$.

c) The numerator of (6.3) is an alternate sum of terms indexed by couples of permutations $w=(w_1,w_2)$. If $w$ does not fix the indeterminates $(x_n,y_n)$, which are specialized to zero, then the corresponding term vanishes after the specialization, because of the claim b) proved above. Consequently, only terms indexed by elements $w$ fixing $(x_n,y_n)$ may give a nonzero contribution. Note that these are actually elements of the group $\mathfrak{S}_{n-1}\times\mathfrak{S}_{n-1}$.

On the other hand, note that

$$V(x_1,\dots,x_{n-1},0)V(y_1,\dots,y_{n-1},0)=V(x_1,\dots,x_{n-1})V(y_1,\dots,y_{n-1})\\ \times x_1\dots x_{n-1}y_1\dots y_{n-1}\,.$$

d) Thus, it remains to check that

$$f_{\mu;a}(x_1,\dots,x_{n-1},0;y_1,\dots,y_{n-1},0)=f_{\mu;a}(x_1,\dots,x_{n-1};y_1,\dots,y_{n-1})\\ \times x_1\dots x_{n-1}y_1\dots y_{n-1}\,.$$

Let us examine the behavior of the expression (6.4) under the specialization $x_n=y_n=0$. The right–hand side of (6.4) consists of three products. The second product, which is equal to

$$\prod_{i=d+1}^{n} x_i^{n-i}y_i^{n-i},$$

turns into the similar expression for $n-1$ multiplied by

$$\prod_{i=d+1}^{n-1} x_iy_i.$$

Consequently, we must prove that the remaining expression in (6.4), which is equal to

$$\prod_{i=1}^{d}(x_i\,|\,a)^{\mu_i-i}x_i^{(n-\mu_i)_+}(y_i\,|\,\widehat{a})^{\mu_i'-i}y_i^{(n-\mu_i')_+}\cdot\prod_{\substack{i,j\le n\\(i,j)\in\mu}}(x_i+y_j),$$

turns into the similar expression for $n-1$ multiplied by

$$\prod_{i=1}^{d}x_iy_i.$$

This is equivalent to the following claim: for any $i=1,\dots,d$,

$$(n-\mu_i)_++\varepsilon_i=(n-1-\mu_i)_++1, \tag{6.5}$$

where

$$\varepsilon_i=\begin{cases}1, & (i,n)\in\mu,\\ 0, & (i,n)\notin\mu,\end{cases}$$

and similarly for $\mu'$.

Consider two cases: $\mu_i\ge n$ and $\mu_i\le n-1$. In the former case, $(n-\mu_i)_+=(n-1-\mu_i)_+=0$ and $\varepsilon_i=1$, so that (6.5) holds. In the latter case, $(n-\mu_i)_+=(n-1-\mu_i)_++1$ and $\varepsilon_i=0$, so that (6.5) is again true. This concludes the proof. □

By Lemma 6.2 and Lemma 6.3, there exists an element $s'_{\mu;a}\in\Lambda$, of degree $|\mu|$ and such that the corresponding supersymmetric polynomials in $n+n$ variables are given by the expressions (6.3) when $n\ge d$ and vanish when $n<d$. We aim to prove that $s'_{\mu;a}=s_{\mu;a}$ by making use of the characterization theorems from §5.

**Lemma 6.4.** *The element $s'_{\mu;a}$ defined above satisfies the vanishing condition of Theorem* 5.5*, i.e.,*

$$s'_{\mu;a}(x(\lambda);y(\lambda))=0 \qquad \textit{unless } \mu\subseteq\lambda.$$

*Proof.* We shall see that the vanishing in question is ensured by the product

$$\prod_{i=1}^{d}(x_i\,|\,a)^{\mu_i-i}(y_i\,|\,\widehat{a})^{\mu'_i-i}$$

entering the expression (6.4).

