A Near-Optimal Sublinear-Time Algorithm for Approximating the Minimum Vertex Cover Size

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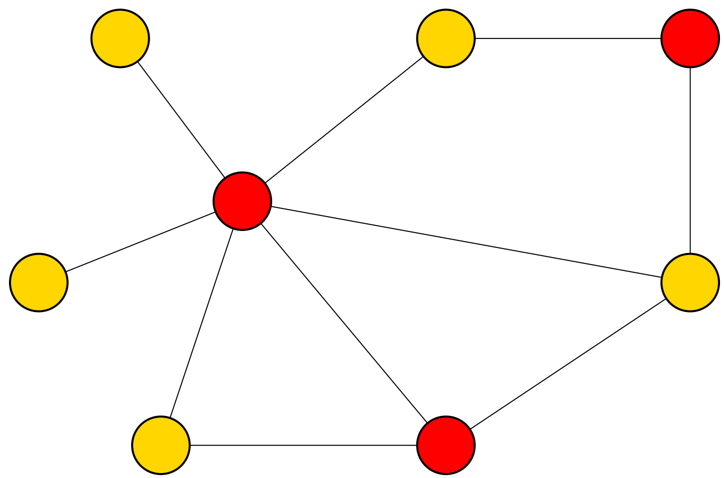
CMU

Joint work with **Dana Ron**, **Michal Rosen**, and **Ronitt Rubinfeld**

The Problem

- Vertex Cover: set S of vertices such that each edge has endpoint in S



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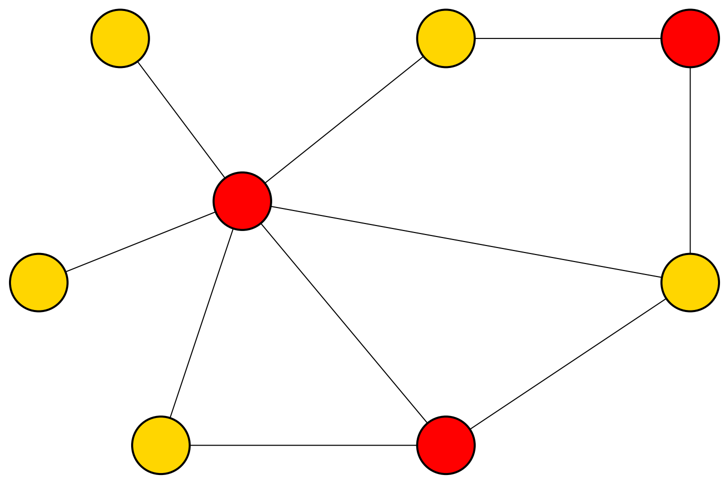
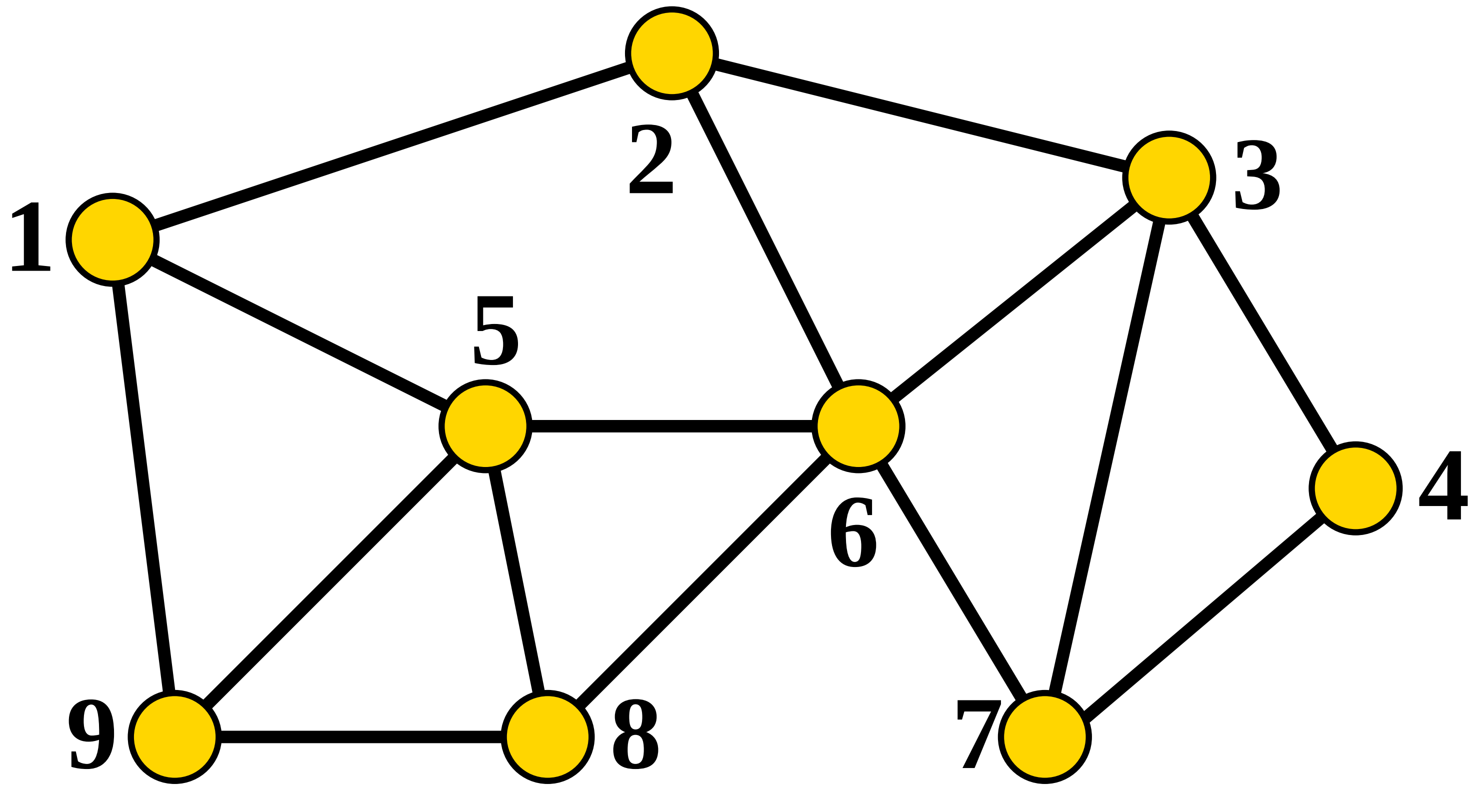


- Our Goal:
  $(2, \epsilon n)$-estimate for the minimum vertex cover size
- X is an (α, β)-estimate for Y if

$$Y \leq X \leq \alpha Y + \beta$$

The Model

Graph G of degree d:



Query access to adjacency list of each node

Query Complexity

Positive results for $(2, \epsilon n)$-estimation:

- Parnas, Ron (2007): $d^{O(\log(d)/\epsilon^3)}$
- Marko, Ron (2007): $d^{O(\log(d/\epsilon))}$
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A negative result due to Parnas and Ron (2007):

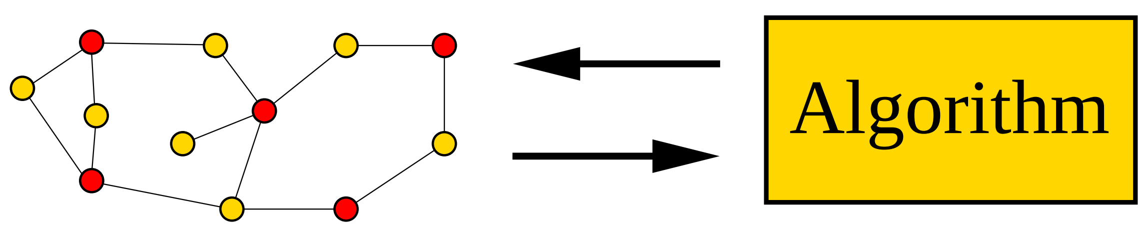
- $(C, \epsilon n)$-estimation requires $\Omega(d)$ queries for any constant C

Quick Review

General Approach

Idea of Parnas and Ron (2007):

- If we had query access to a small vertex cover, we could approximate its size up to $\pm\epsilon n$ by sampling $O(1/\epsilon^2)$ vertices



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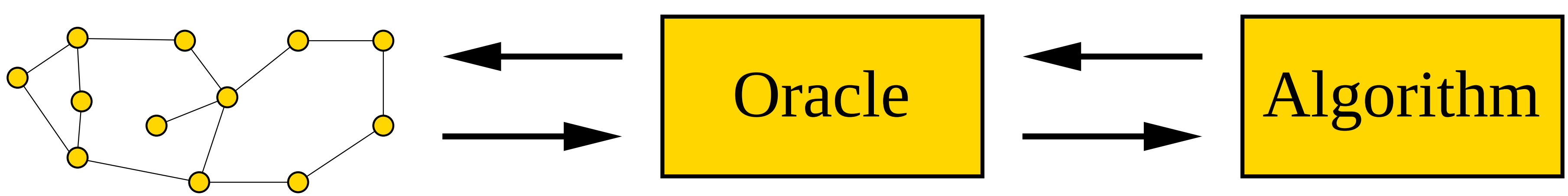

