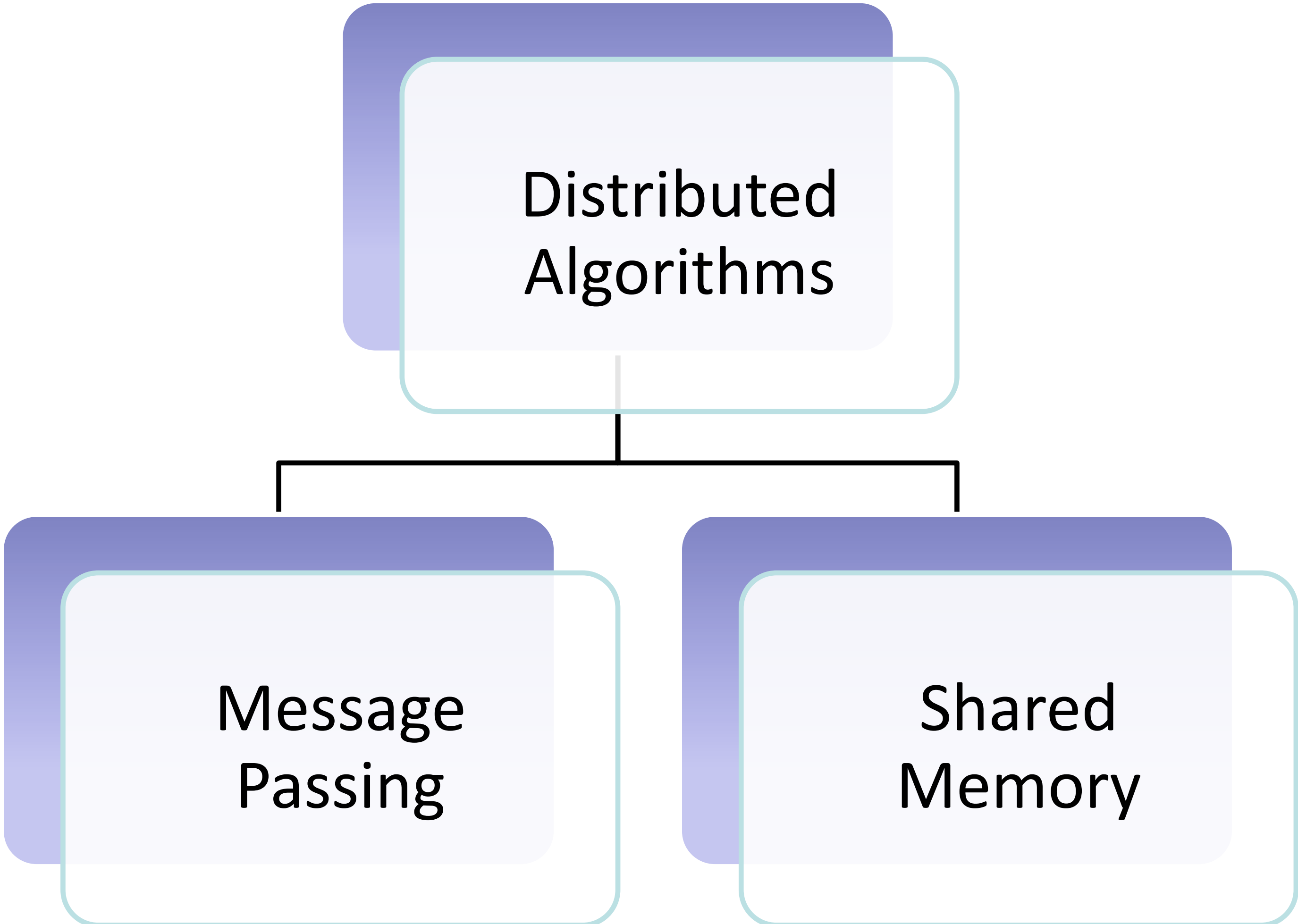
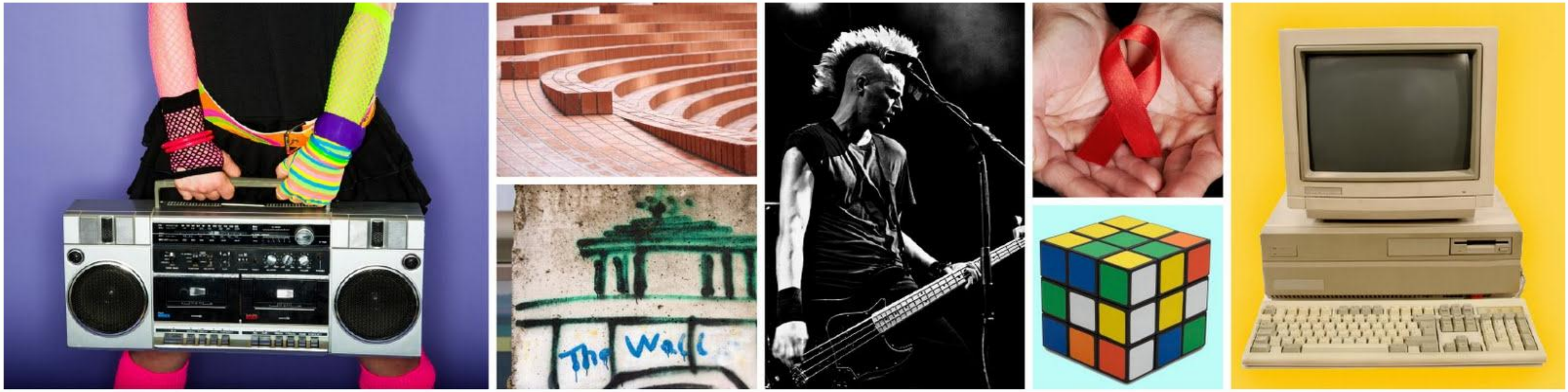
Distributed Algorithms

Tutorial

Roger Wattenhofer

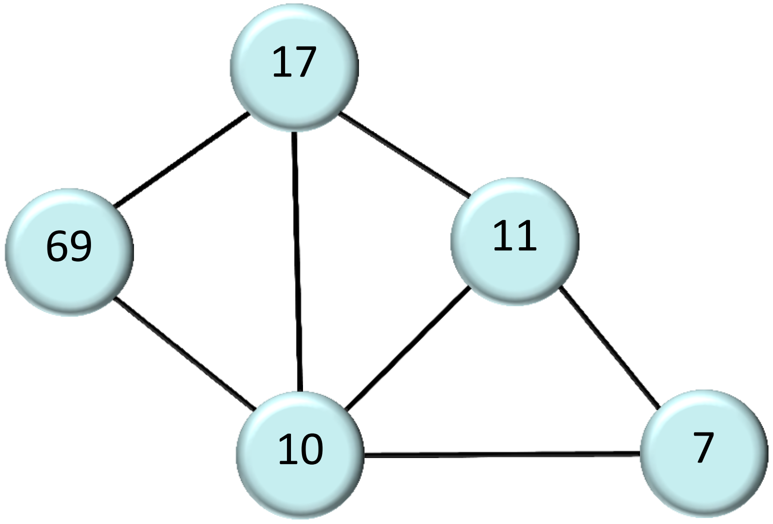

Distributed Algorithms
Message Passing
Shared Memory



THE 1980s

Example: Maximal Independent Set (MIS)

- Given a network with *n* nodes, nodes have unique IDs.
- Find a Maximal Independent Set (MIS)
  - a non-extendable set of pair-wise non-adjacent nodes



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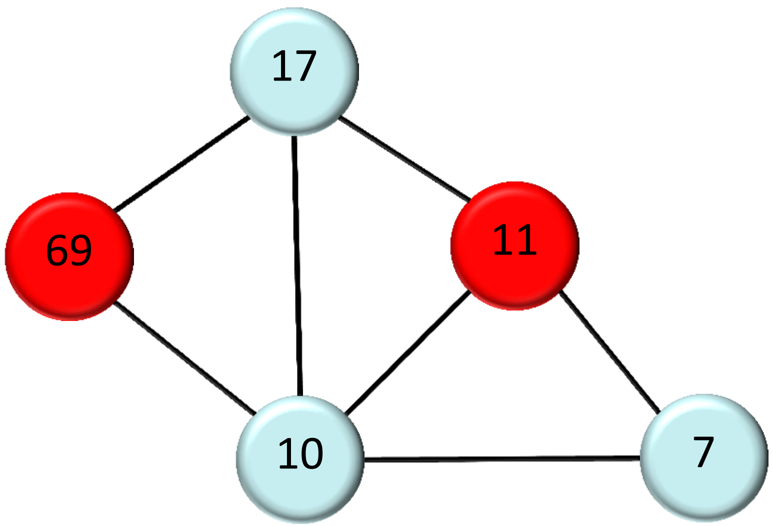

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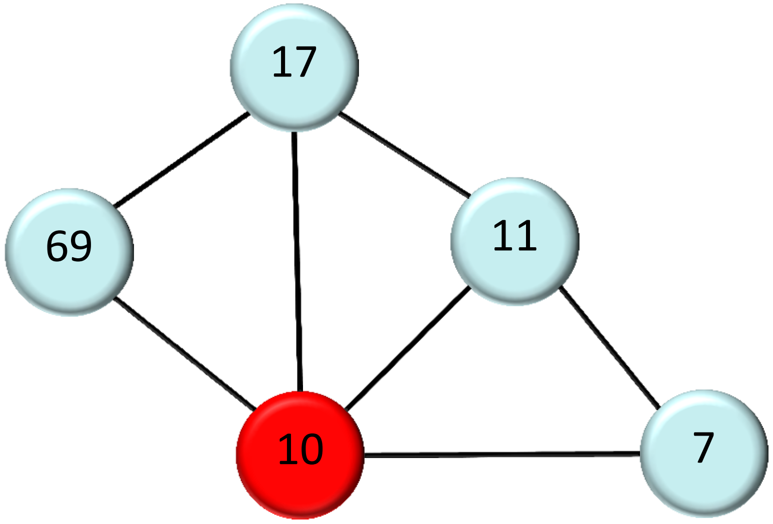

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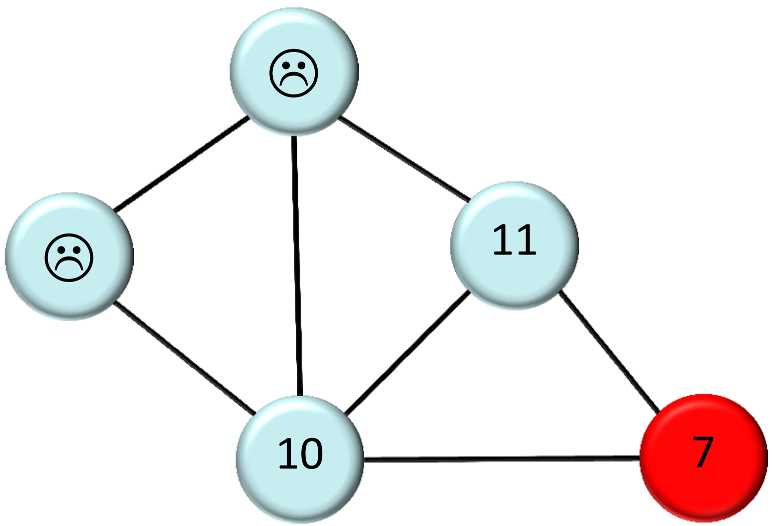