Write both diagrams in Frobenius notation,

$$\mu=(p_1,\dots,p_d\,|\,q_1,\dots,q_d), \qquad \lambda=(P_1,\dots,P_D\,|\,Q_1,\dots,Q_D),$$

and recall that

$$x(\lambda)_i=a_{P_i+1}, \quad y(\lambda)_i=\widehat{a}_{Q_i+1}, \quad 1\le i\le D,$$
$$x(\lambda)_i=y(\lambda)_i=0, \quad i>D.$$

By the definition of $s'_{\mu;a}$, vanishing holds if $D<d$, because $x(\lambda)$ and $y(\lambda)$ have at most $D$ nonzero coordinates, so that we may assume $D\ge d$.

Abbreviate

$$x_i=x(\lambda)_i=a_{P_i+1}, \quad y_i=y(\lambda)_i=\widehat{a}_{Q_i+1}, \quad 1\le i\le D.$$

It suffices to prove that for any $w=(w_1,w_2)\in\mathfrak{S}_D\times\mathfrak{S}_D$,

$$w\left[\prod_{i=1}^{d}(x_i\,|\,a)^{p_i}(y_i\,|\,\widehat{a})^{q_i}\right]=0$$

unless $P_i\ge p_i$ and $Q_i\ge q_i$ for all $i=1,\dots,d$. By the symmetry $x\leftrightarrow y$, $a\leftrightarrow\widehat{a}$, it suffices to prove that for any $w_1\in\mathfrak{S}_D$,

$$w_1\left[\prod_{i=1}^{d}(x_i\,|\,a)^{p_i}\right]=0 \tag{6.6}$$

unless $P_i\ge p_i$ for all $i=1,\dots,d$.

The left–hand side in (6.6) has the form

$$\prod_{i=1}^{d}(a_{P_{j_i}+1}\,|\,a)^{p_i}=\prod_{i=1}^{d}(a_{P_{j_i}+1}-a_1)\dots(a_{P_{j_i}+1}-a_{p_i}),$$

where $j_1,\dots,j_d$ is a certain $d$–tuple of pairwise distinct indices from $\{1,\dots,D\}$. If (6.6) does not hold then

$$P_{j_1}\ge p_1,\dots,P_{j_d}\ge p_d. \tag{6.7}$$

Let us check that this implies

$$P_1 \geq p_1, \dots, P_d \geq p_d. \tag{6.8}$$

Indeed, recall that

$$P_1 > \dots > P_d, \qquad p_1 > \dots > p_d.$$

Together with (6.7) this implies that among the numbers $P_1, \dots, P_d$, there are at least one number $\geq p_1$ (namely, $P_{j_1}$), at least two numbers $\geq p_2$ (namely, $P_{j_1}, P_{j_2}$), and so on, which implies (6.8). $\square$

Now we are in a position to prove the equality $s'_{\mu;a} = s_{\mu;a}$. Comparing the formula (6.3)–(6.4) defining $s'_{\mu;a}$ with the Sergeev–Pragacz formula (6.1)–(6.2) for the Schur function $s_\mu$ we see that the top degree homogeneous component of $s'_{\mu;a}$ coincides with $s_\mu$. By Lemma 6.4, $s'_{\mu;a}$ possesses the same vanishing property as $s_{\mu;a}$. Consequently, by Theorem 5.5, $s'_{\mu;a} = s_{\mu;a}$. Note that in Theorem 5.5, the numbers $a_i$ are required to be pairwise distinct but one may attain this by making use of the continuity argument, because both $s'_{\mu;a}$ and $s_{\mu;a}$ depends on the parameters continuously.

An alternative way is to apply Theorem 5.6. Then we must verify for $s'_{\mu;a}$ the three properties listed in the statement of this theorem. The first property (control of degree) is obvious. The third property (vanishing) is ensured by Lemma 6.4. The second property (required value at $(x(\mu); y(\mu))$) is verified as follows. From the proof of Lemma 6.4 one sees that in formula (6.3) applied to the variables $(x(\mu); y(\mu))$, all terms with $w \neq e$ are zero. Next, it is readily seen that the term with $w = e$ leads to the expression (5.12'), which, by Proposition 5.4, coincides with $s_{\mu;a}(x(\mu); y(\mu))$.