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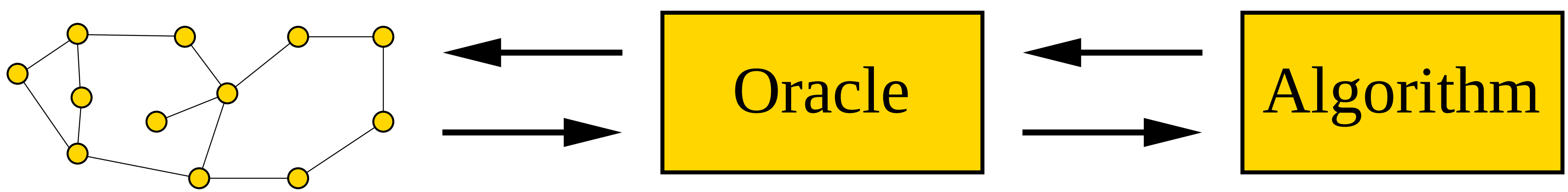
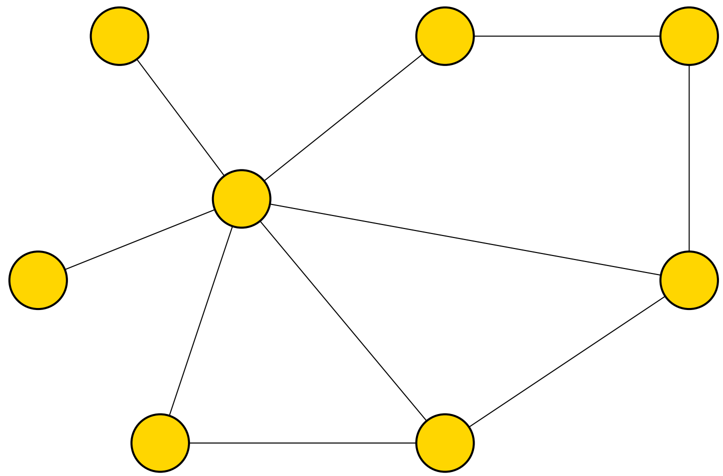
- Parnas and Ron's construction:
  simulation of local distributed algorithms of Kuhn, Moscibroda, and Wattenhofer (2006)



Simulation of the Greedy Algorithm

Classical 2-approximation algorithm [Gavril, Yannakakis]:

- Greedily find a maximal matching M
- Output the set of nodes matched in M



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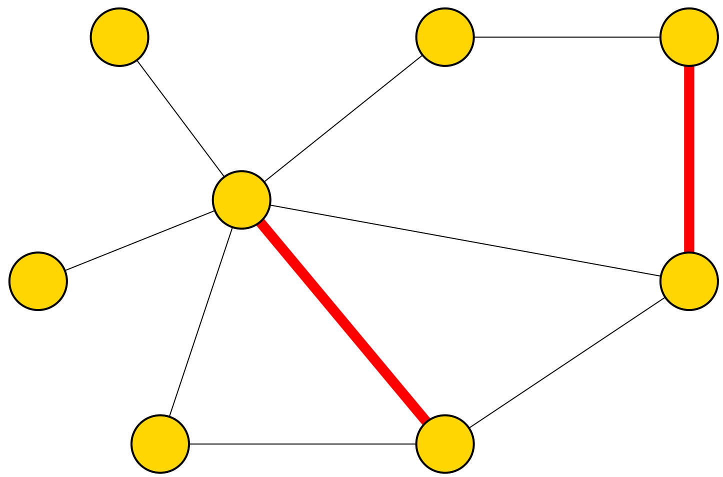

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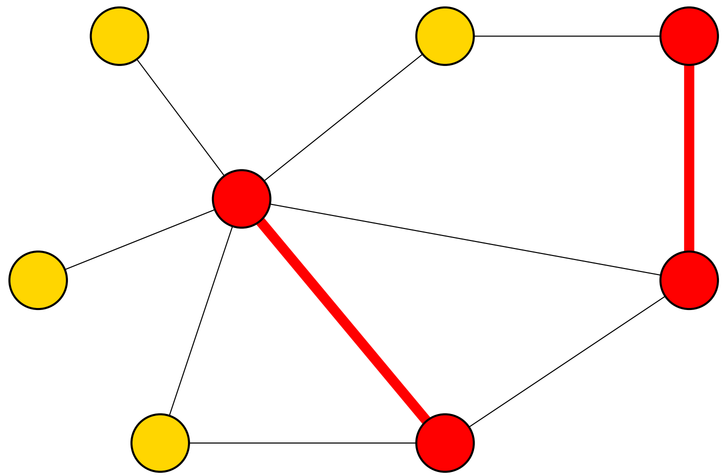

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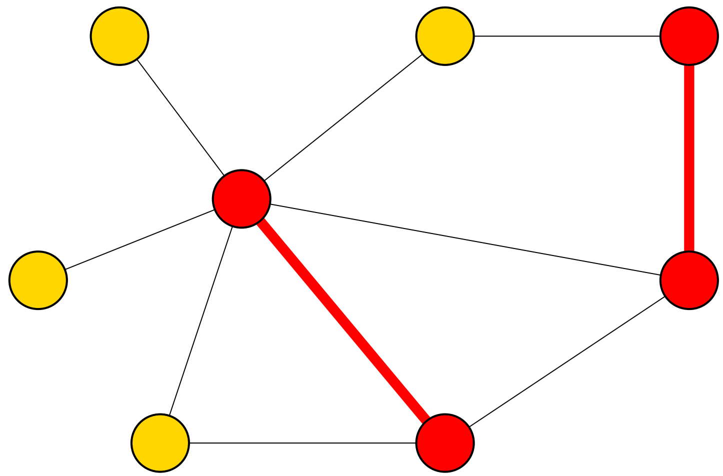

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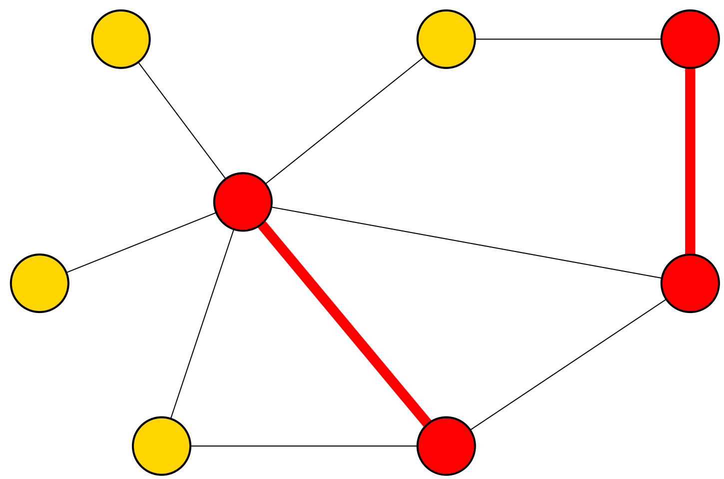

The idea from Nguyen and O. (2008):

- Construction of M: consider edges in random order
- (Try to) locally check if an edge belongs to M