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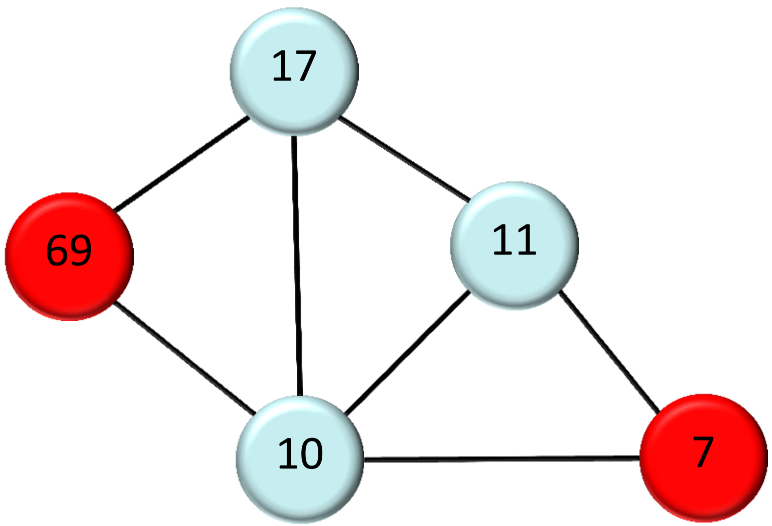

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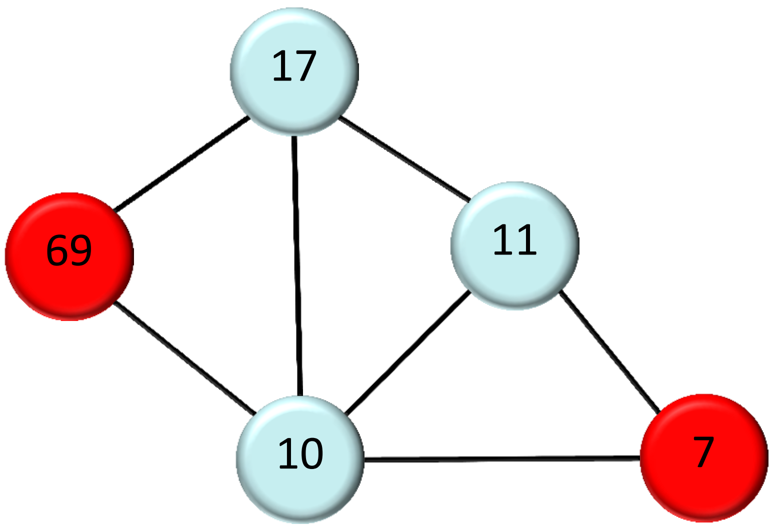
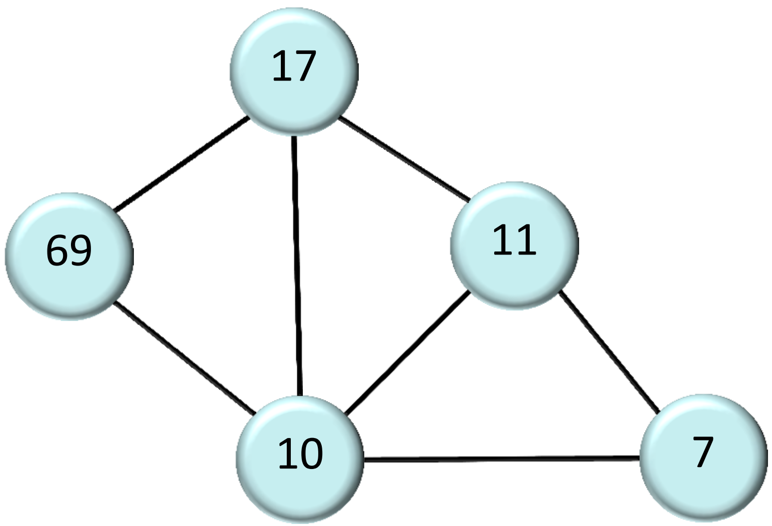


- Traditional (sequential) computation:
  The simple greedy algorithm finds MIS (in linear time)

What about a Distributed Algorithm?

- Nodes are agents with unique ID's that can communicate with neighbors by sending messages. In each synchronous round, every node can send a (different) message to each neighbor.



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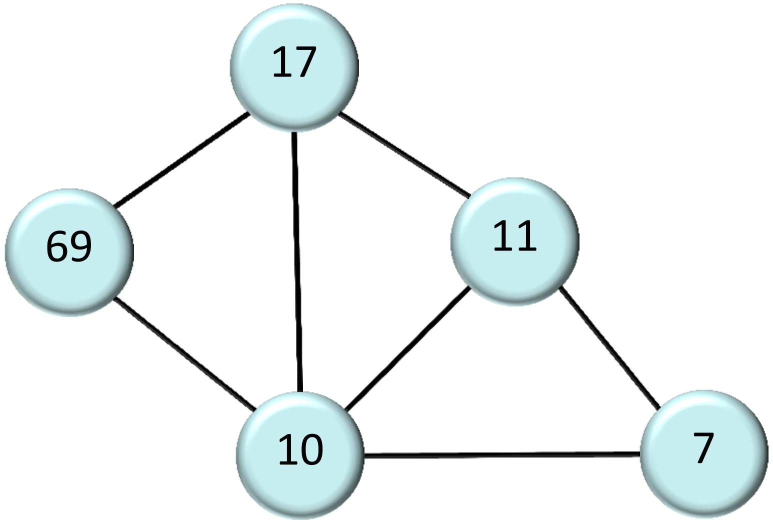
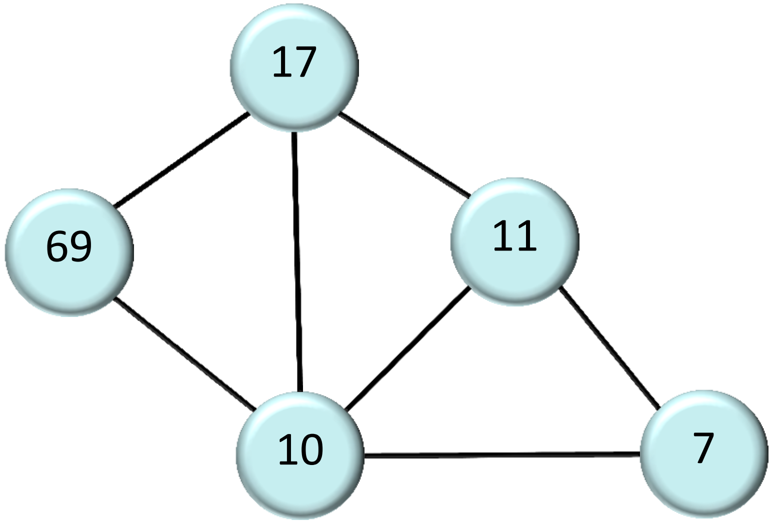



```
each round:
every node:
1. send msgs
2. rcv msgs
3. compute
```

A Simple Distributed Algorithm

- Wait until all neighbors with higher ID decided
- If no higher ID neighbor is in MIS → join MIS

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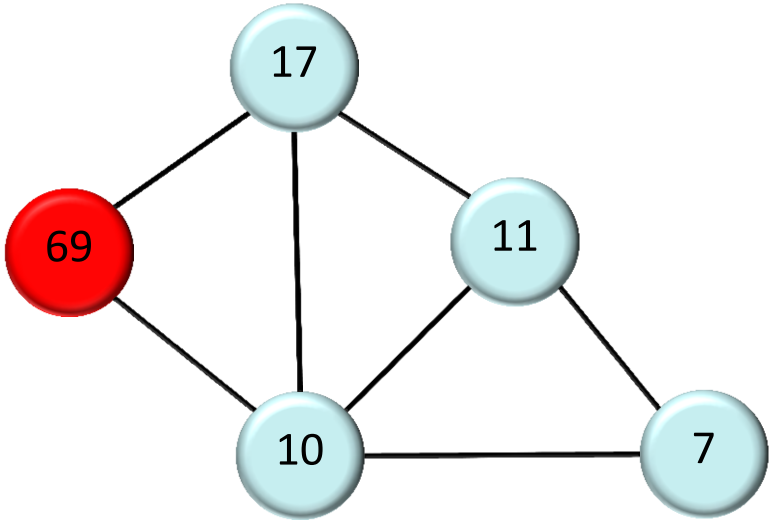


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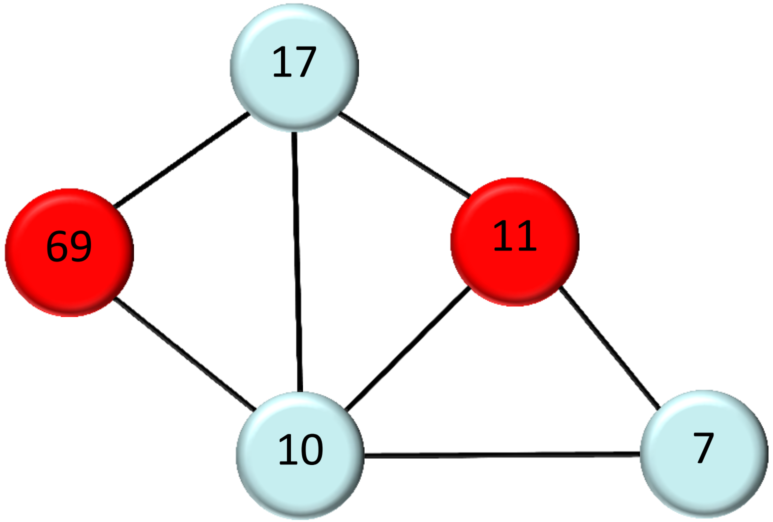


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- What's the problem with this distributed algorithm?

Example

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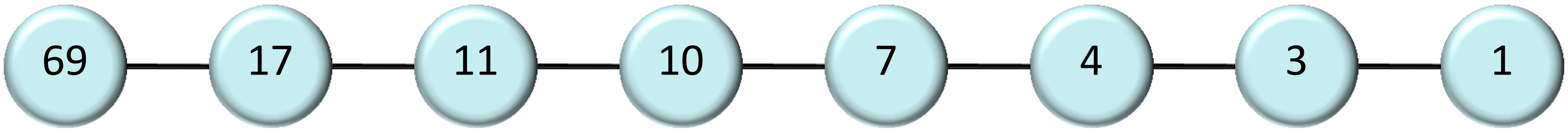

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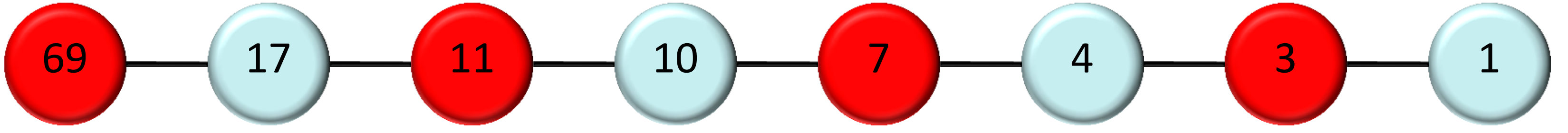

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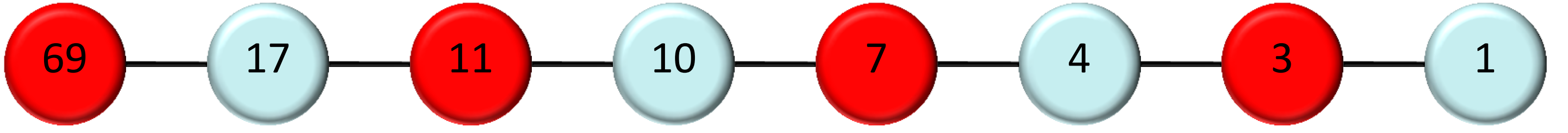
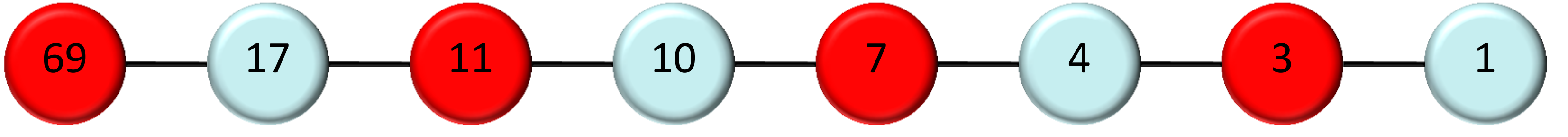


- What if we have minor changes?



