What to do
When all you have is Brownian motion

Student probability day VII,
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Conformal mapping
Warning

This presentation shows graphical images of Brownian motion.

Viewer discretion is advised.
The Skorokhod embedding problem
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- Life gives you Brownian motion.
The Skorokhod embedding problem

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- But you do not want Brownian motion. You want to sample from a distribution $\mu$.

- How do you sample from $\mu$ using your Brownian motion?
The Skorokhod embedding problem

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• If you choose $T$ in a special way, then perhaps $B_T$ distributes as $\mu$?
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**Uniform distribution on $[-1,1]$?**
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  • Also Gross 19.
Solution 1: Dubins

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• Not a clever generalization of the Bernoulli \( \pm 1 \) method.

• Rather, uses conformal mappings
A conformal solution

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**Theorem:** Let $\mu$ have 0 mean and finite variance. Let $B_t$ be a planar Brownian motion. There exists a simply connected domain $\Omega$ such that when $B_t$ exits $\Omega$, its $x$ coordinate distributes as $\mu$. 
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Here is its inverse cdf, $F_\mu^{-1}$.

The inverse mapping theorem says that if $U$ is uniform, then

$$F_\mu^{-1}(U) \sim \mu$$

Any $T$ satisfying $T(U) \sim \mu$ must satisfy

$$F_\mu(x) = \mathbb{P}[X \leq x] = \mathbb{P}[T(U) \leq x] = \mathbb{P}[U \leq T^{-1}(x)] = T^{-1}(x)$$
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• How convenient! Brownian motion is conformally invariant

• That is, the image of Brownian motion under a conformal map is a (time-changed) Brownian motion as well
A conformal solution

How convenient! Brownian motion is uniform on the circle

That is, the image of Brownian motion under a conformal map is a \( f(z) \) Brownian motion as well.
A conformal solution
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• Since $\arg(B_{T_{\text{circle}}})$ is uniform, all we need is a conformal map $\psi$ which has

$$\text{Re}\{\psi(e^{i\theta})\} = F_{\mu}^{-1}(\theta)$$
A conformal solution

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  \[
  \text{Re}\{\psi(e^{i\theta})\} = F_{\mu}^{-1}(\theta)
  \]

- Luckily for us, on the unit circle, a Fourier series and a power series are the same thing.

\[
\begin{align*}
  f(z) &= \sum_{n=0}^{\infty} a_n z^n \\
  &= \sum_{i=0}^{\infty} a_n e^{i n \theta} \\
  &= \sum_{i=0}^{\infty} a_n (\cos n \theta + i \sin n \theta)
\end{align*}
\]
A conformal solution

• Let $\varphi_\mu(\theta) = F_\mu^{-1}\left(\frac{\theta}{\pi}\right)$
A conformal solution

- Let $\varphi_\mu(\theta) = F^{-1}_\mu\left(\frac{|\theta|}{\pi}\right)$

- Expand $\varphi_\mu(\theta)$ as an even Fourier series:

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- On the unit circle, the x-coordinate (i.e., real part) of \( \psi_\mu \) agrees with \( \varphi_\mu \)
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  \text{Re}\{\psi_\mu(e^{i\theta})\} = \sum_{n=0}^{\infty} \hat{\varphi}_\mu(n) \cos n\theta = \varphi_\mu(\theta)
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- Expand $\varphi_\mu(\theta)$ as an even Fourier series:
  \[ \varphi_\mu(\theta) = \sum_{n=0}^{\infty} \hat{\varphi}_\mu(n) \cos n\theta \]
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  \[ \psi_\mu(z) = \sum_{n=0}^{\infty} \hat{\varphi}_\mu(n) z^n \]
- On the unit circle, the x-coordinate (i.e. real part) of $\psi_\mu$ agrees with $\varphi_\mu$
  \[ \text{Re}\{\psi_\mu(e^{i\theta})\} = \sum_{n=0}^{\infty} \hat{\varphi}_\mu(n) \cos n\theta = \varphi_\mu(\theta) \]
- The Fourier coefficients decay, so $\psi_\mu$ is analytic inside the unit disc
A potential problem

• Even though Brownian motion is preserved under analytic maps, in order to transform boundary to boundary we must be one-to-one

• Otherwise:

\[ f(z) = z^2 \]
A potential problem

• For “nice” enough $\mu$, this is not a problem
• If $\mu$ is bounded and $F_\mu$ is strictly monotone increasing then $F_\mu^{-1}$ is continuous and bounded.
• $\psi_\mu$ then maps the circle’s boundary to a simple closed loop, which is the boundary of $\Omega$
A potential problem

- For nasty $\mu$, we don’t have that luxury
  - E.g: an atomic distribution with finite weight on every rational
  - E.g: The Cantor distribution
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• In this case the mapping may diverge
A solution

- **Theorem:** Let \( \{f_k(z)\} \) be a series of one-to-one functions on a domain \( D \) which converge uniformly on every compact subset of \( D \) to a function \( f \). Then \( f \) is either one-to-one or constant.

- If we take nice smooth functions \( F_k \) (not necessarily CDFs) which converge to \( F_k \), we’ll get \( \psi_k \) s which are one-to-one and which will converge to \( \psi_\mu \).
Examples

Gaussian

Uniform
Examples

$$P[k] \sim \frac{1}{2^k}$$

Cantor
For more information, call 1-800-https://arxiv.org/abs/1905.00852

Thanks!