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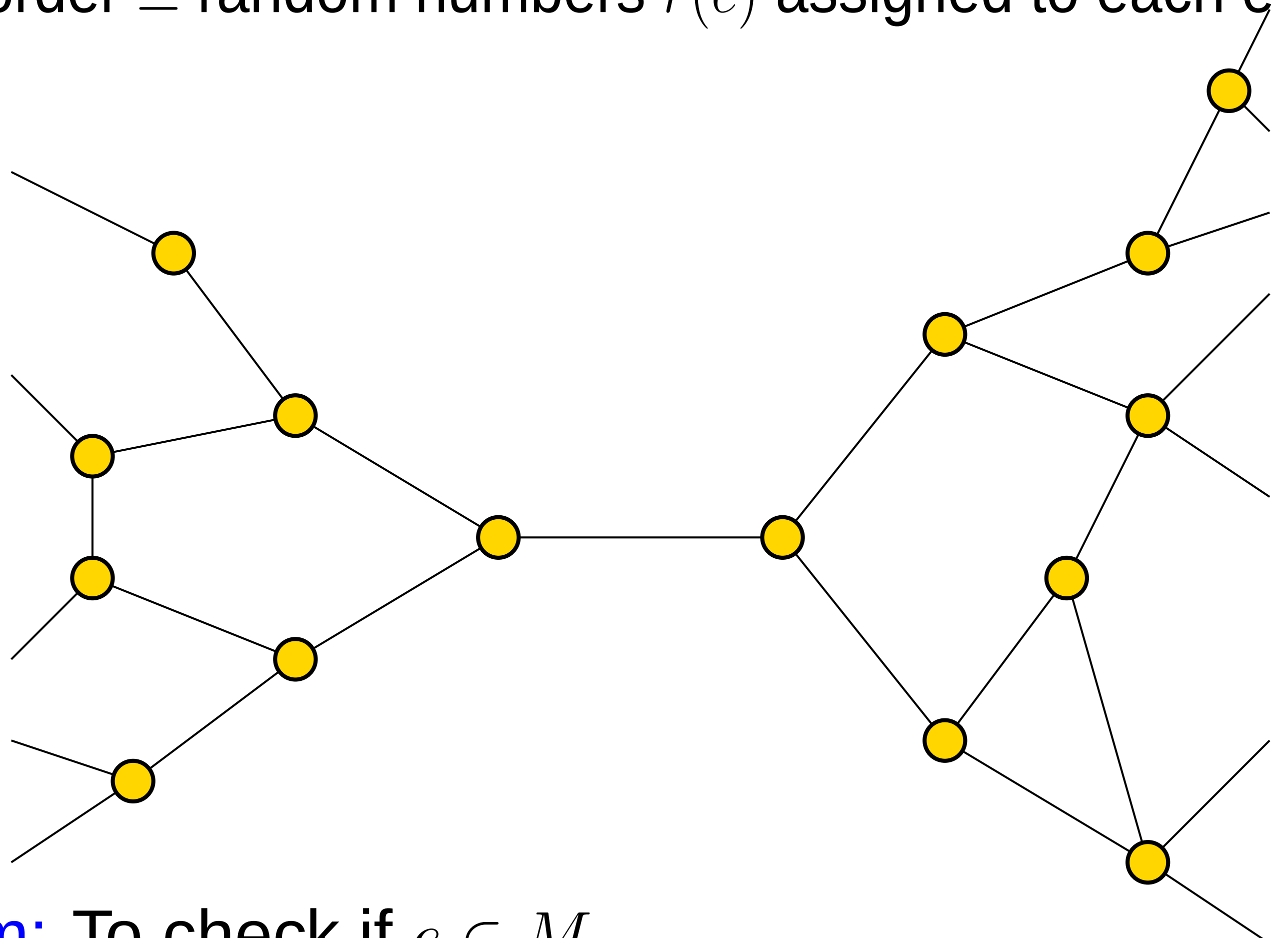
Random order $\equiv$ random numbers $r(e)$ assigned to each edge



Algorithm: To check if $e \in M$

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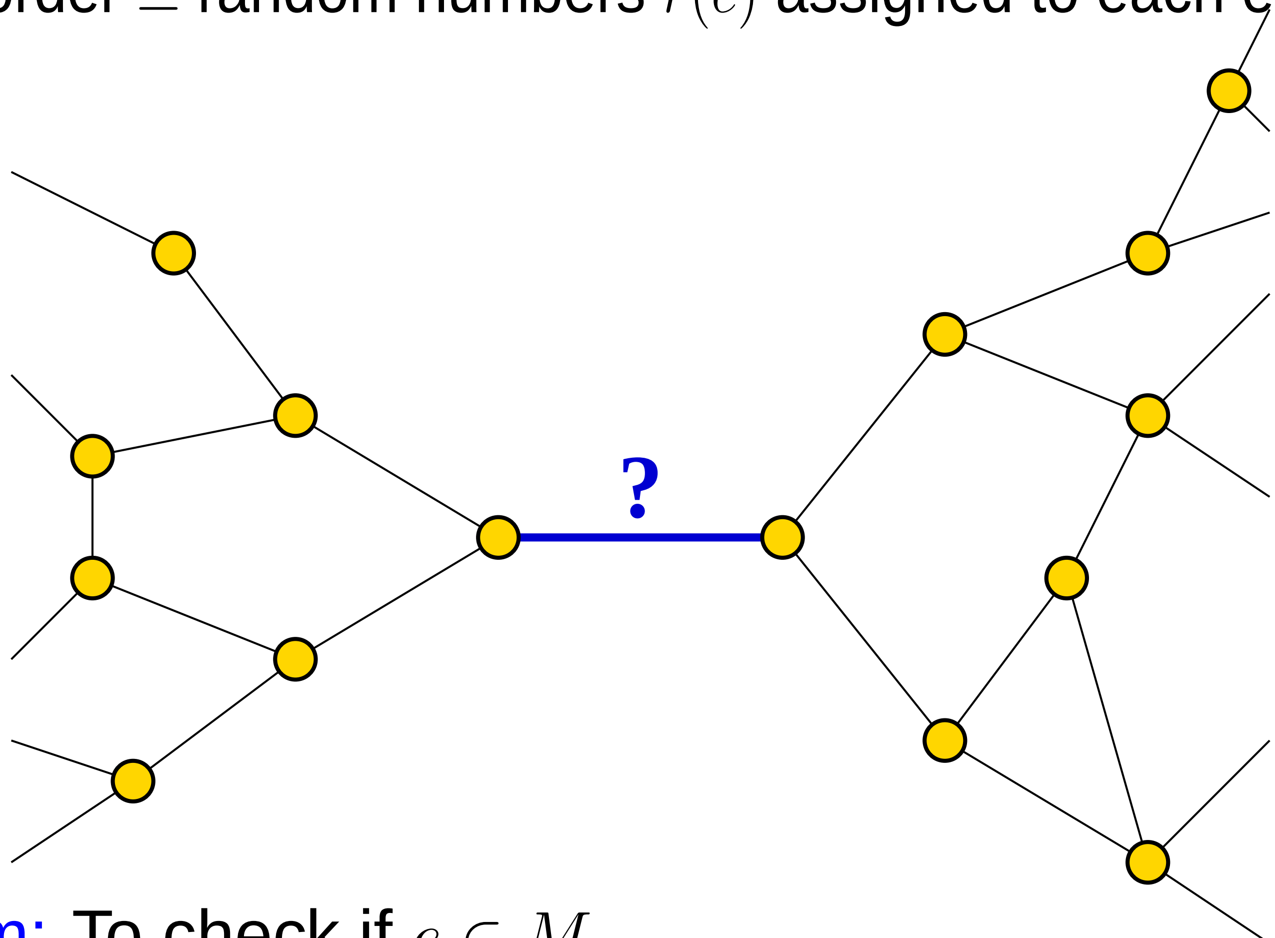

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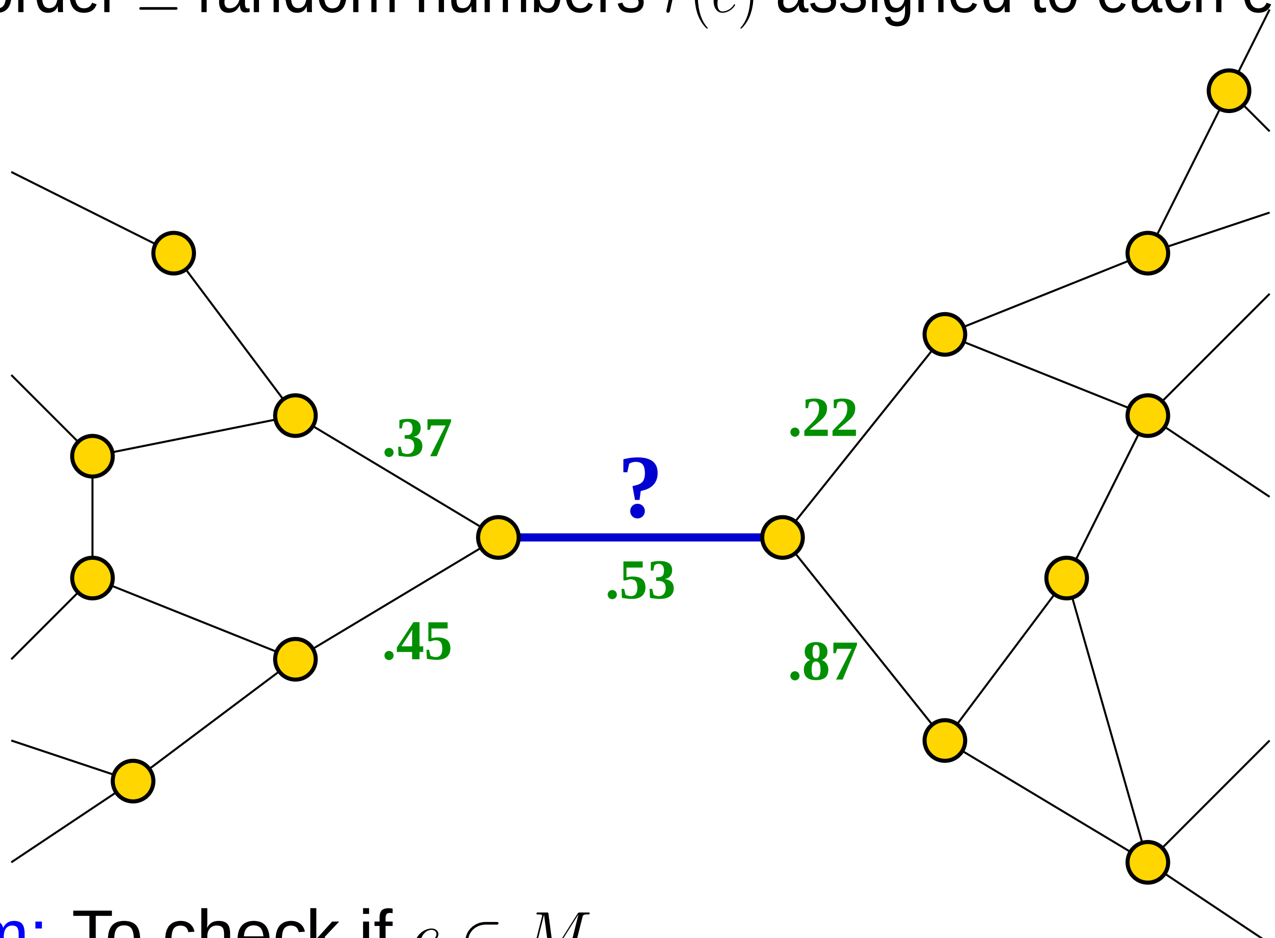