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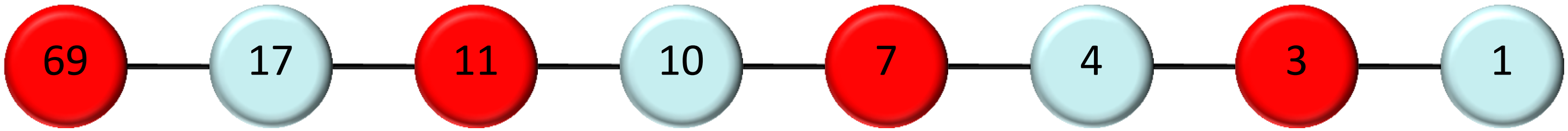

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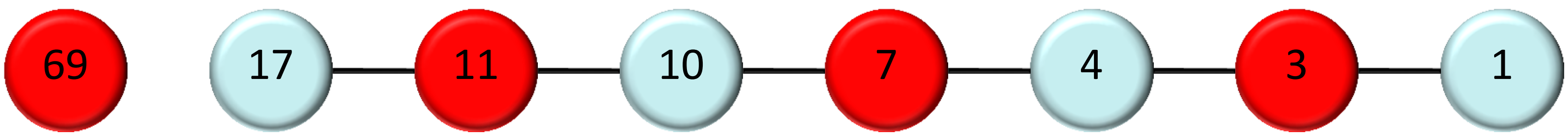

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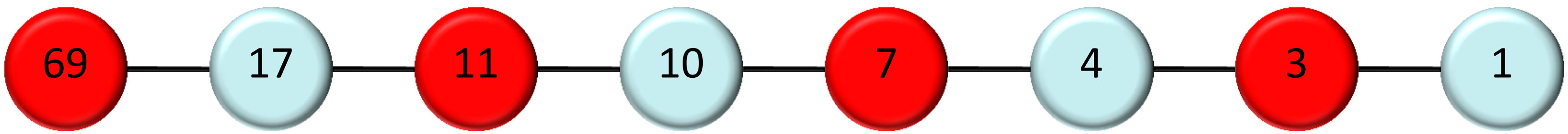

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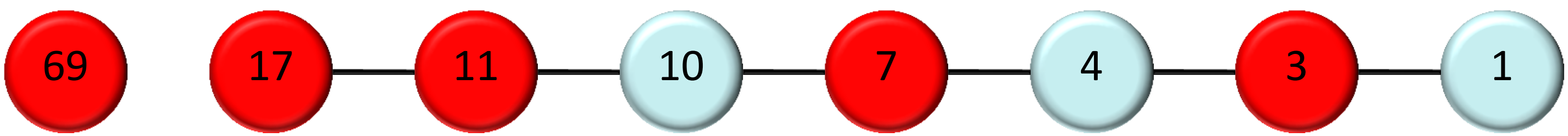

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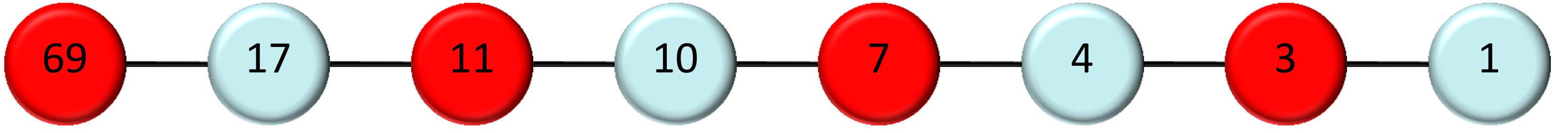

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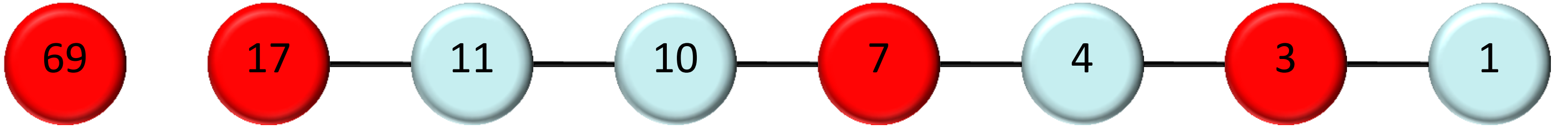

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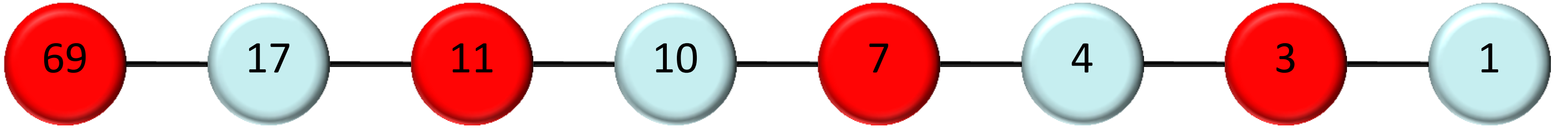

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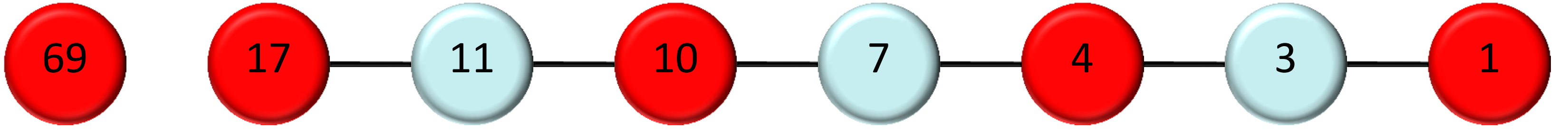

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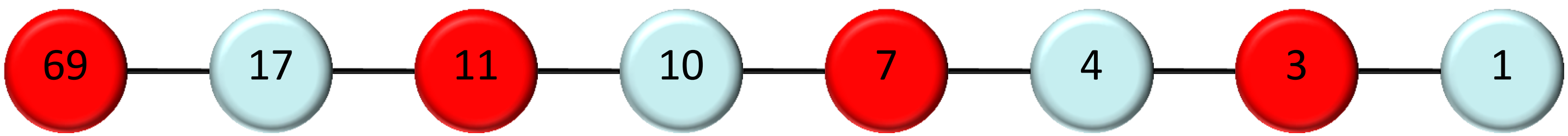

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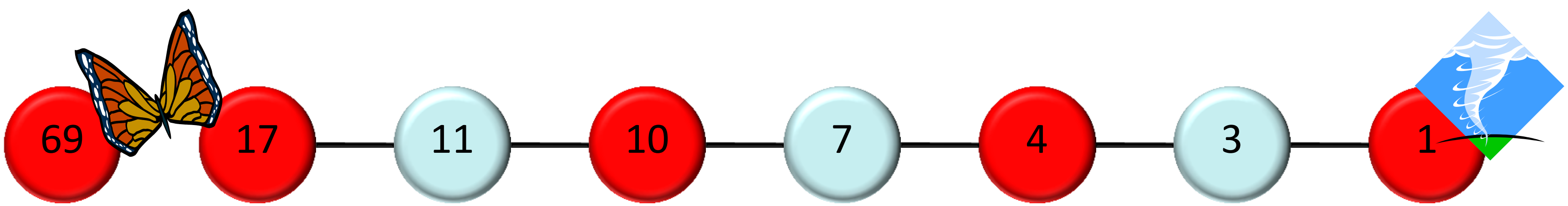
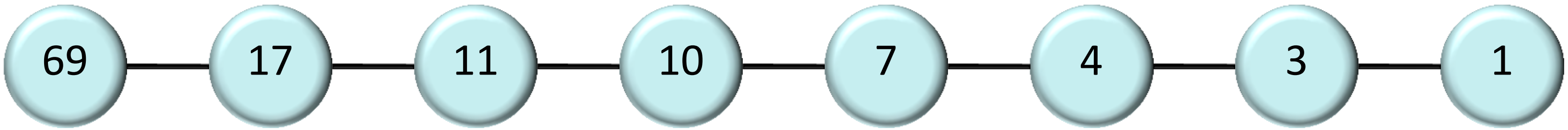
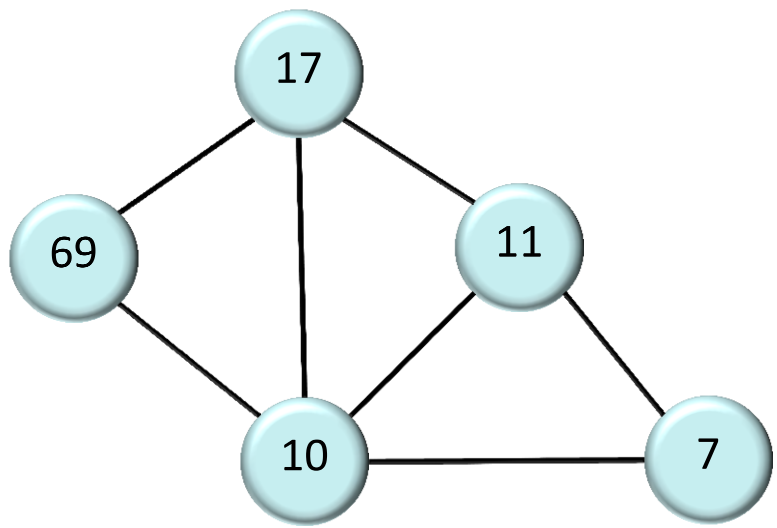


- Proof by animation: In the worst case, the algorithm is slow (linear in the number of nodes). In addition, we have a terrible „butterfly effect“.

What about a Fast Distributed Algorithm?

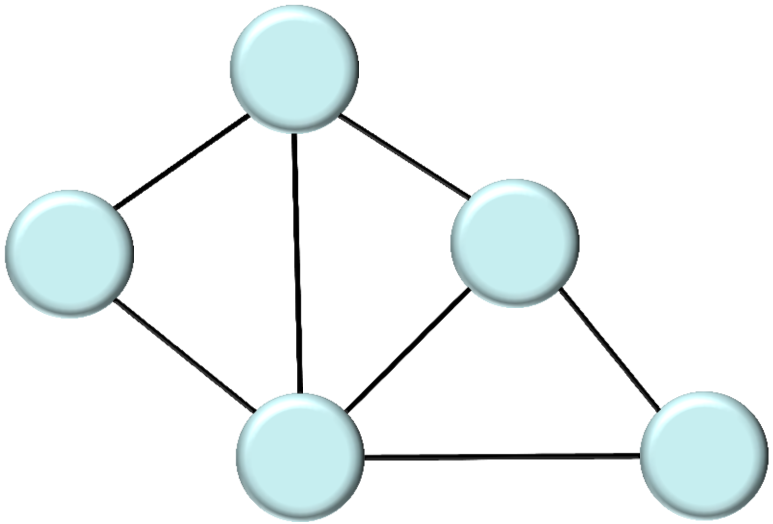
- Can you find a distributed algorithm that is polylogarithmic in the number of nodes *n*, for any graph?





What about a Fast Distributed Algorithm?

- Surprisingly, for deterministic distributed algorithms, this is an Open problem!
- However, randomization helps! In each synchronous round, nodes should choose a random value. If your value is larger than the value of your neighbors, join MIS!