This concludes the proof of Theorem 6.1. $\square$

The following result is an analog of the Berele–Regev factorization property for the supersymmetric Schur functions [BR], [Ma1, I.3, Ex. 23].

**Corollary 6.5.** *Let $\mu = (p_1, \dots, p_d \,|\, q_1, \dots, q_d)$ be a Young diagram of depth $d$. Then*

$$s_{\mu;a}(x_1, \dots, x_d; y_1, \dots, y_d) = \frac{\det[(x_i \,|\, a)^{p_j}]_{i,j=1}^d}{V(x_1, \dots, x_d)} \cdot \frac{\det[(y_i \,|\, \widehat{a})^{q_j}]_{i,j=1}^d}{V(y_1, \dots, y_d)} \cdot \prod_{i,j=1}^{d} (x_i + y_j)\,.$$

*Proof.* We apply formulas (6.3) and (6.4) for $n = d$. Then the last product in (6.4) coincides with the product above, which is invariant under permutations $w$. Consequently, the alternate sum in (6.3) becomes the product of two determinants. Note also that for each $i = 1, \dots, d$, we have $\mu_i - i = p_i$, $(n - \mu_i)_+ = (d - \mu_i)_+ = 0$, and, similarly, $\mu'_i - i = q_i$, $(n - \mu'_i)_+ = 0$. $\square$

## 7. Transition coefficients

Formulas (3.1) and (4.9) yield generating series for the one–row and one–column multiparameter Schur functions; let us rewrite them in slightly different notation:

$$\begin{aligned} 1+\sum_{p=0}^{\infty}\frac{s_{(p\,|\,0);a}}{(u\,|\,a)^{p+1}} &= H(u) = 1+\sum_{p=0}^{\infty}\frac{s_{(p\,|\,0)}}{u^{p+1}}\,, \\ 1+\sum_{q=0}^{\infty}\frac{s_{(0\,|\,q);a}}{(v\,|\,\widehat{a})^{q+1}} &= E(v) = 1+\sum_{q=0}^{\infty}\frac{s_{(0\,|\,q)}}{v^{q+1}}\,. \end{aligned} \tag{7.1}$$

The next proposition yields a generating series for the hook functions $s_{(p\,|\,q);a}$; this series is an element of $\Lambda[[u^{-1},v^{-1}]]$.

**Proposition 7.1.** *We have*

$$1+(u+v)\sum_{p,q=0}^{\infty}\frac{s_{(p\,|\,q);a}}{(u\,|\,a)^{p+1}(v\,|\,\widehat{a})^{q+1}} = H(u)E(v).$$

*Proof.* This is a generalization of Theorem 2.3, and we shall argue as in the proof of that theorem. By virtue of (7.1), the equality in question is equivalent to

$$1+(u+v)\sum_{p,q=0}^{\infty}\frac{s_{(p\,|\,q);a}}{(u\,|\,a)^{p+1}(v\,|\,\widehat{a})^{q+1}} = \left(1+\sum_{p=0}^{\infty}\frac{s_{(p\,|\,0);a}}{(u\,|\,a)^{p+1}}\right)\left(1+\sum_{q=0}^{\infty}\frac{s_{(0\,|\,q);a}}{(v\,|\,\widehat{a})^{q+1}}\right).$$