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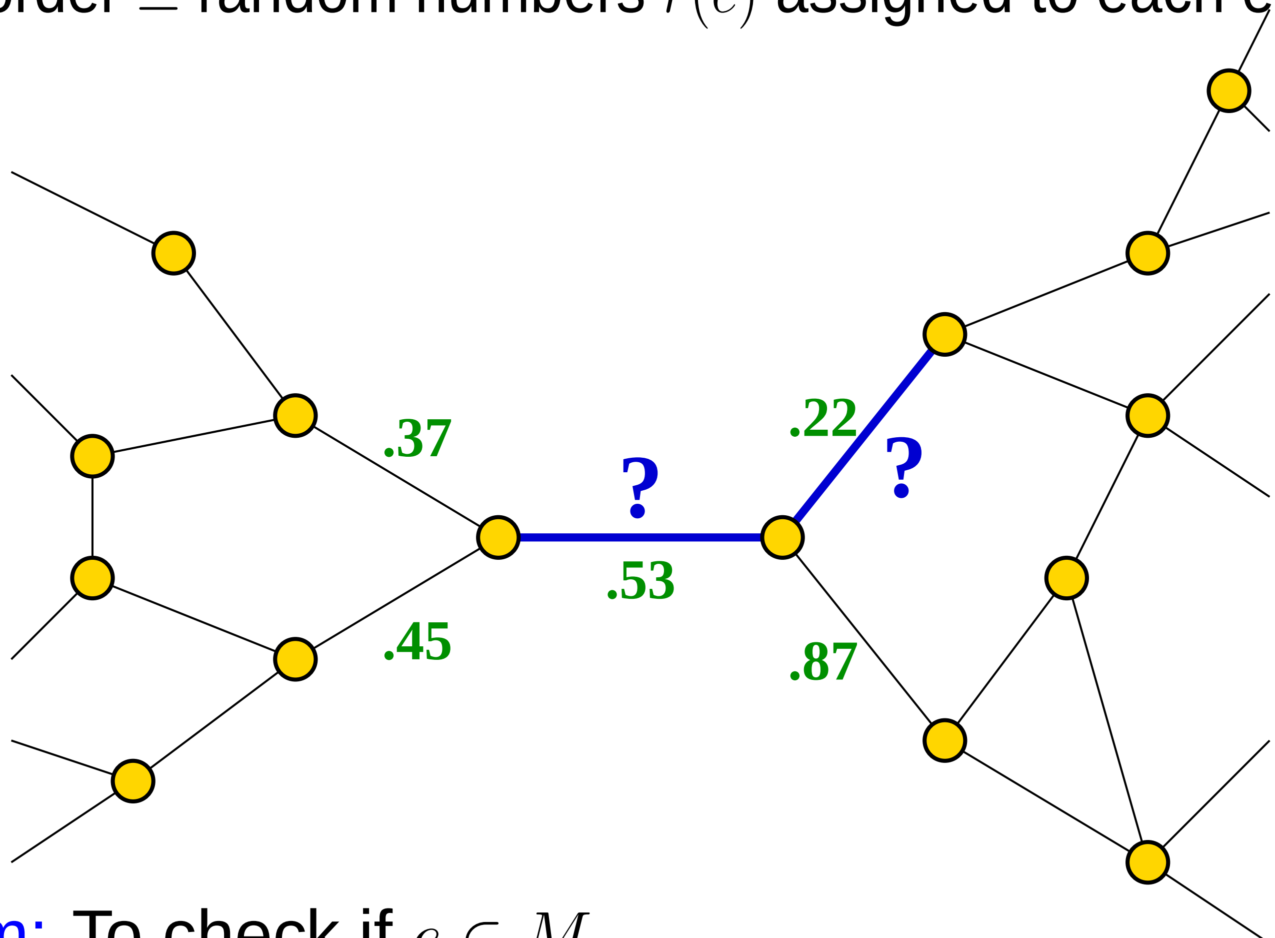

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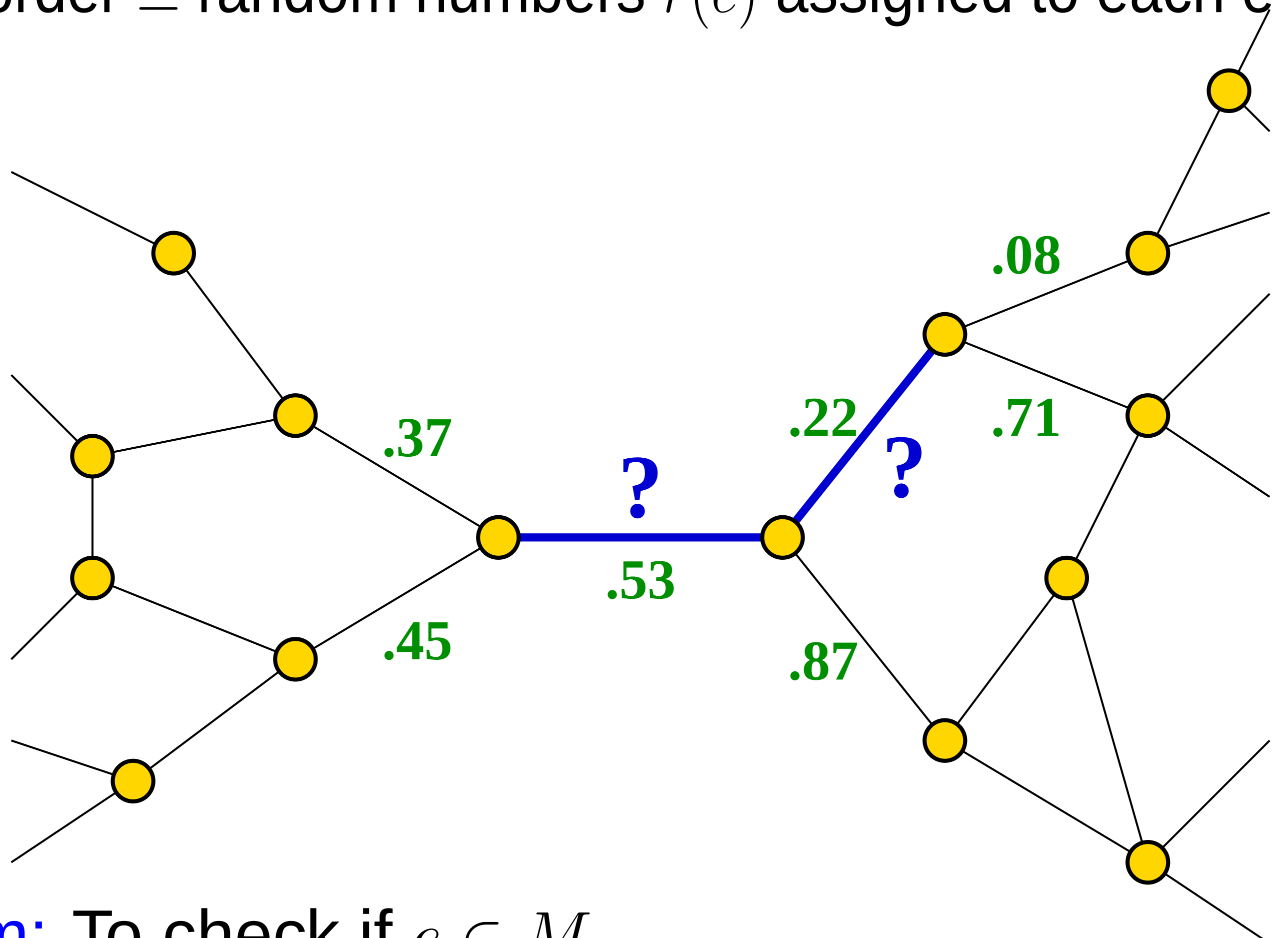