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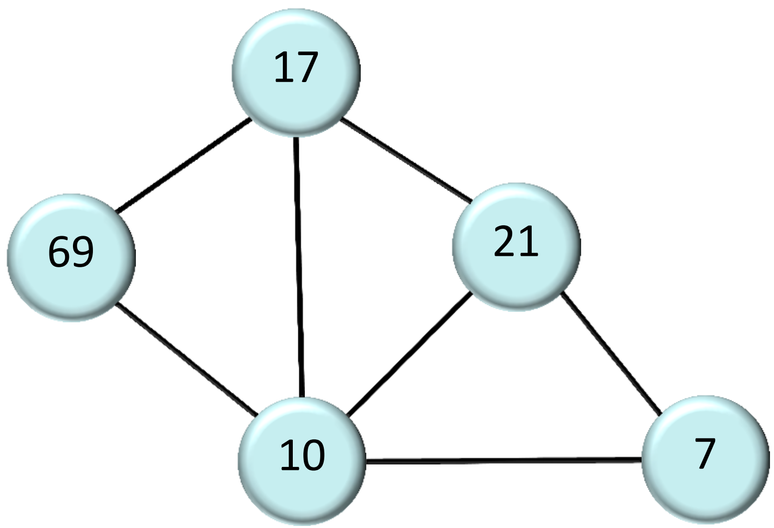

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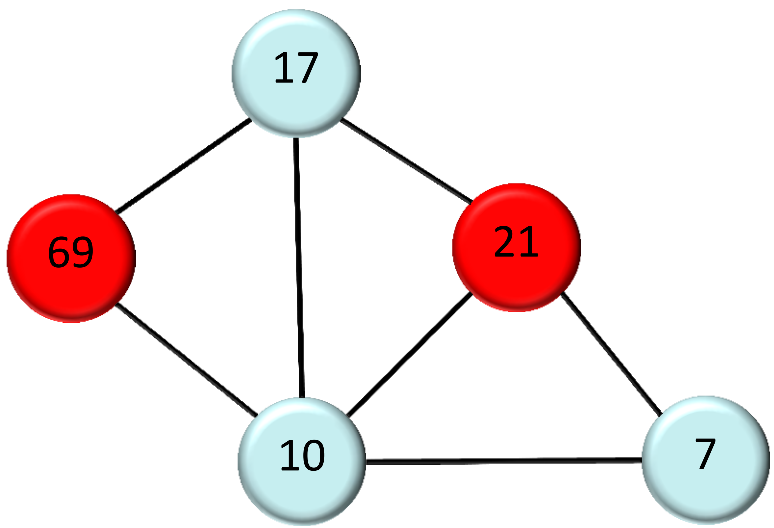

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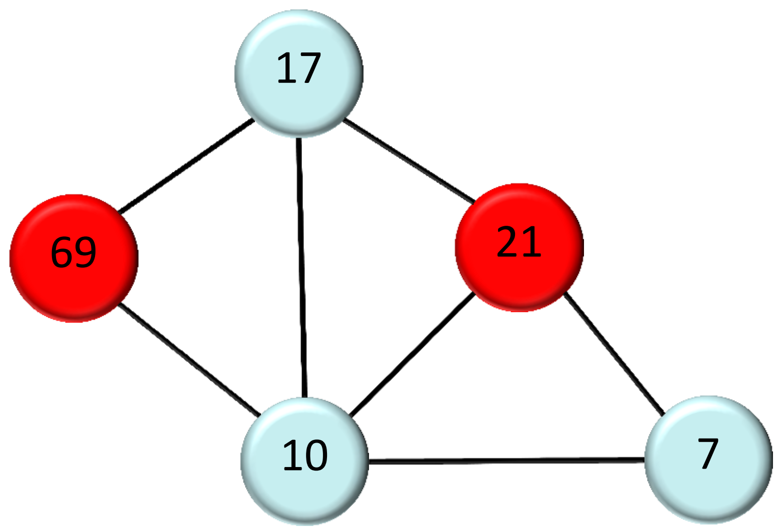
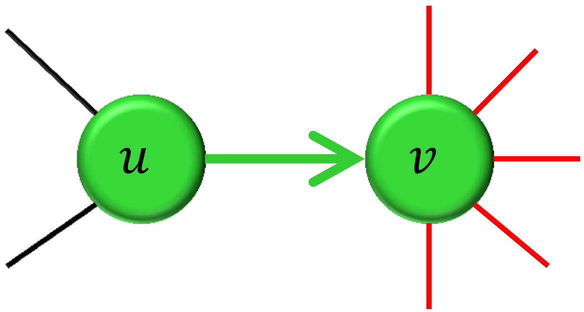


- How many synchronous rounds does this take in expectation (or whp)?

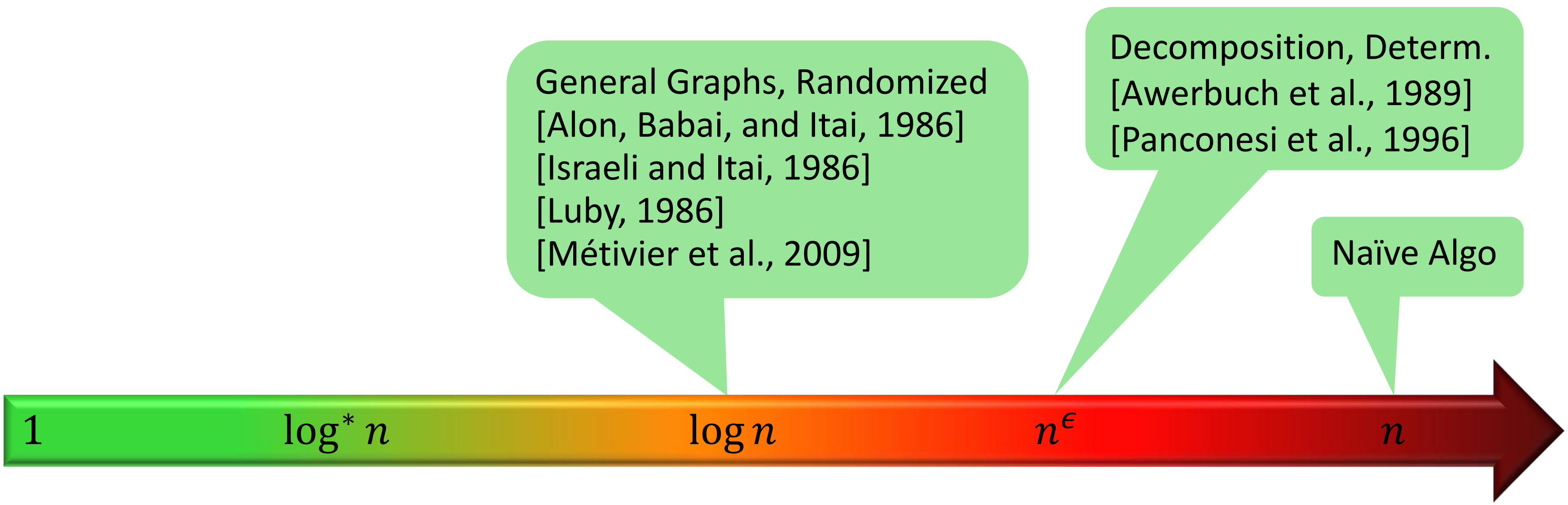
Analysis

- Event $(u \to v)$: node u got largest random value in combined neighborhood $N_u \cup N_v$.
- We only count edges of v as deleted.



- Similarly event $(v \to u)$ deletes edges of u.
- We only double-counted edges.
- Using linearity of expectation, in expectation at least half of the edges are removed in each round.
- In other words, whp it takes $O(\log n)$ rounds to compute an MIS.

Results: MIS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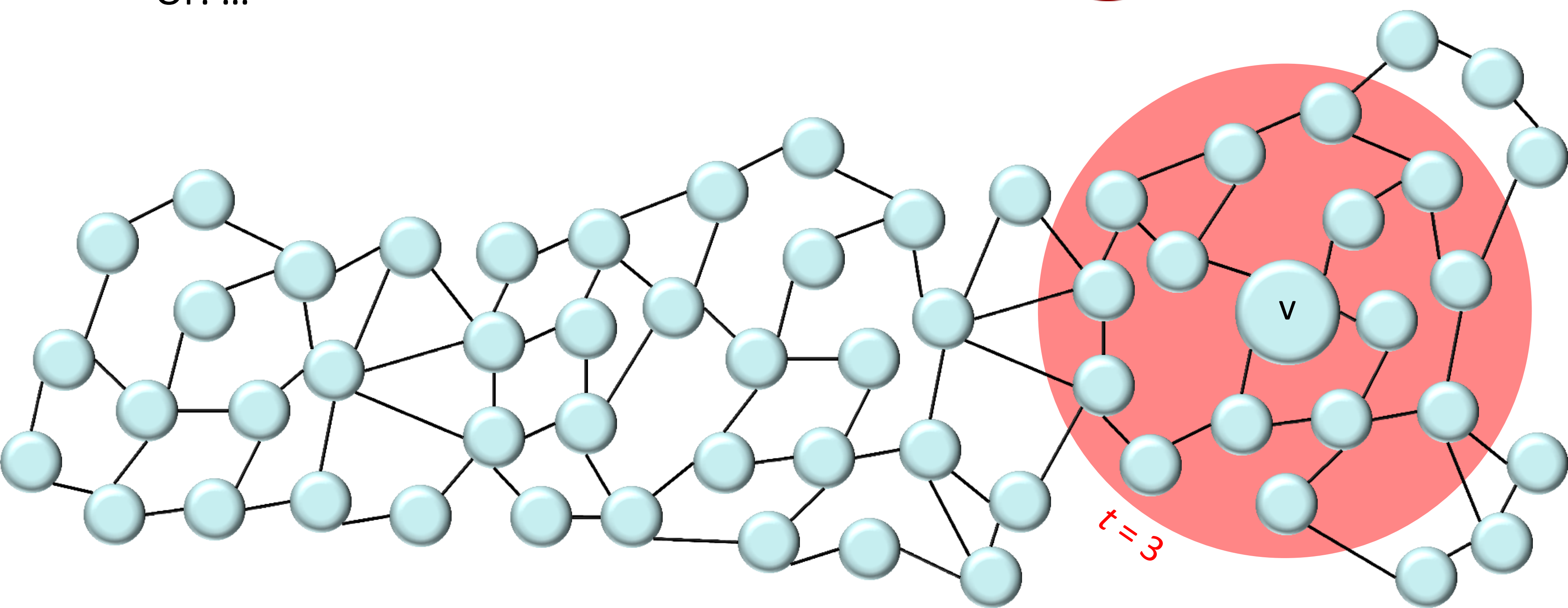
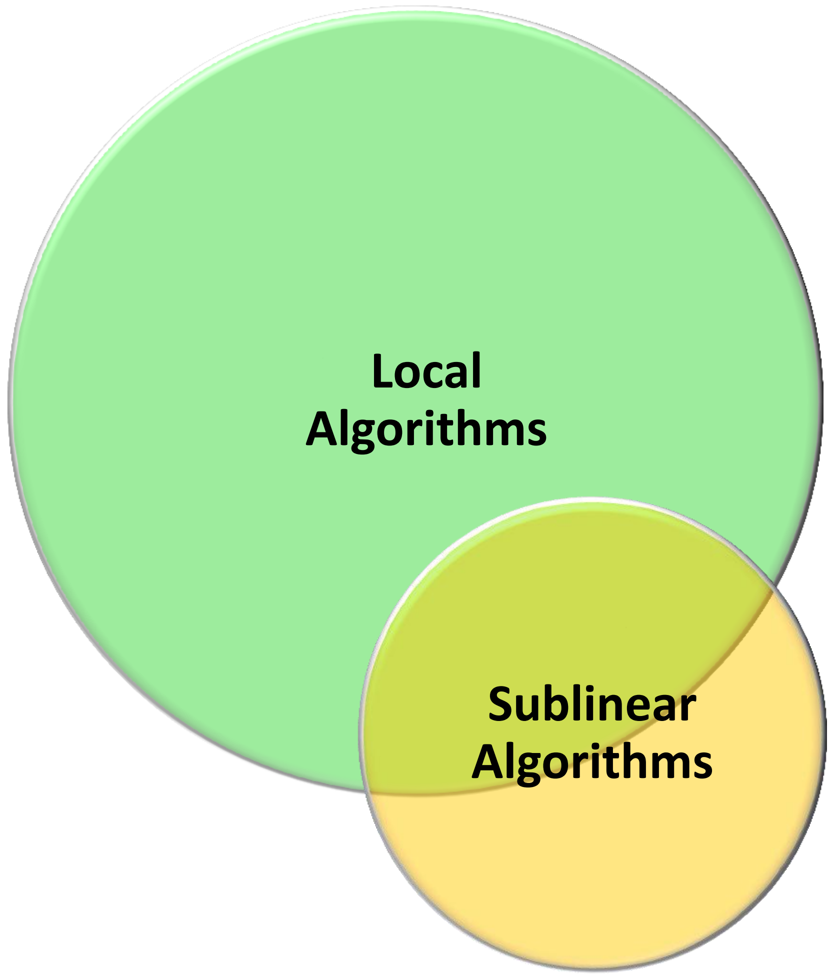
Local Algorithms