Using the identity

$$\begin{aligned} \frac{u+v}{(u\,|\,a)^{p+1}(v\,|\,\widehat{a})^{q+1}} &= \frac{(u-a_{p+1})+(v-\widehat{a}_{q+1})+(a_{p+1}+\widehat{a}_{q+1})}{(u\,|\,a)^{p+1}(v\,|\,\widehat{a})^{q+1}} \\ &= \frac{1}{(u\,|\,a)^{p}(v\,|\,\widehat{a})^{q+1}}+\frac{1}{(u\,|\,a)^{p+1}(v\,|\,\widehat{a})^{q}}+\frac{a_{p+1}+\widehat{a}_{q+1}}{(u\,|\,a)^{p+1}(v\,|\,\widehat{a})^{q+1}} \end{aligned}$$

we reduce this to the following system of relations, where $p,q=0,1,\dots$:

$$s_{(p+1\,|\,q);a}+s_{(p\,|\,q+1);a}+(a_{p+1}+\widehat{a}_{q+1})s_{(p\,|\,q);a} = s_{(p\,|\,0);a}s_{(0\,|\,q);a}\,.$$

By virtue of (1.4), the expansion of the product $s_{(p\,|\,0);a}s_{(0\,|\,q);a}$ into a linear combination of the functions $s_{\nu;a}$ has the form

$$s_{(p\,|\,0);a}s_{(0\,|\,q);a} = s_{(p+1\,|\,q);a}+s_{(p\,|\,q+1);a}+\sum_{\nu:\,|\nu|\le p+q+1}c(\nu)s_{\nu;a}\,.$$

Let $X$ be the set of those $\nu$'s which enter this sum with nonzero coefficients $c(\nu)$. We claim that $X$ contains at most the diagram $(p\,|\,q)$; here we will use the fact that $|\nu|\le p+q+1$. Indeed, let $\lambda$ be a minimal (with respect to inclusion) diagram in $X$ and let us evaluate both sides at $(x(\lambda);y(\lambda))$. On the right, the result is nonzero, because $s_{\lambda;a}(x(\lambda);y(\lambda))\ne 0$[8] while all other terms on the right have zero

[8] Here we tacitly assume that the sequence $a=(a_i)$ has no repetitions in order to apply Corollary 5.3. To cover the case when repetitions are present, one can use the continuity argument.

contributions, because neither a diagram $\nu \neq \lambda$ from $X$ nor $(p+1 \,|\, q)$ and $(p \,|\, q+1)$ are contained in $\lambda$. So, the result of evaluation on the left is nonzero, too. This implies that $\lambda$ contains both $(p \,|\, 0)$ and $(0 \,|\, q)$, which is only possible for $\lambda = (p \,|\, q)$.

Thus, the above expansion takes the form

$$s_{(p \,|\, 0);a} s_{(0 \,|\, q);a} = s_{(p+1 \,|\, q);a} s_{(p \,|\, q+1);a} + \text{const } s_{(p \,|\, q);a} \,.$$

Evaluating both sides at

$$(x(p \,|\, q); y(p \,|\, q)) = (a_{p+1}, 0, 0, \dots ; \widehat{a}_{q+1}, 0, 0, \dots)$$

we get

$$\text{const} = \frac{s_{(p \,|\, 0);a}(x(p \,|\, q); y(p \,|\, q))\, s_{(0 \,|\, q);a}(x(p \,|\, q); y(p \,|\, q))}{s_{(p \,|\, q)}(x(p \,|\, q); y(p \,|\, q))} \,.$$

The right–hand side can be readily evaluated by making use of Corollary 6.5. This gives the desired result: const $= a_{p+1} + \widehat{a}_{q+1}$. $\square$

Let $b = (b_i)_{i \in \mathbb{Z}}$ be another sequence of parameters. We aim to express the functions $s_{\mu;a}$ through the functions $s_{\nu;b}$. In particular, we are interested in expansion on the conventional supersymmetric Schur functions $s_\nu$, which correspond to $b \equiv 0$.