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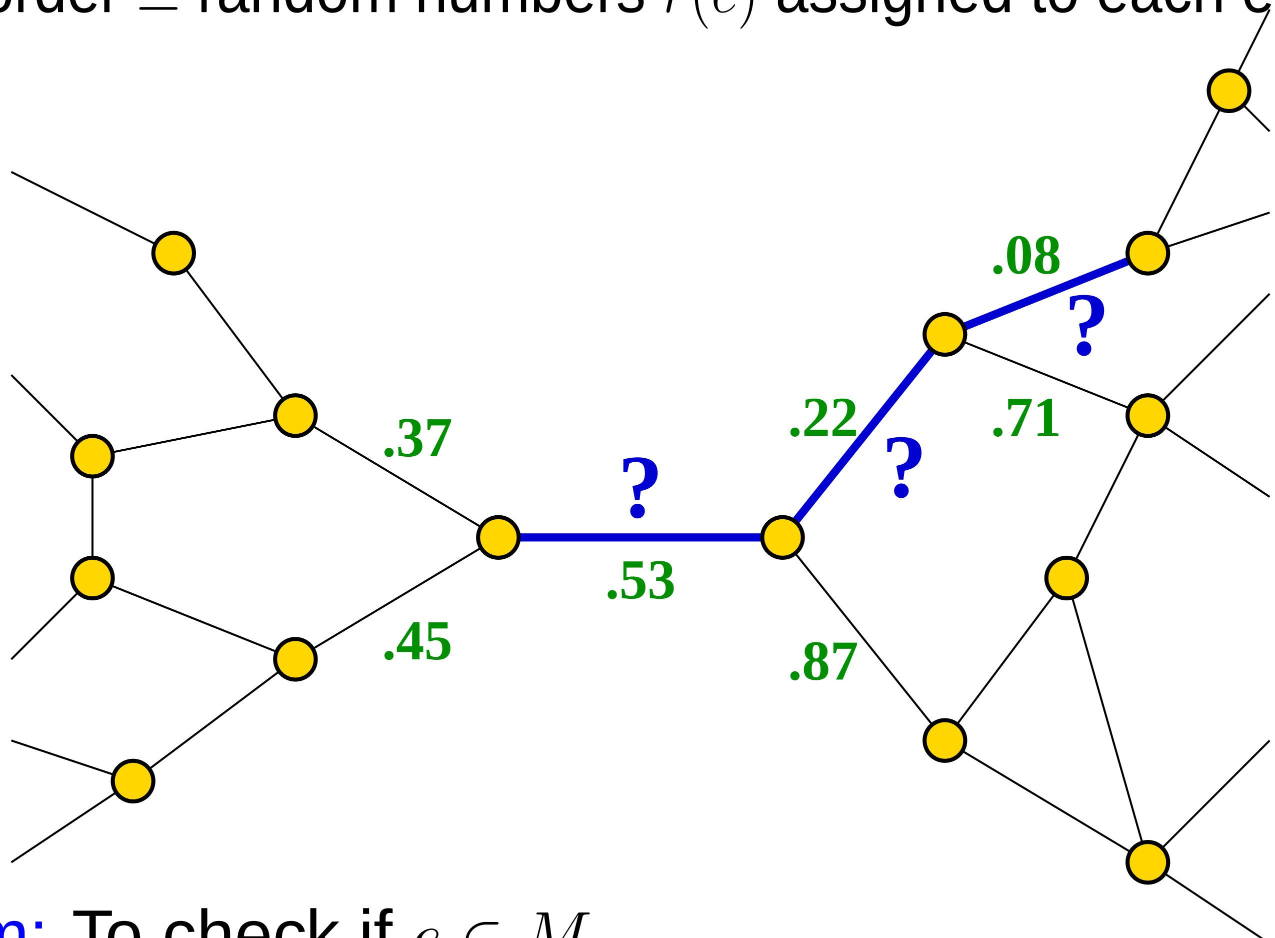

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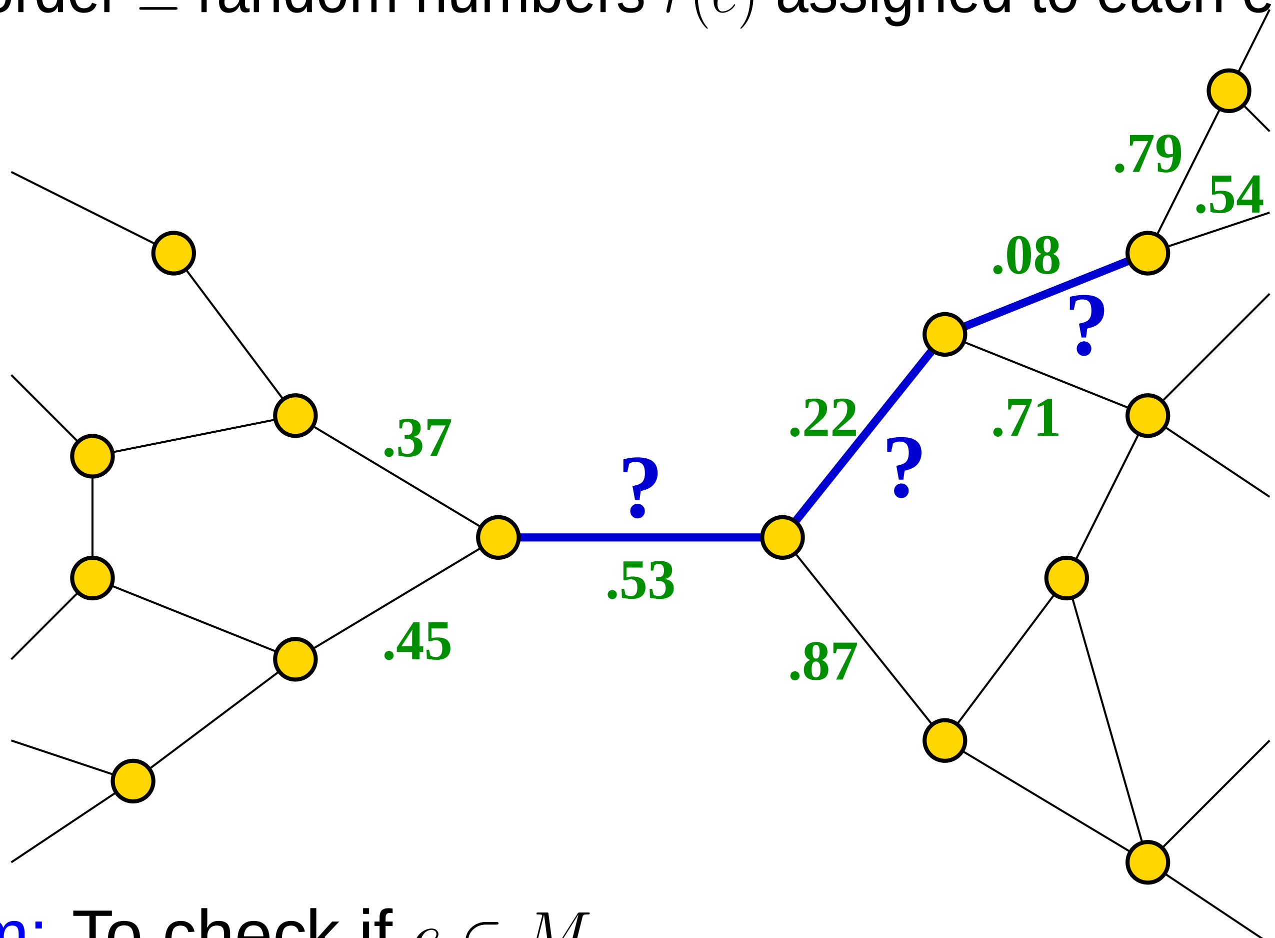