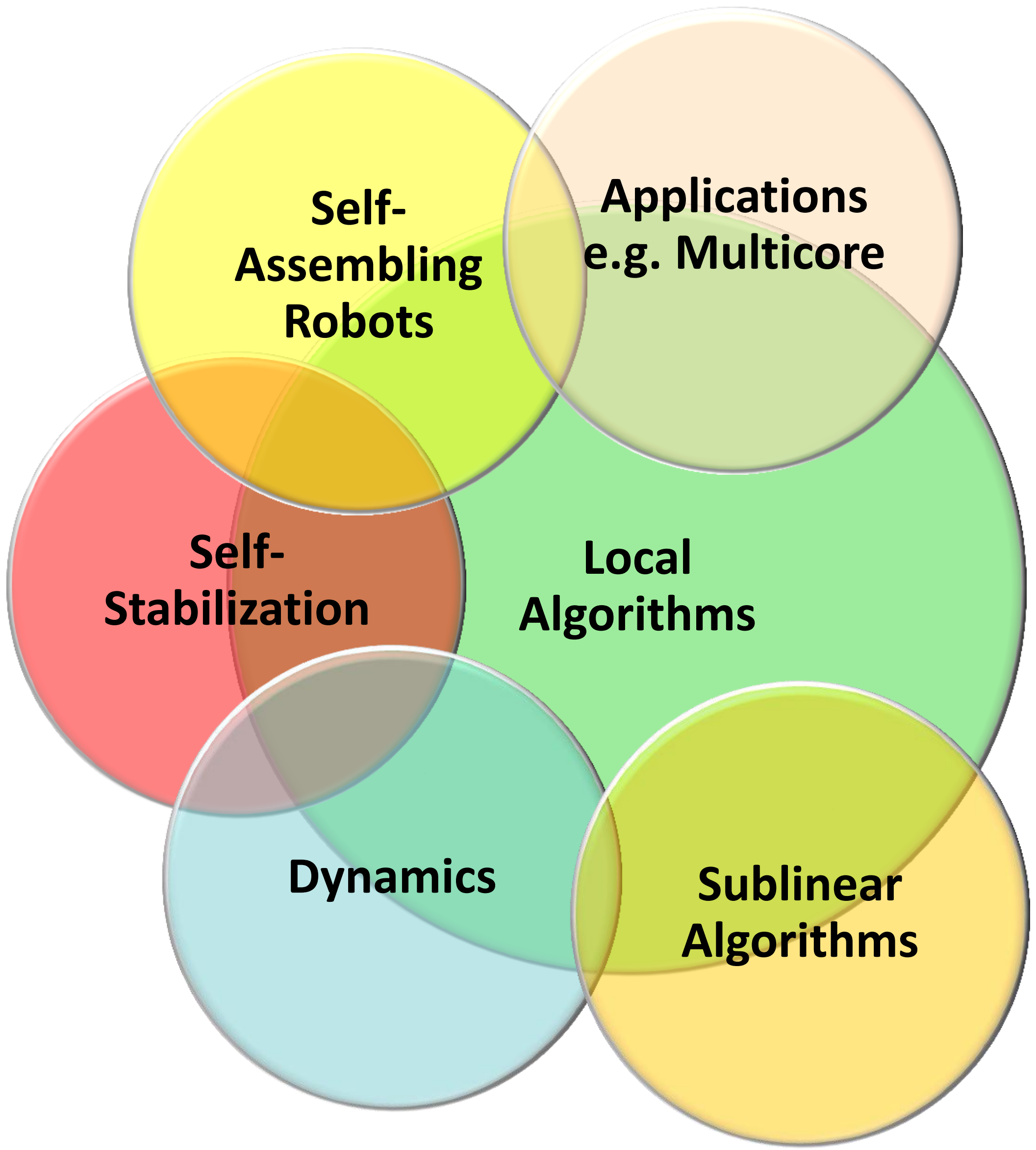
- Each node can exchange a message with all neighbors, for *t* communication rounds, and must then decide.
- Or: Given a graph, each node must determine its decision as a function of the information available within radius *t* of the node.
- Or: Change can only affect nodes up to distance *t*. 
- Or: …



Locality



Locality is Everywhere!



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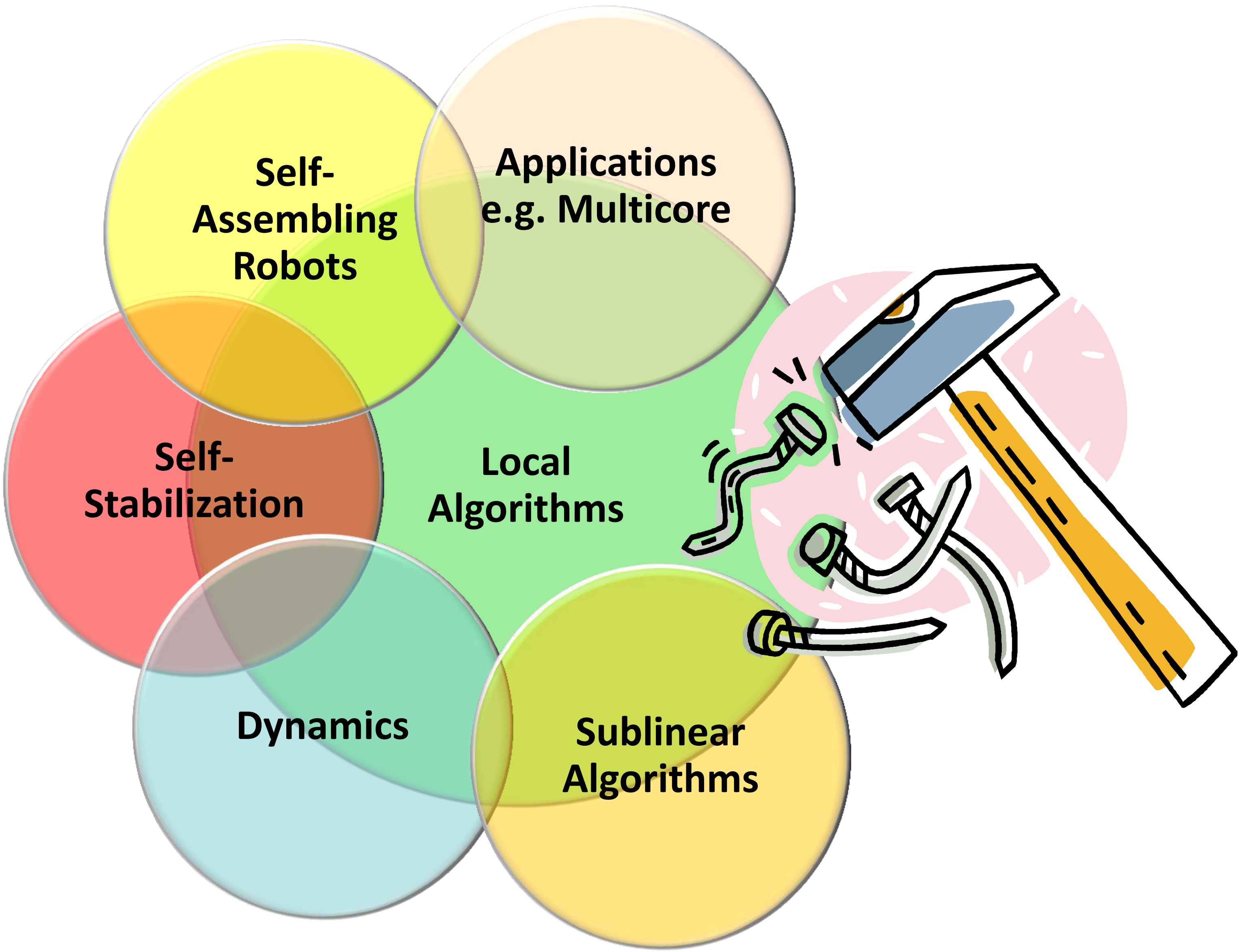
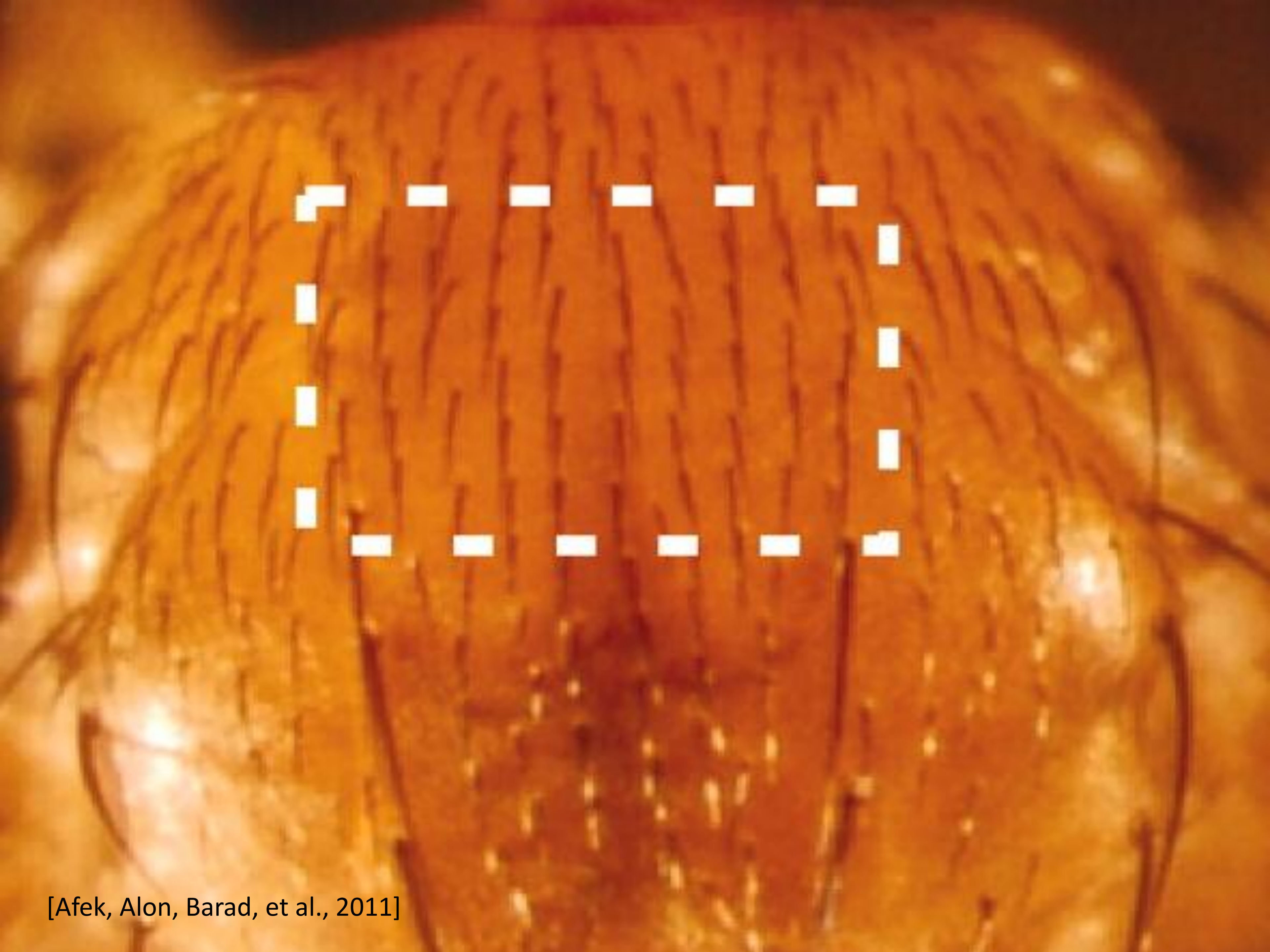
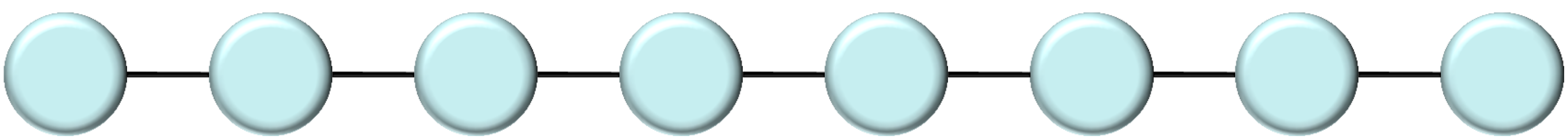