For $p \geq p' \geq 0$ set

$$c_{pp'}(a,b) = h_{p-p'}(b_1, \dots, b_{p'+1}; -a_1, \dots, -a_p), \tag{7.2}$$

where $h_{p-p'}$ is the conventional supersymmetric $h$-function of degree $p - p'$. In particular,

$$c_{pp'}(a,0) = h_{p-p'}(0, \dots, 0; -a_1, \dots, -a_p) = (-1)^{p-p'} e_{p-p'}(a_1, \dots, a_p), \tag{7.3}$$

cf. (2.9). Note that $c_{pp}(a,b) = 1$.

**Proposition 7.2** (cf. Proposition 2.4). *We have*

$$s_{(p \,|\, q);a} = \sum_{p'=0}^{p} \sum_{q'=0}^{q} c_{pp'}(a,b)\, c_{qq'}(\widehat{a}, \widehat{b})\, s_{(p' \,|\, q');b}. \tag{7.4}$$

*Proof.* We have the identity

$$(u+v) \sum_{p,q=0}^{\infty} \frac{s_{(p \,|\, q);a}}{(u \,|\, a)^{p+1} (v \,|\, \widehat{a})^{q+1}} = (u+v) \sum_{p,q=0}^{\infty} \frac{s_{(p \,|\, q);b}}{(u \,|\, b)^{p+1} (v \,|\, \widehat{b})^{q+1}} \,,$$

because, by Proposition 7.1, both sides are equal to $H(u)E(v) - 1$. Then we argue exactly as in the proof of Proposition 2.4. $\square$

**Theorem 7.3** (cf. Theorem 2.6). *Let $a = (a_i)_{i\in\mathbb{Z}}$ and $b = (b_i)_{i\in\mathbb{Z}}$ be two sequences of parameters. In the expansion*

$$s_{\mu;a} = \sum_{\nu} c_{\mu\nu}(a, b)\, s_{\nu;b} \tag{7.5}$$

*the coefficients $c_{\mu\nu}(a, b)$ vanish unless $\nu \subseteq \mu$ and $d(\nu) = d(\mu)$.*

*Assume $d(\nu) = d(\mu)$ and write both diagrams in Frobenius notation,*

$$\mu = (p_1, \dots, p_d \,|\, q_1, \dots, q_d), \quad \nu = (p'_1, \dots, p'_d \,|\, q'_1, \dots, q'_d).$$

*Then*

$$c_{\mu\nu}(a, b) = \det[c_{p_i,p'_j}(a, b)]\ \det[c_{q_i,q'_j}(\widehat{a}, \widehat{b})], \tag{7.6}$$

*where the determinants are of order $d$ and the coefficients in the right–hand side are defined by* (7.2).

*Proof.* The same argument as in the proof of Theorem 2.6 reduces the desired claim to the special case $d = 1$, which was the subject of Proposition 7.2. □

## Appendix by Vladimir Ivanov: Proof of the combinatorial formula for multiparameter Schur functions

The aim of this appendix is to prove Theorem 4.6, which is restated below as Theorem A.6. Here we are using a slightly different notation for the combinatorial sum, the $h$–functions and $s$–functions.

**Definition A.1.** Suppose $(a_i)_{i\in\mathbb{Z}}$ is an arbitrary sequence of complex numbers. For $k = 0, 1, 2, \dots$, set

$$(x \mid a)^k = \prod_{i=1}^{k}(x - a_i).$$

**Definition A.2.** Fix $n = 1, 2, \dots$ and denote by $\mathbb{A}_n$ the ordered alphabet $\{1' < 1 < 2' < 2 < \cdots < n' < n\}$. Put $|i'| = |i| = i$ for $i = 1, 2, \dots, n$.

**Definition A.3.** Fix a skew Young diagram $\lambda/\mu$. A map $T : \lambda/\mu \to \mathbb{A}_n$ is called *a tableau of shape $\lambda/\mu$ and order $n$* if the following conditions hold:

1. $T(i, j) \le T(i, j + 1)$;
2. $T(i, j) \le T(i + 1, j)$;
3. for each $i = 1, 2, \dots, n$, there is at most one symbol $i'$ in each row;
4. for each $i = 1, 2, \dots, n$, there is at most one symbol $i$ in each column.