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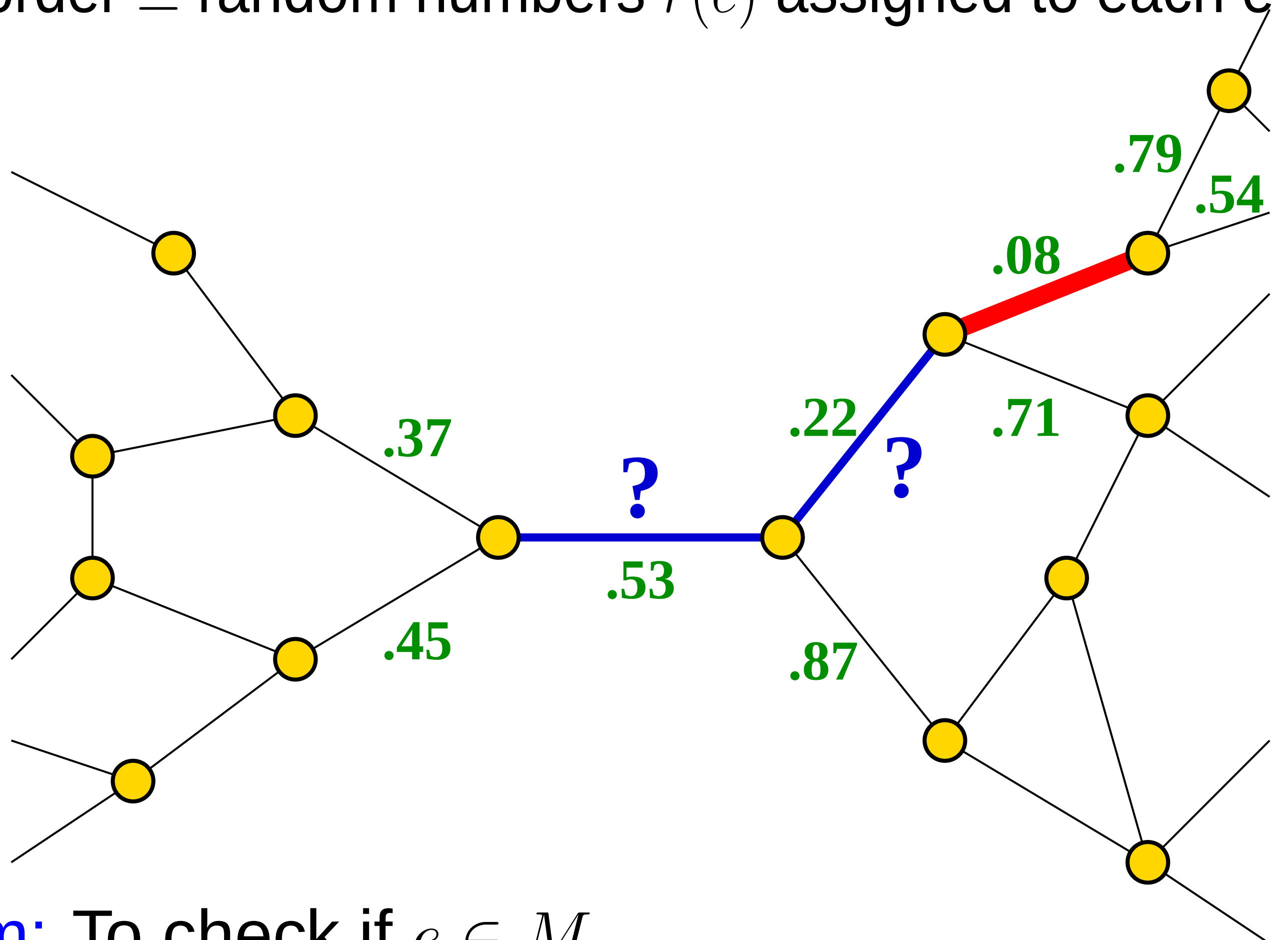

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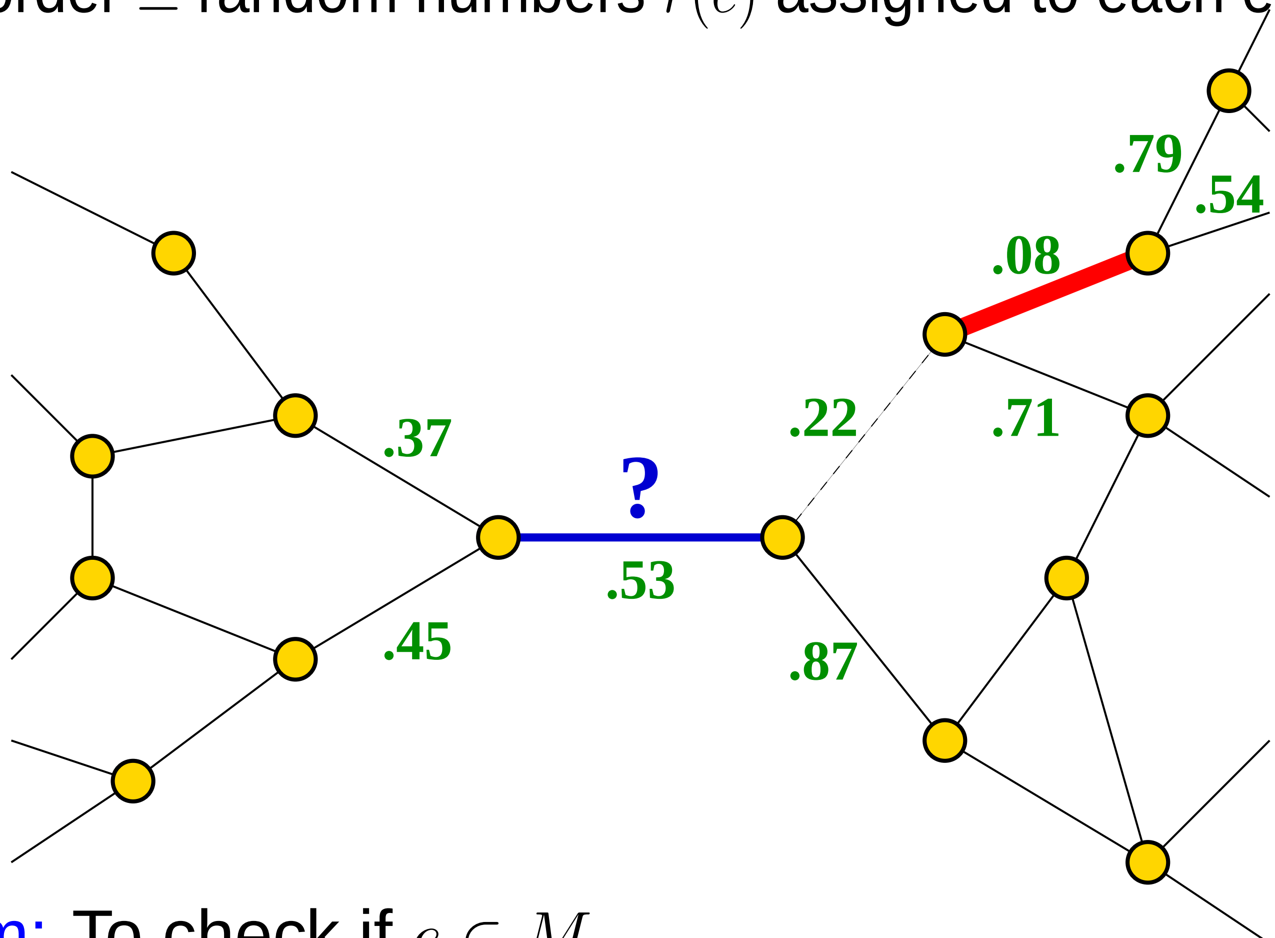

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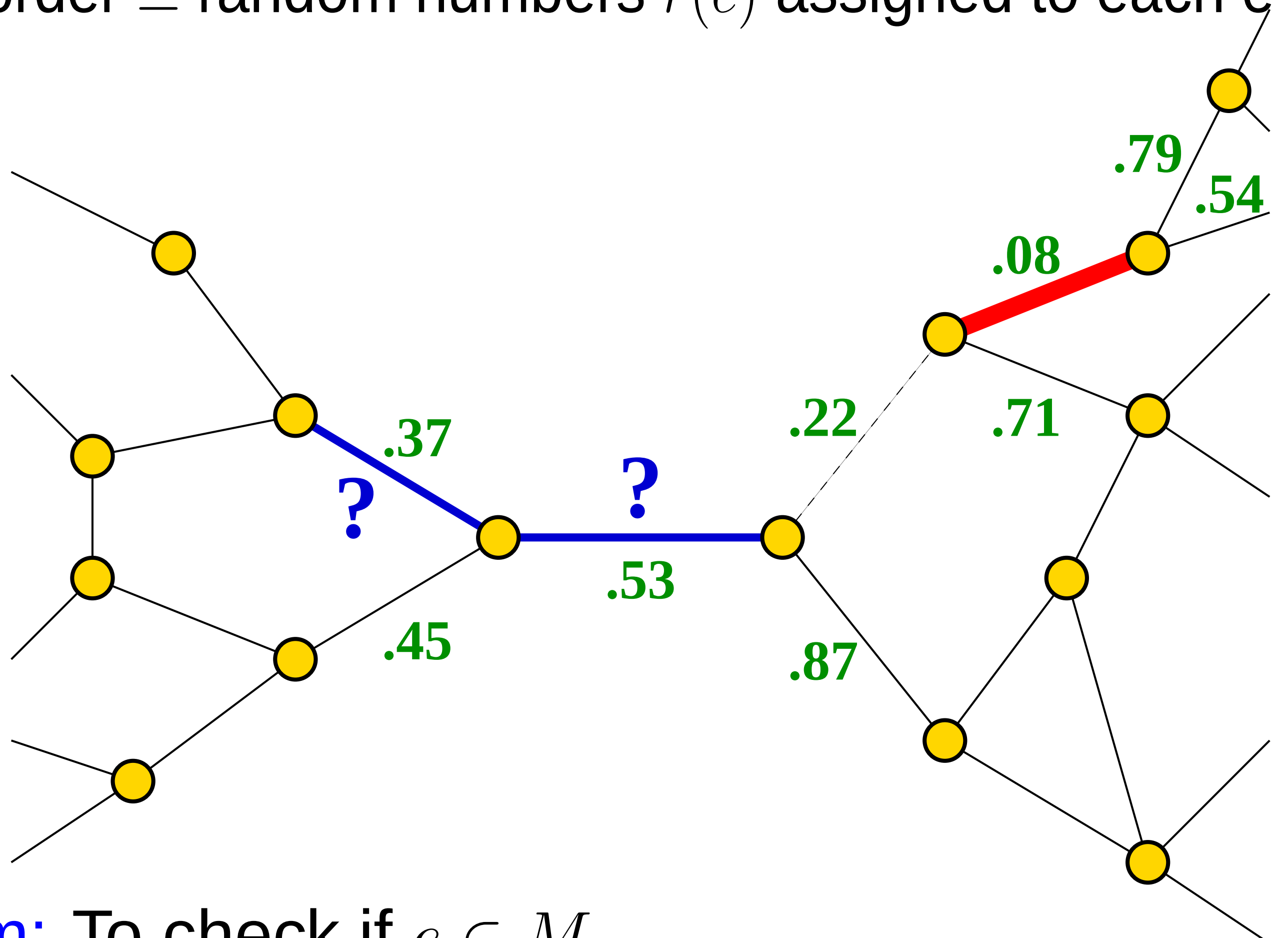