[Afek, Alon, Barad, et al., 2011]

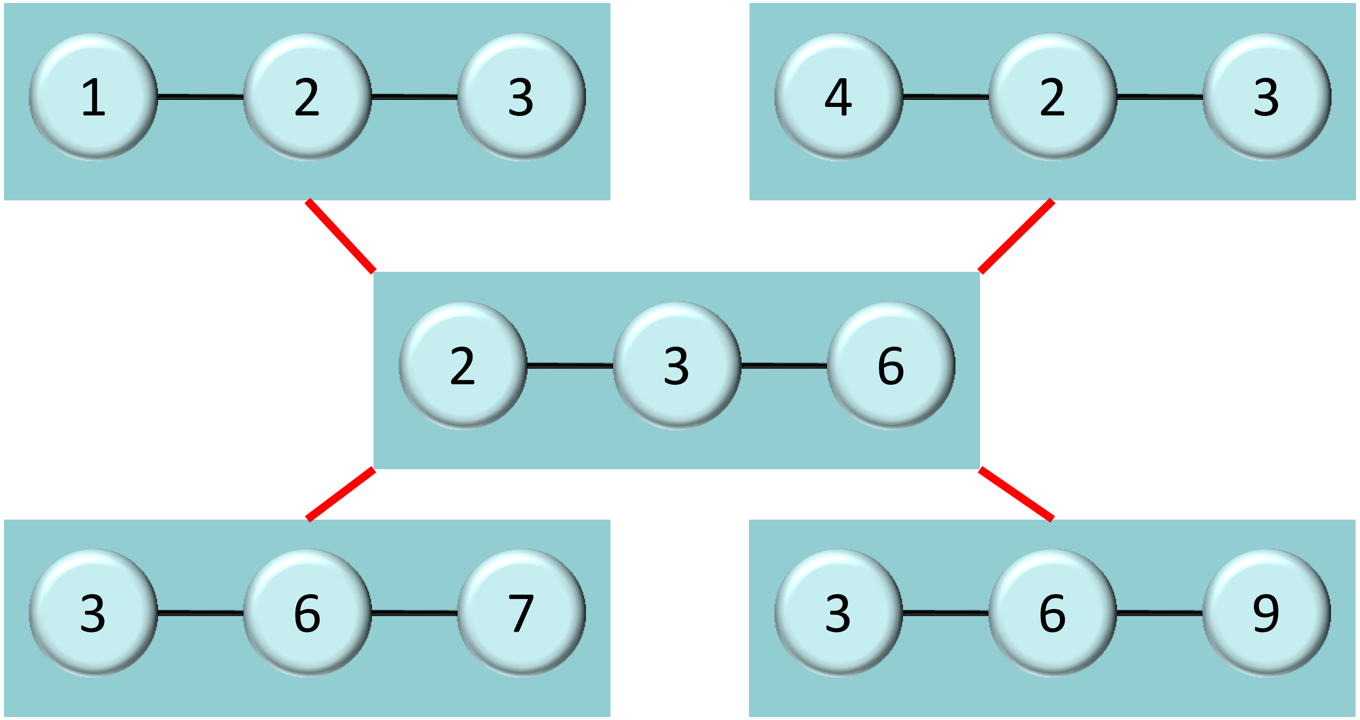
What about an Even Faster Distributed Algorithm?

- Since the 1980s, nobody was able to improve this simple algorithm.
- What about lower bounds?
- There is an interesting lower bound, essentially using a Ramsey theory argument, that proves that an MIS needs at least $\Omega(\log^* n)$ time.
  - log* is the so-called iterated logarithm – how often you need to take the logarithm until you end up with a value smaller than 1.
  - This lower bound already works on simple networks such as the linked list



Coloring Lower Bound on Oriented Ring

- Build graph G_t, where nodes are possible views of nodes for distributed algorithms of time t. Connect views that could be neighbors in ring.
- Here is for instance of G_1:



- Chromatic number of G_t is exactly minimum possible colors in time t.

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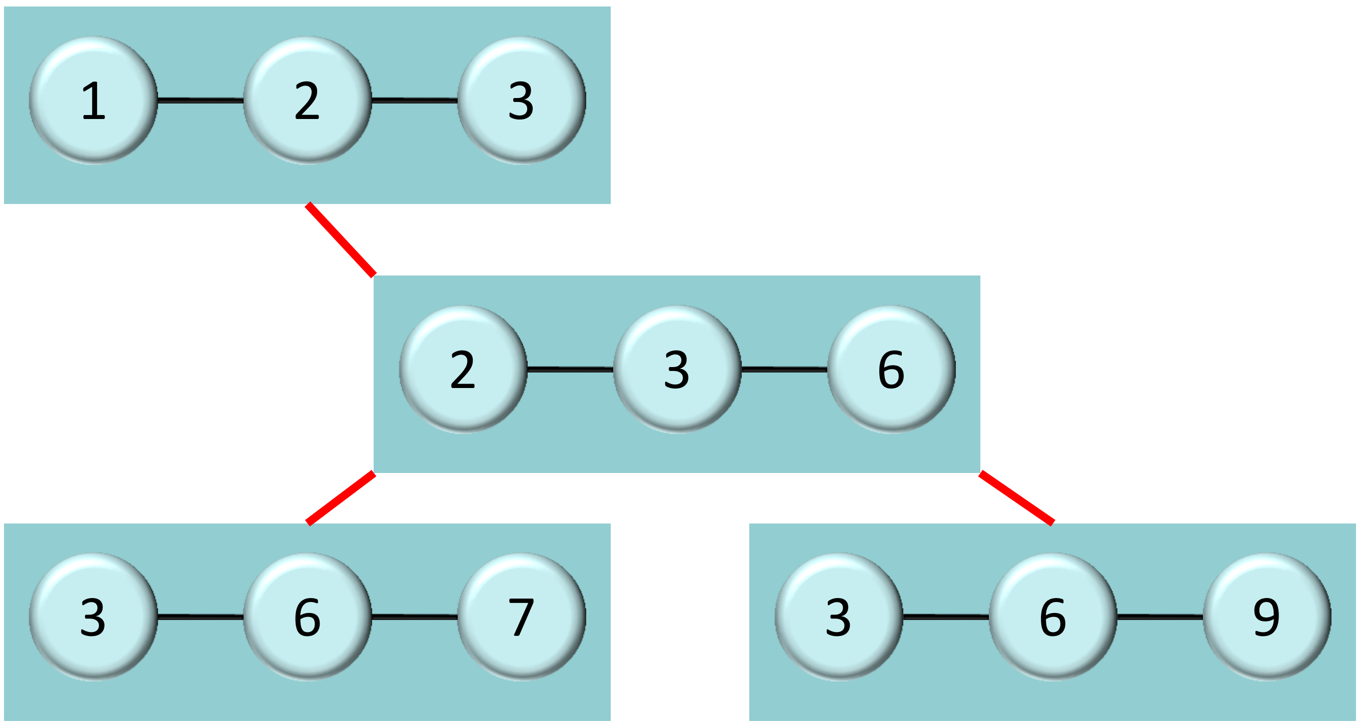