Let us denote by $\mathrm{Tab}(\lambda/\mu, n)$ the set of all tableaux of shape $\lambda/\mu$ and order $n$.

**Definition A.4.** Suppose that $x_1, x_2, \dots, y_1, y_2, \dots$ are variables, $(a_i)_{i\in\mathbb{Z}}$ is an arbitrary sequence, $\mu \subset \lambda$ are Young diagrams. Set

$$\Sigma_{\lambda/\mu|n}(x; y \mid a) = \sum_{T\in\mathrm{Tab}(\lambda/\mu,n)} \prod_{\substack{\square\in\lambda/\mu \\ T(\square)=1,2,\dots,n}} (x_{T(\square)} - a_{c(\square)}) \prod_{\substack{\square\in\lambda/\mu \\ T(\square)=1',2',\dots,n'}} (y_{T(\square)} + a_{c(\square)}),$$

where $c(i, j) = j - i$, cf. (4.7).

**Definition A.5.** Let $\mu \subset \lambda$. Set

$$s_{\lambda/\mu}(x_1, x_2, \dots, x_n; y_1, y_2, \dots, y_n \mid a) = \det[h_{\lambda_i - \mu_j + j - i}(x_1, x_2, \dots, x_n; y_1, y_2, \dots, y_n \mid \tau^{\mu_j - j + 1} a)]_{1 \le i,j \le \ell(\lambda)},$$

where the polynomials $h_k(x_1, \dots, x_n; y_1, \dots, y_n \mid a)$ are given by the generating series

$$1 + \sum_{k=1}^{\infty} \frac{h_k(x_1, \dots, x_n; y_1, \dots, y_n \mid a)}{(u \mid a)^k} = \prod_{i=1}^{n} \frac{u + y_i}{u - x_i},$$

$h_k \equiv 0$ if $k < 0$, and $h_0 \equiv 1$, cf. (3.4).

**Theorem A.6.**

$$s_{\lambda/\mu}(x_1, x_2, \dots, x_n; y_1, y_2, \dots, y_n \mid a) = \Sigma_{\lambda/\mu|n}(x; y \mid a).$$

*Proof.* First let us prove this claim when $\lambda = (k), \mu = \varnothing$. Then $s_{\lambda/\mu}(x; y \mid a) = h_k(x; y \mid a)$ and our claim is equivalent to the following branching rule (see Proposition 4.3):

$$h_k(x_1, \dots, x_{n-1}, x_n; y_1, \dots, y_{n-1}, y_n \mid a) = h_k(x_1, \dots, x_{n-1}; y_1, \dots, y_{n-1} \mid a) + (x_n + y_n) \sum_{r=0}^{k-1} h_r(x_1, \dots, x_{n-1}; y_1, \dots, y_{n-1} \mid a) \prod_{t=1}^{k-1-r} (x_n - a_{k-t}).$$

Let us denote $h_k(x_1, \dots, x_{n-1}, x_n; y_1, \dots, y_{n-1}, y_n \mid a)$ by $h_{k|n}(x; y \mid a)$. From the definition of $h_k$ we obtain

$$1 + \sum_{k=1}^{\infty} \frac{h_{k|n}(x; y \mid a)}{(u \mid a)^k} = \prod_{i=1}^{n} \frac{u + y_i}{u - x_i} = \frac{u + y_n}{u - x_n} + \sum_{k=1}^{\infty} \frac{h_{k|n-1}(x; y \mid a)}{(u \mid a)^k} \frac{u + y_n}{u - x_n}. \quad \text{(A.1)}$$

For an arbitrary natural $k$ we have

$$\frac{u + y}{u - x} = 1 + (x + y) \sum_{r=1}^{\infty} \frac{(x \mid \tau^k a)^{r-1}}{(u \mid \tau^k a)^r}. \quad \text{(A.2)}$$