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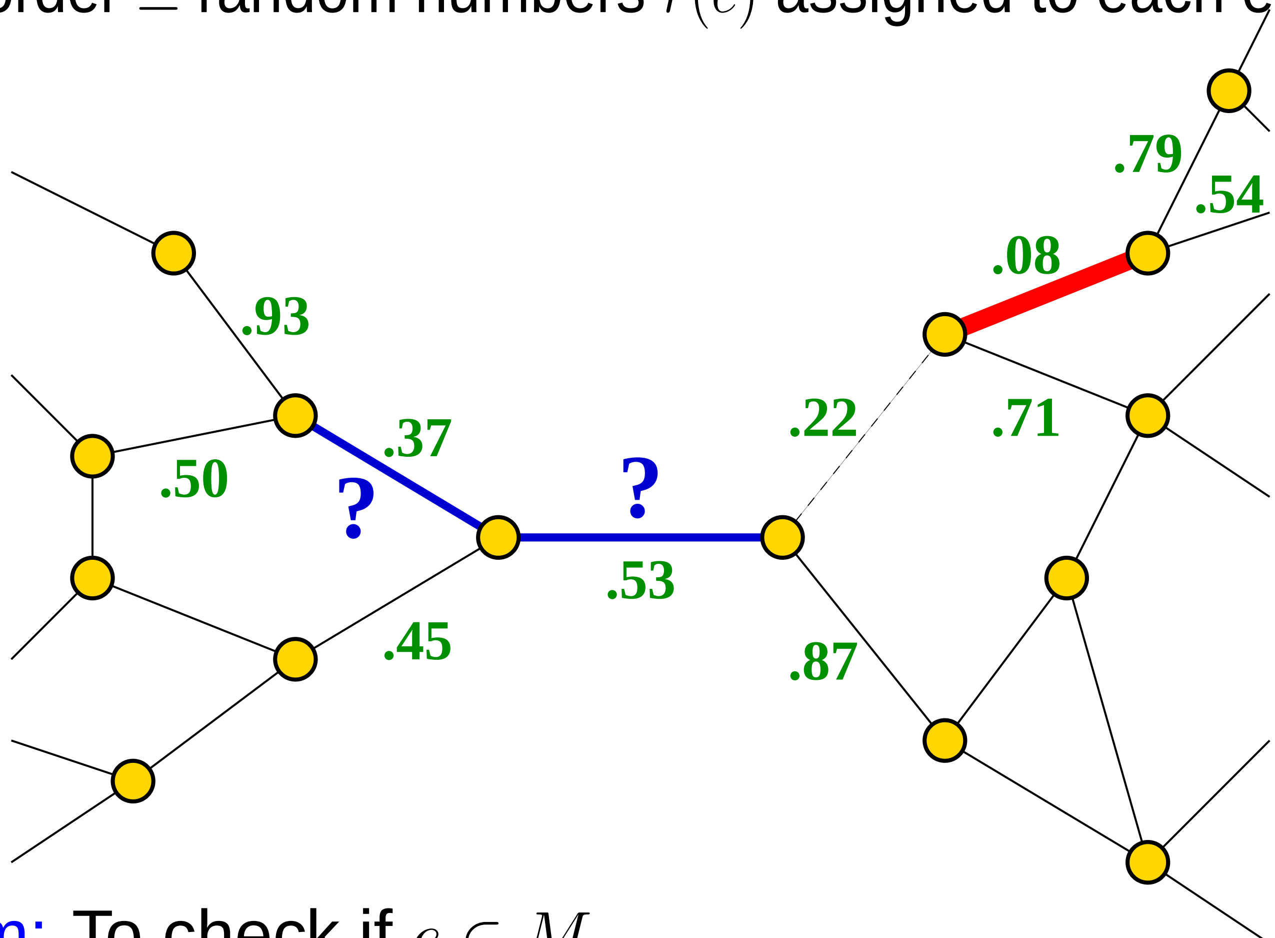

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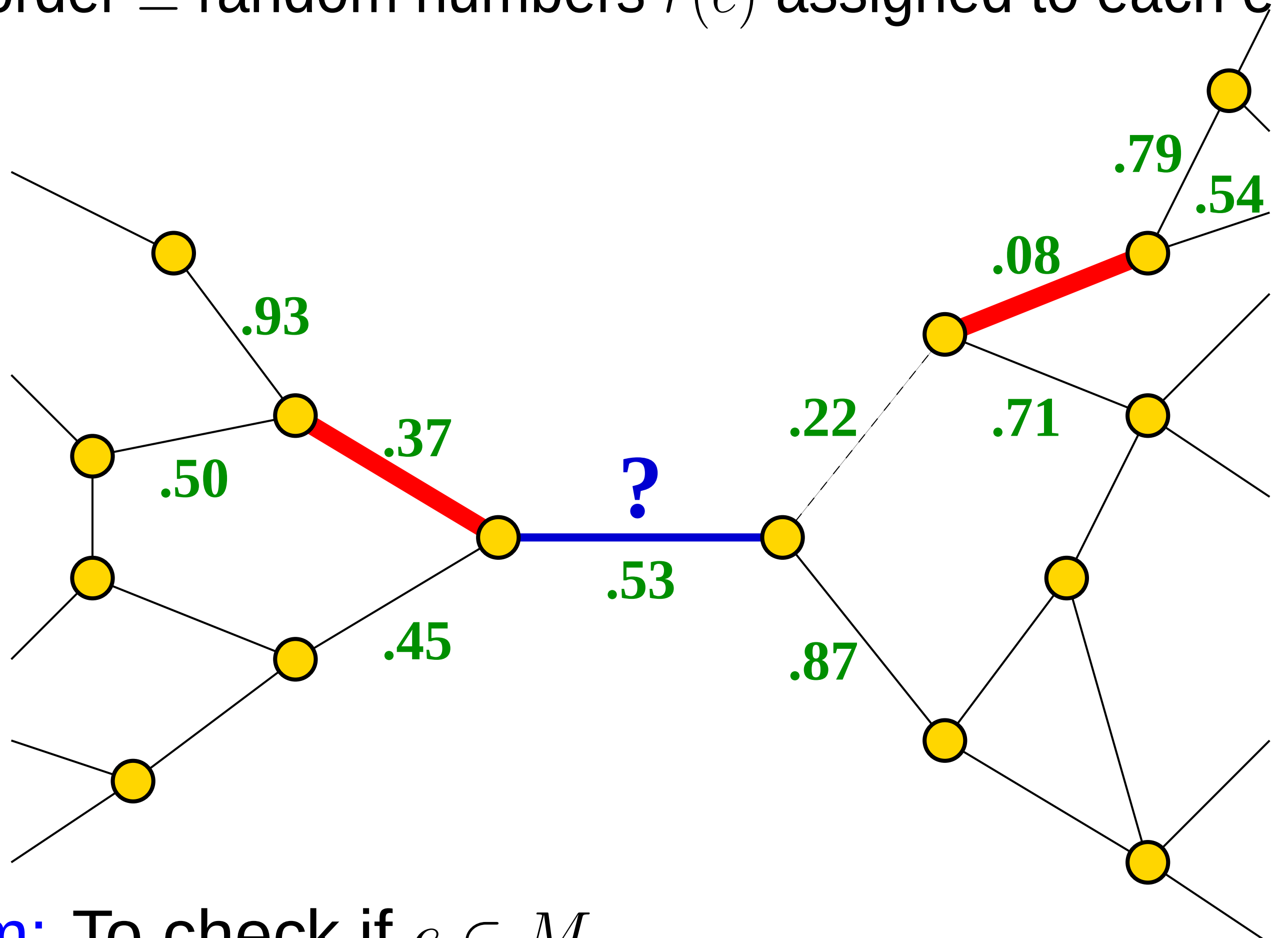