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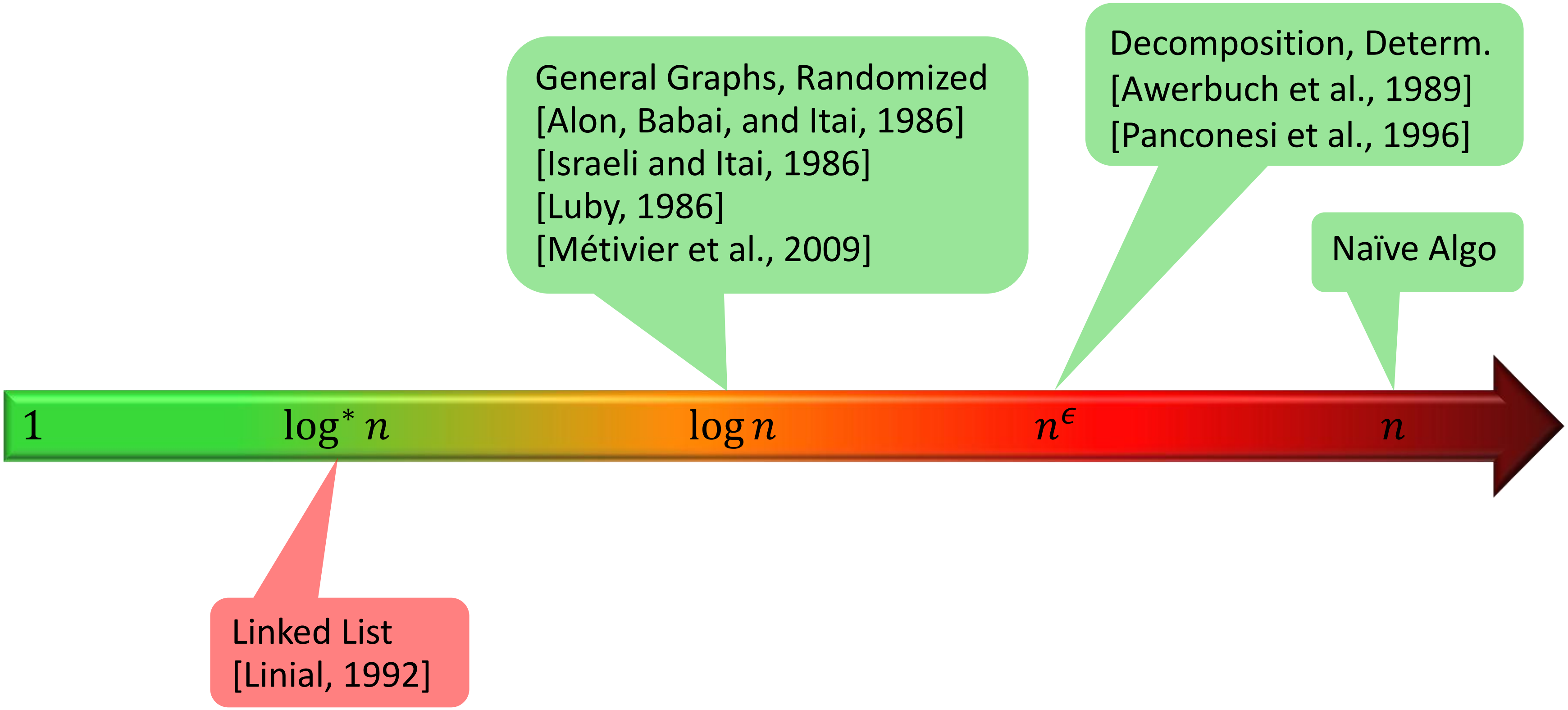
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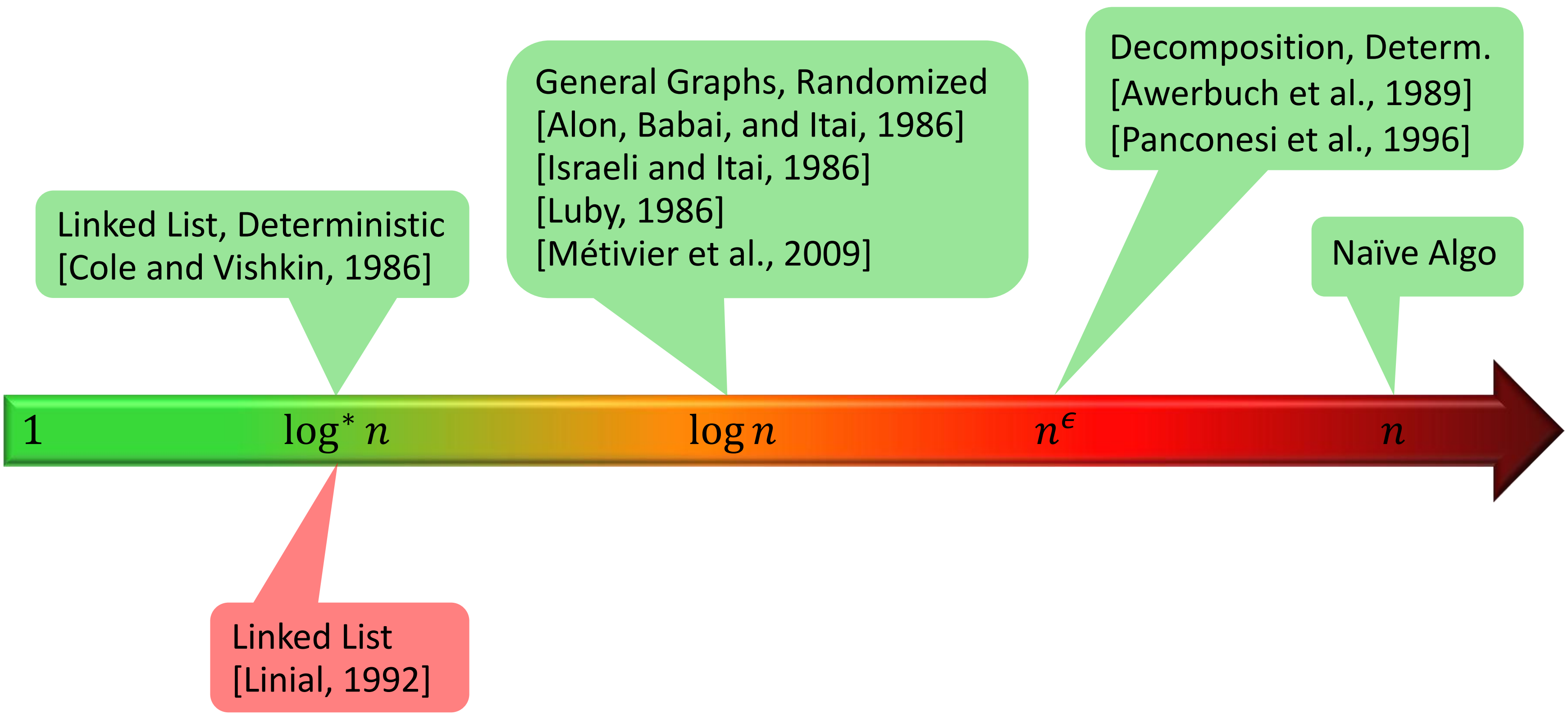
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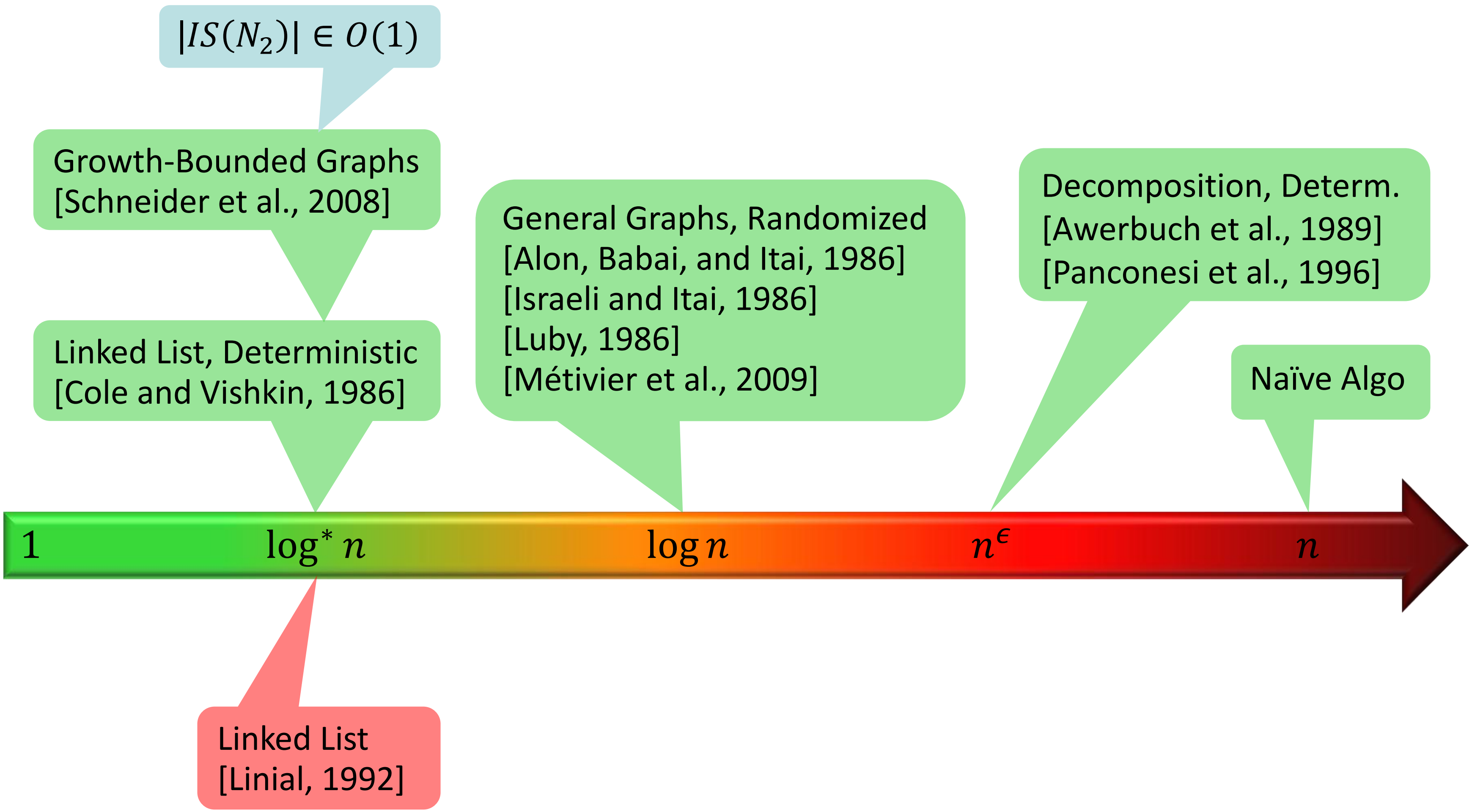

$|IS(N_2)| \in O(1)$
Growth-Bounded Graphs
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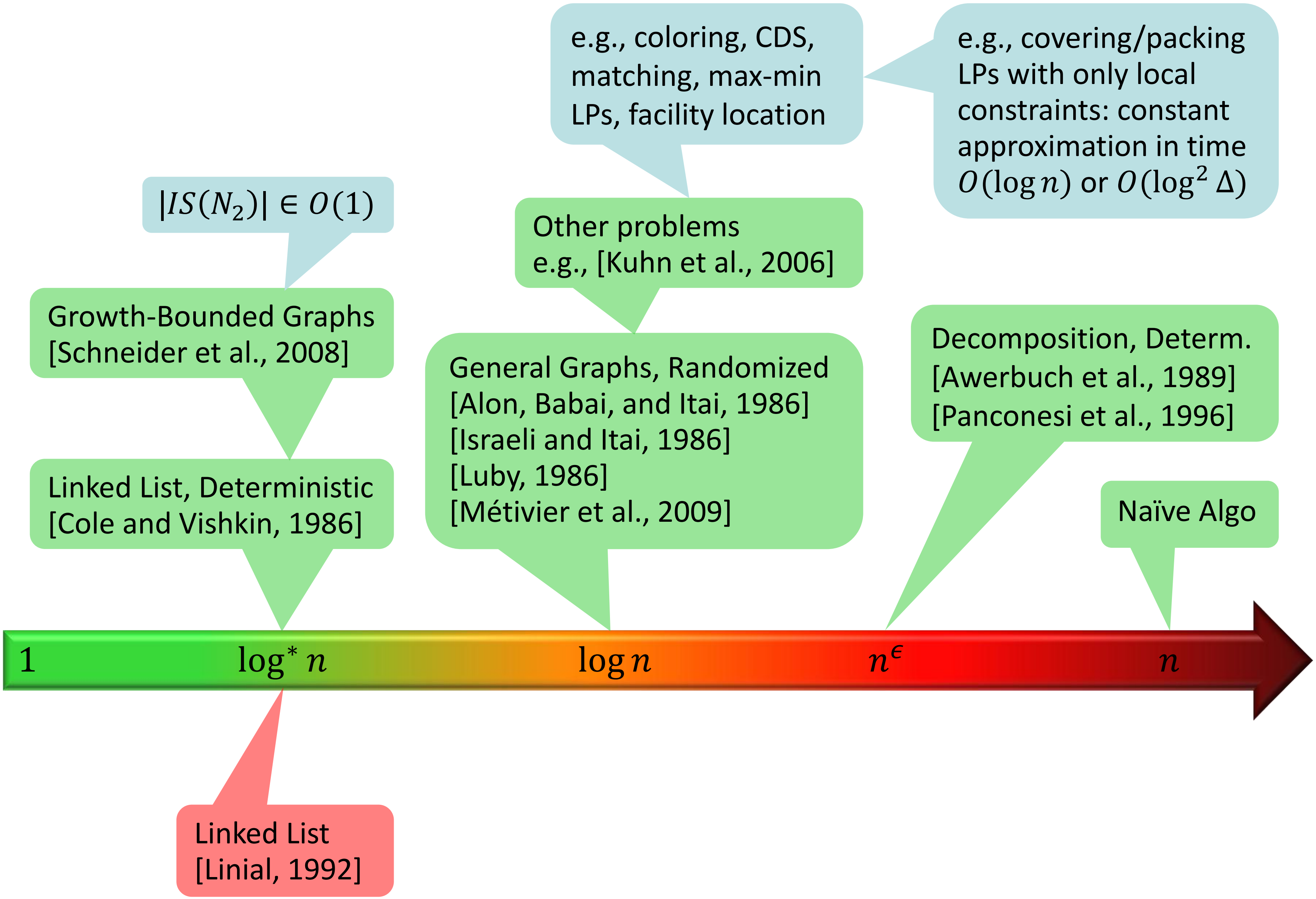