This follows, for example, from Molev's results [Mo, Prop. 1.2, Theorem 2.1] if we put $m = n = 1$. Using (A.1) and (A.2), we obtain

$$1 + \sum_{k=1}^{\infty} \frac{h_{k|n}(x; y \mid a)}{(u \mid a)^k} = \frac{u + y_n}{u - x_n} + \sum_{k=1}^{\infty} \frac{h_{k|n-1}(x; y \mid a)}{(u \mid a)^k} \frac{u + y_n}{u - x_n}$$

$$= 1 + (x_n + y_n) \sum_{k=1}^{\infty} \frac{(x_n \mid a)^{k-1}}{(u \mid a)^k} + \sum_{k=1}^{\infty} \frac{h_{k|n-1}(x; y \mid a)}{(u \mid a)^k} \cdot \left( 1 + (x_n + y_n) \sum_{r=1}^{\infty} \frac{(x_n \mid \tau^k a)^{r-1}}{(u \mid \tau^k a)^r} \right)$$

$$= 1 + \sum_{k=1}^{\infty} \frac{h_{k|n-1}(x; y \mid a) + (x_n + y_n) \sum_{r=0}^{k-1} h_{r|n-1}(x; y \mid a) \prod_{t=1}^{k-1-r} (x_n - a_{k-t})}{(u \mid a)^k}.$$

Therefore, when $\lambda = (k)$ and $\mu = \varnothing$, the assertion of the theorem is proved:

$$h_{k|n}(x; y \mid a) = \sum_{T \in \mathrm{Tab}((k),n)} \prod_{\substack{\square \in \lambda/\mu \\ T(\square)=1,2,\dots,n}} (x_{T(\square)} - a_{c(\square)}) \prod_{\substack{\square \in \lambda/\mu \\ T(\square)=1',2',\dots,n'}} (y_{T(\square)} + a_{c(\square)}). \tag{A.3}$$

Now suppose that $\lambda/\mu$ is an arbitrary skew Young diagram. Let us show that

$$\Sigma_{\lambda/\mu|n}(x; y \mid \tau^{\mu_j - j+1} a) = \det[h_{\lambda_i - \mu_j + j - i|n}(x; y \mid a)]_{1 \le i,j \le \ell(\lambda)}.$$

We use an appropriate modification of the Gessel-Viennot method (see [Sa, §4.5] for a detailed description of this method).

Suppose we have $2n$ variables $x_1, \dots, x_n, y_1, \dots, y_n$. We define a *path* as a finite sequence $p = (p(0), p(1), \dots, p(l))$ of points in the $(x, y)$–plane, such that:

- Each point $p(t)$ belongs to $\mathbb{Z} \times \{0, 1, 2, \dots, n\}$.
- The path starts on the horizontal line $y = 0$ and ends on the line $y = n$, i.e., $p(0) = (m_1, 0)$ and $p(l) = (m_2, n)$ with some $m_1, m_2 \in \mathbb{Z}$.
- Each step $p(t+1) - p(t)$ is of the form $(1, 0)$, $(1, 1)$ or $(0, 1)$ (in accordance with these three options, we speak of a *horizontal, diagonal* or *vertical* step, respectively).
- The first step $p(1) - p(0)$ is either diagonal or vertical but not horizontal.

We shall deal with collections of paths $L = (p_i)_{1 \le i \le \ell(\lambda)}$ such that $p_i$ starts at $(\mu_i - i, 0)$, ends at $(\lambda_i - i, n)$, and $p_i \cap p_j = \varnothing$ if $i \ne j$. The set of all such collections will be denoted by $L(\lambda/\mu)$.