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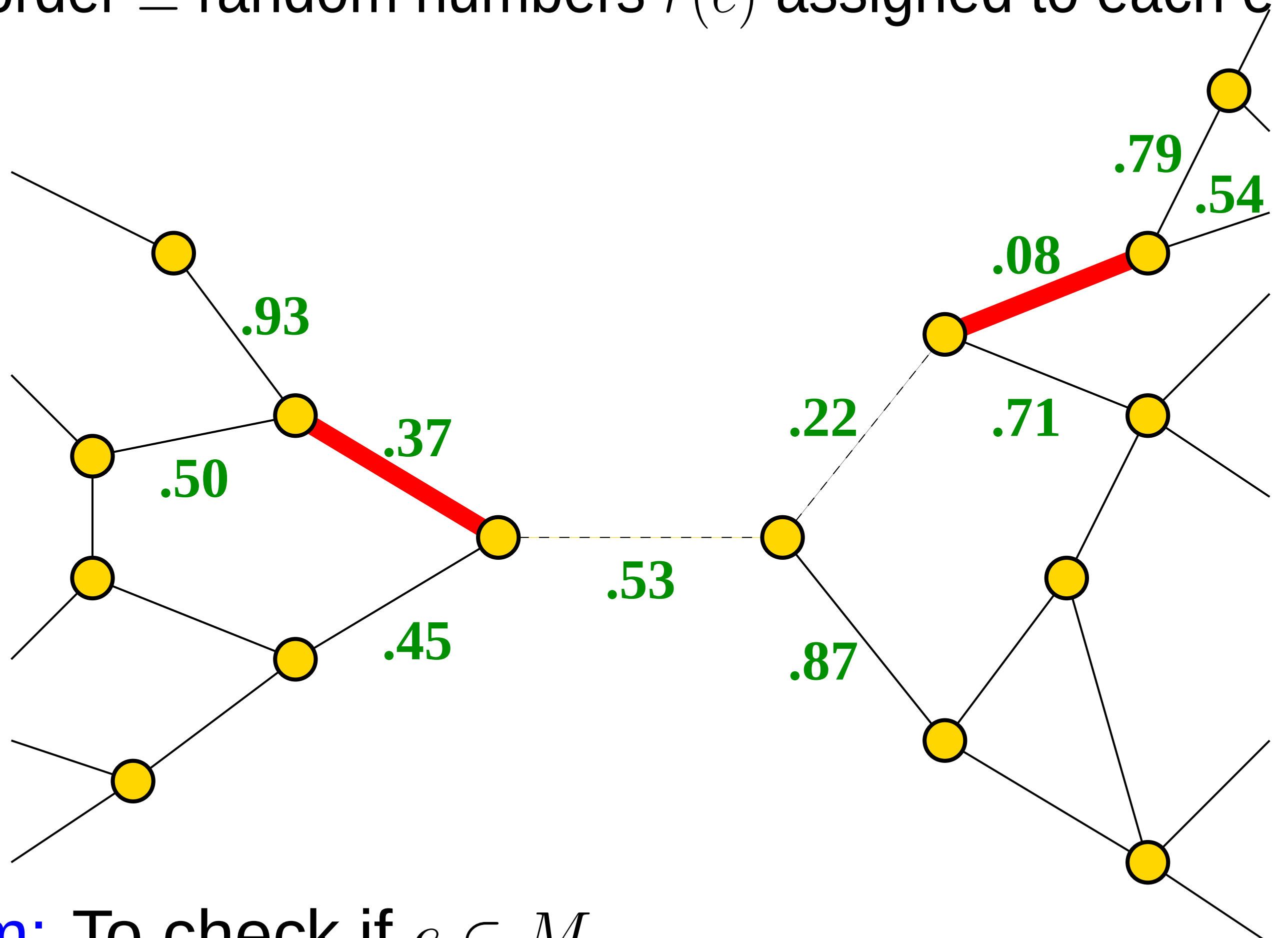

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- Yoshida, Yamamoto, Ito (2009):

  The expected number of recursive calls is $O(d)$ for a random edge

Our New Algorithm
(Part 1)

Overview

What happens to three factors of d?

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In this talk:

- Item 2 in Part 1
- Item 3 in Part 2

Our Exploration Method

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Our bound:

The expected number of visited edges for a random vertex is

$$O\left(\text{average\_degree} \cdot \frac{\text{maximum\_degree}}{\text{minimum\_degree}}\right)$$

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- This suffices to inductively obtain a sufficiently good upper-bound on $X_k(e)$

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- Non-regular graphs: can “regularize” on the fly
- For each recursive call, the query complexity is bounded by $O(d)$
- Total: $O(d^2/\operatorname{poly}(\epsilon))$ queries

Our New Algorithm
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Limiting the Exploration of Neighbor Sets

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- Simplest attempt:
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- Problem:
  - An edge can have different numbers assigned at the endpoints
  - This could result in an inconsistent execution of the algorithm
  - Hard to predict results

Our Approach

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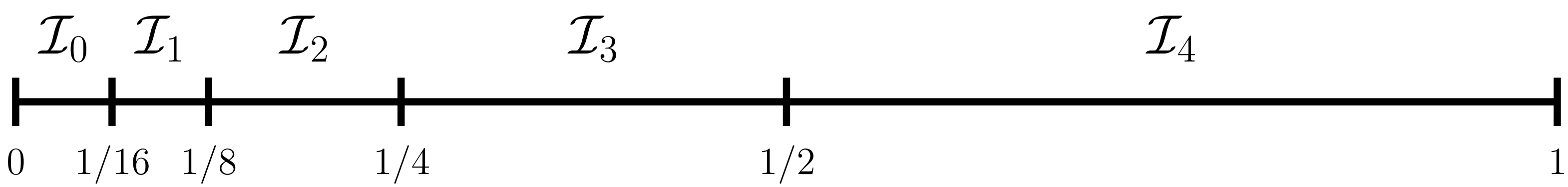
- For each edge (u, w), $D[u]$ ad $D[w]$ may communicate to fix the random number assigned to (u, w).

How we implement this:

- Each $D[v]$ tries to discover only the necessary head of the list
- We partition the range $[0, 1]$ into a logarithmic number of "layers"
- The algorithm discovers edges in the next layer, only if need be

Selecting a Random Number

- Partition $(0, 1]$ into $\Theta(\log n)$ ranges:
  - $(0, 2^{-\log n}]$
  - $(2^{-i}, 2^{-i+1}]$ for $1 \leq i \leq \log n$



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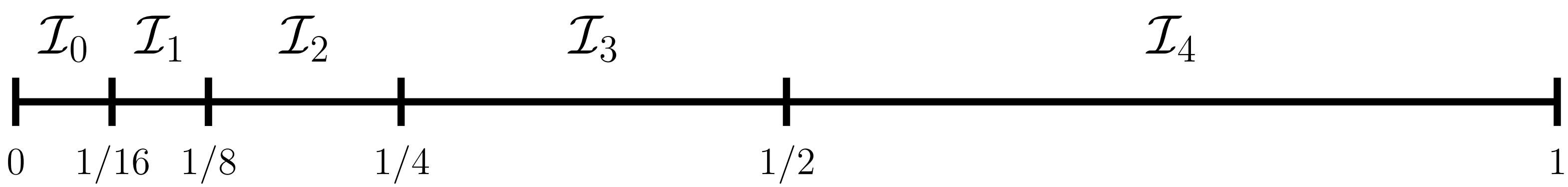

- To assign a random number, consider ranges from left to right:

for $i = 0$ **to** k:
with probability $\frac{|\mathcal{I}_i|}{\sum_{j=i}^{k} |\mathcal{I}_j|}$
return random number in $\mathcal{I}_i$

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- Each iteration of the loop simulated simultaneously for all incident edges

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Extending to the entire graph:

- The same iteration of the loop may be executed by both u and v for an edge (u, v)
- We make sure that the decision made in the first execution is in effect by making $D[u]$ and $D[v]$ talk to each other

Reducing the Query Complexity

How do we reduce the number of queries?

- For an edge (u, v) as long as $D[u]$ and $D[v]$ don't assign a specific number:
  - Their decisions are consistent
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Note: To reduce the running time, quickly select the edges chosen for the currently selected range

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    (see [Nguyen, O. 2008] and [Yoshida, Yamamoto, Ito 2009])

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    Is there a $\mathrm{poly}(d/\epsilon)$-time algorithm?

    (see [Nguyen, O. 2008] and [Yoshida, Yamamoto, Ito 2009])
  - Perhaps not.
    Is there a $\mathrm{poly}(1/\epsilon)$-time algorithm for planar graphs?

    (see [Hassidim, Kelner, Nguyen, O. 2009])

Thank You