Formula Satisfaction
Testing

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Monotone Binary Read-Once Boolean Formula $\Phi$

Goal: Given full knowledge of the formula, $\epsilon > 0$ and oracle access to an assignment to $\Phi$, 
“evaluate” $\Phi$
Related Results

- “Some 3CNF properties are hard to test”

- “Testing Properties of Constraint-Graphs”

- “Languages that can be accepted by Read-Once Bounded Width Branching Programs are Testable with a Constant Number of Queries”

- “There exists a language that can be accepted by Read-Twice Bounded Width Branching Program whose query complexity is $\Omega(n)$”

- “Regular Languages are Testable with a Constant Number of Queries”.
Some of Our Results

- Read-\textit{once}, binary, Boolean formula
  non-adaptive query complexity: quasi polynomial in $1/\varepsilon$

- Read-$k$-times, \textbf{monotone}, binary Boolean formula
  non-adaptive query complexity: quasi polynomial in $k/\varepsilon$

- Read-\textit{once}, binary, formula over alphabet size four,
  query complexity may depend on formula size
Testing satisfaction ("exponential algorithm")

Assumption: formula consists of interleaved AND, OR layers
Token Phase

- Put a token on output

- For a gate $v$ at depth at most $\text{poly}(1/\varepsilon)$ that have a token
  - OR with at most $1/\varepsilon$ children children, each one of its children is given a token.
  - AND, select a child so that the probability of a child $u$ being selected is $|u|/|v|$. Repeat independently $O(1/\varepsilon)$ times.
Evaluation Phase

- Variable with token - Query

- Evaluate to $1$:
  - Gate with a token and all of its children without a token,
  - OR with a child that has less than $\varepsilon$ fraction of its parents variables,

- AND with token yet all of its children without a token – ignore children without a token

The rest of the evaluation is done in the standard way. The result is the output.
Analysis

• Query complexity \( \left(\frac{1}{\varepsilon}\right)^O\left(\frac{1}{\varepsilon}\right) \)

• A 0 evaluation, is a witness that the formula is not satisfied

• If formula satisfied then there is no 0-witness

? Assignment is far from satisfying the formula then w.h.p a 0-witness is found
Assume assignment is $\epsilon$–far from satisfying the output

- Output AND, then with w.h.p. the assignment is $\epsilon(1 - \epsilon/10)$ from satisfying one of its children that got a token

- Output OR, then for each one of its children $u$ the assignment is $\epsilon(1+\epsilon)$ far from satisfying $u$
Testing satisfaction, $0$-witness

Lemma:

- If the assignment is $\varepsilon$–far from satisfying the formula then w.h.p a $0$-witness is found

Proof

- If $\varepsilon > \frac{1}{2}$

- Assume true for every $\varepsilon>\delta>1/2$

- Let $\varepsilon > \delta(1-\delta/2)$
  
  - If the output is an OR gate then for each one of its children $u$ the assignment is $\delta$ far from satisfying $u$

  - If the output is an AND gate the assignment is $\varepsilon(1-\varepsilon/10)$ from satisfying one of its children that got a token. For that child we are back to the OR case (if it is a variable we are done)
Token Phase Revisit

- Put a token on output

- For a gate $v$ at depth at most $\text{poly}(1/\varepsilon)$ that have a token
  - OR with at most $1/\varepsilon$ children, each one of its children is given a token.
  - AND, select a child so that the probability of a child $u$ being selected is $|u|/|v|$. Repeat independently $O(1/\varepsilon)$ times.
Quasi Polynomial test

Idea:

Critical variable
Quasi Polynomial test

Idea:

Critical variable exists

- Largest child $\geq (1+\epsilon)2\epsilon/3$
- Largest child $\geq (1+\epsilon)^22\epsilon/3$
- Largest child $\geq 2\epsilon/3$

- $\geq 4\epsilon/3$
- $\geq 2\epsilon/3$
- $\geq 0$
- $>4\epsilon/3$
Enough Critical

Lemma:
- If assignment is “$2\varepsilon/3$ - far” from satisfying $\Phi$ then there exists a critical variable for the assignment.

Observation:
- If the assignment is “$\varepsilon$ - far” from satisfying $\Phi$ then $\varepsilon/3$ of the variables are critical for the assignment.
Monotone Ternary Gates

$$\text{Maj}(A, B, C) = (A \land (B \lor C)) \lor (B \land C)$$
Open Problems

- Improve upper bound or prove lower bound (ideally $\text{poly}(1/\varepsilon)$ upper bound)
- More type of gates in the non-binary case
- Lower bound for ternary alphabet
Thank You