With each collection $L = (p_i)_{1 \le i \le \ell(\lambda)} \in L(\lambda/\mu)$ we associate a tableau $T$ of shape $\lambda/\mu$ and of order $n$ by the following rules:

$$\{p_i(t) = (m-1, k), \quad p_i(t+1) = (m, k)\} \quad \Rightarrow \quad T(i, m+i) = k;$$
$$\{p_i(t) = (m-1, k-1), \quad p_i(t+1) = (m, k)\} \quad \Rightarrow \quad T(i, m+i) = k'.$$

That is, the $i$th path codes the filling of the $i$th row in $\lambda/\mu$, each square $(i, j)$ being associated with a nonvertical step in $p_i$. If the endpoint of a nonvertical step has coordinates $(m, k)$ then $j = i + m$, and we have $T(i, j) = k$ or $T(i, j) = k'$ according to whether the step is horizontal or diagonal.

To verify that $T$ is indeed a tableau of shape $\lambda/\mu$ and of order $n$ we check the four properties of Definition A.3.

The properties (1) and (3) follow from the definition of the path $p_i$.

Let us prove the property (2): if $p_i(t+1) = (m, k)$ and $p_{i+1}(s) = (m-1, r)$, then $r \ge k$. Moreover, if $p_i(t) = (m-1, k)$, then $r > k$.

Now let us check the property (4): if $p_i(t) = (m-1, k), p_i(t+1) = (m, k)$, and $p_{i+1}(s) = (m-1, r)$, then $r > k$.

One can easily check that the correspondence $L \to T$ is a bijection between $L(\lambda/\mu, n)$ and $\mathrm{Tab}(\lambda/\mu, n)$.

Next, we assign to an arbitrary path $p$ its *weight* $\Pi(p)$ as follows: the weight of $p$ is the product of the weights of its steps, where

- a horizontal step with endpoint $(m, k)$ has weight $x_k - a_m$;
- a diagonal step with endpoint $(m, k)$ has weight $y_k + a_m$;
- any vertical step has weight 1.

For an arbitrary collection $L = (p_i)_{1\le i\le \ell(\lambda)} \in L(\lambda/\mu)$ we set

$$\Pi_L = \prod_{i=1}^{\ell(\lambda)} \Pi(p_i).$$

Then we get

$$\Pi_L = \prod_{\substack{\square\in\lambda/\mu\\ T(\square)=1,2,\dots,n}} (x_{T(\square)} - a_{c(\square)}) \prod_{\substack{\square\in\lambda/\mu\\ T(\square)=1',2',\dots,n'}} (y_{T(\square)} + a_{c(\square)}),$$

where $T \leftrightarrow L$. It follows that

$$\Sigma_{\lambda/\mu|n} = \sum_{L\in L(\lambda/\mu,n)} \Pi_L. \tag{A.4}$$

The standard argument of the Gessel–Viennot theory (see [Sa, §4.5]) shows that

$$\sum_{L\in L(\lambda/\mu,n)} \Pi_L = \det[\Sigma(i,j)]_{1\le i,j\le \ell(\lambda)}, \tag{A.5}$$

where $\Sigma(i,j) = \sum_p \Pi(p)$, summed over all paths $p$ starting at $(\mu_j - j, 0)$ and ending at $(\lambda_i - i, n)$. From (A.3) it follows that

$$\Sigma(i,j) = h_{\lambda_i-\mu_j+j-i|n}(x;y \mid \tau^{\mu_j-j+1}a). \tag{A.6}$$

From (A.4),(A.5) and (A.6) we obtain that

$$\Sigma_{\lambda/\mu|n}(x;y\mid a) = \det[h_{\lambda_i-\mu_j+j-i|n}(x;y\mid \tau^{\mu_j-j+1}a)]_{1\le i,j\le\ell(\lambda)}$$

and, consequently,

$$s_{\lambda/\mu}(x_1,x_2,\dots,x_n;y_1,y_2,\dots,y_n\mid a) = \Sigma_{\lambda/\mu|n}(x;y\mid a).$$

□

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