e.g., coloring, CDS, matching, max-min LPs, facility location
e.g., covering/packing LPs with only local constraints: constant approximation in time $O(\log n)$ or $O(\log^2 \Delta)$
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Other problems e.g., [Kuhn et al., 2006]
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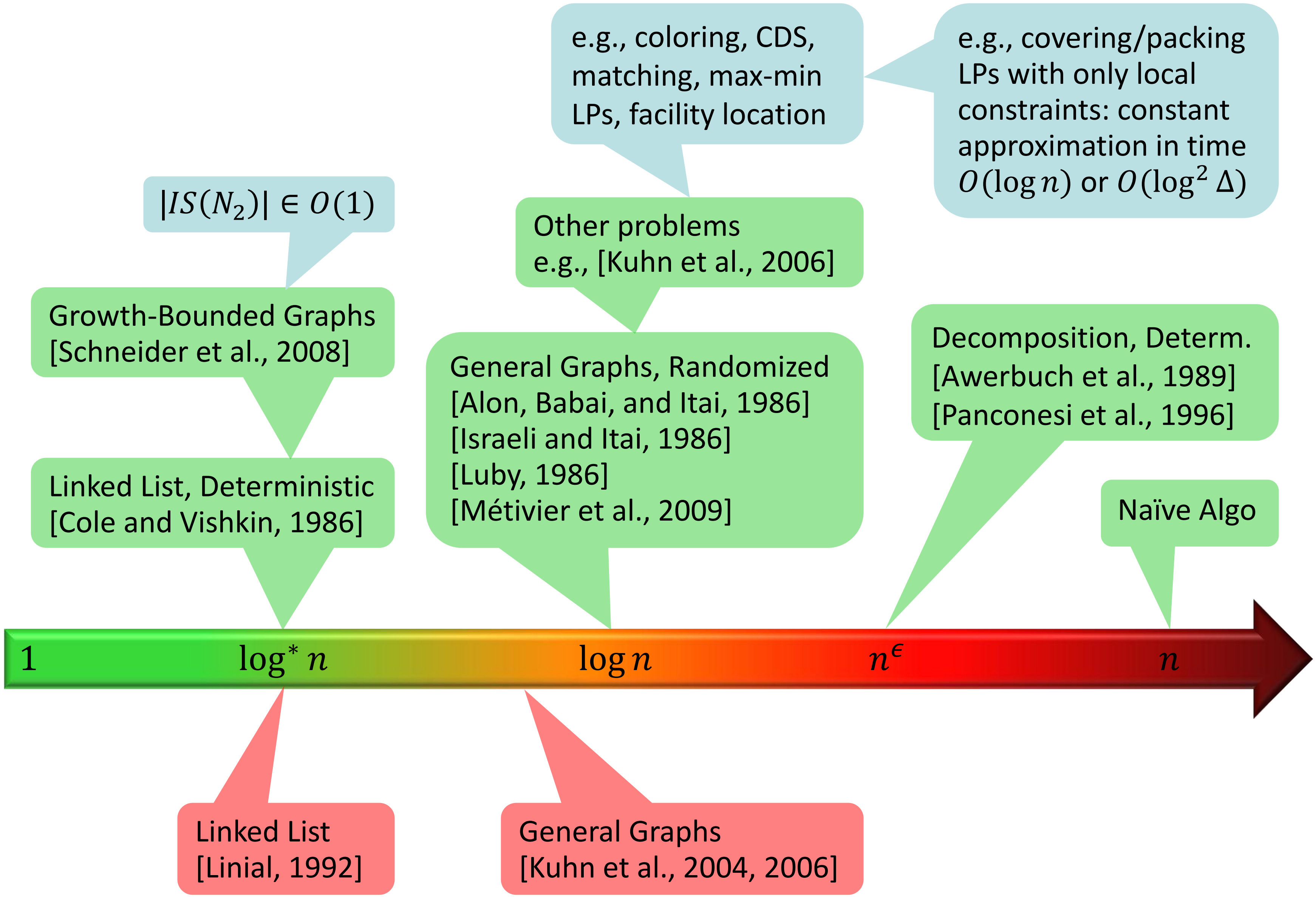

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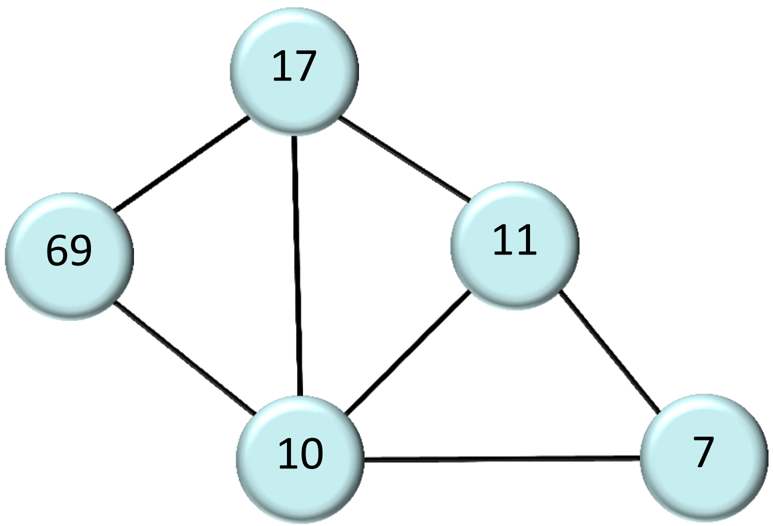
General Graphs [Kuhn et al., 2004, 2006]

Example: Minimum Vertex Cover (MVC)

- Given a network with *n* nodes, nodes have unique IDs.
- Find a Minimum Vertex Cover (MVC)
  - a minimum set of nodes such that all edges are adjacent to node in MVC



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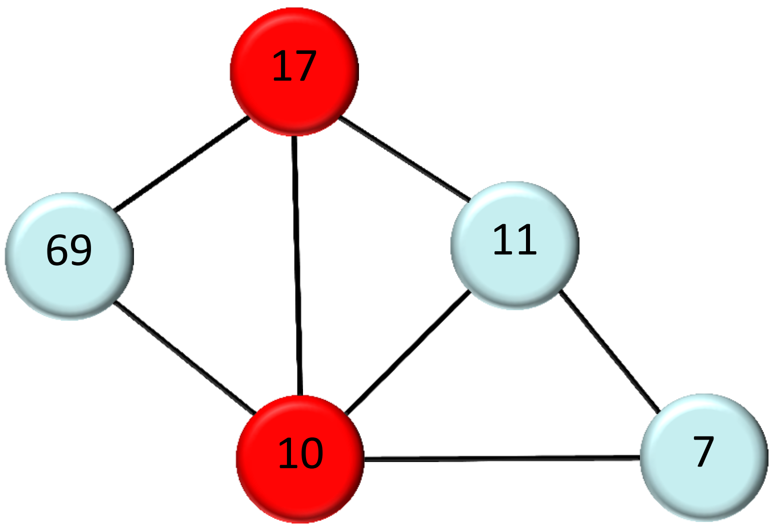
- Given a network with *n* nodes, nodes have unique IDs.
- Find a Minimum Vertex Cover (MVC)
  - a minimum set of nodes such that all edges are adjacent to node in MVC



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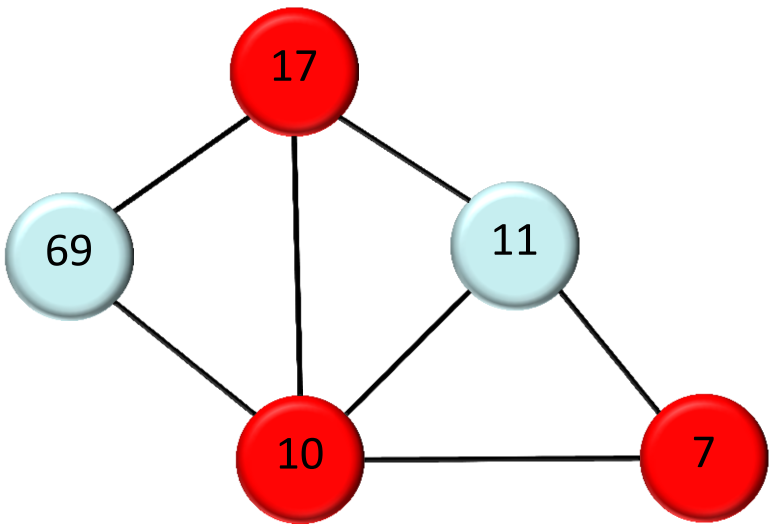
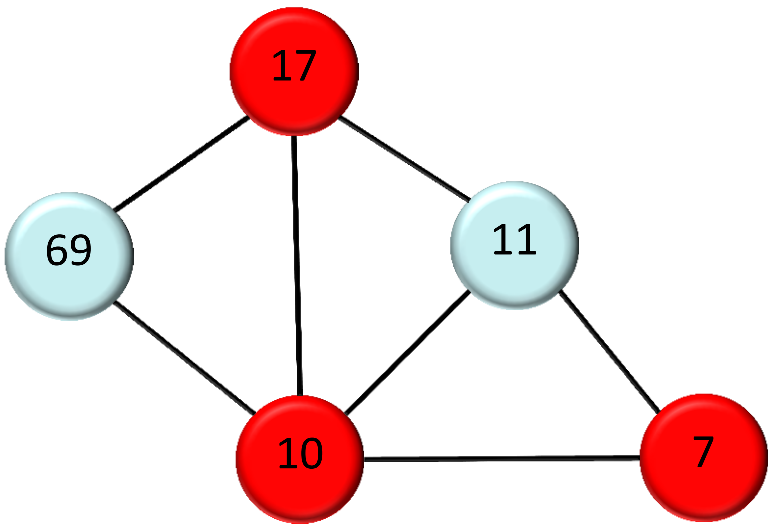


Differences between MIS and MVC

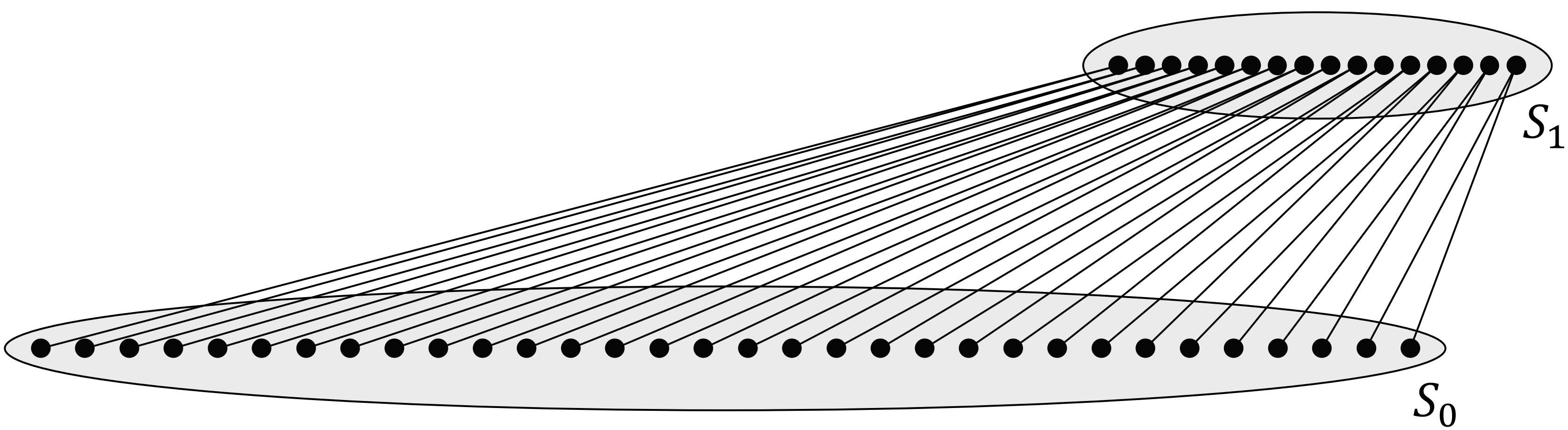
- Central (non-local) algorithms: MIS is trivial, whereas MVC is NP-hard
- Instead: Find an MVC that is “close” to minimum (approximation)
- Trade-off between time complexity and approximation ratio



- MVC: Various simple (non-distributed) 2-approximations exist!
- What about distributed algorithms?!?

Finding the MVC (by Distributed Algorithm)

- Given the following bipartite graph with $|S_0| = \delta\ |S_1|$
- The MVC is just all the nodes in S_1
- Distributed Algorithm…



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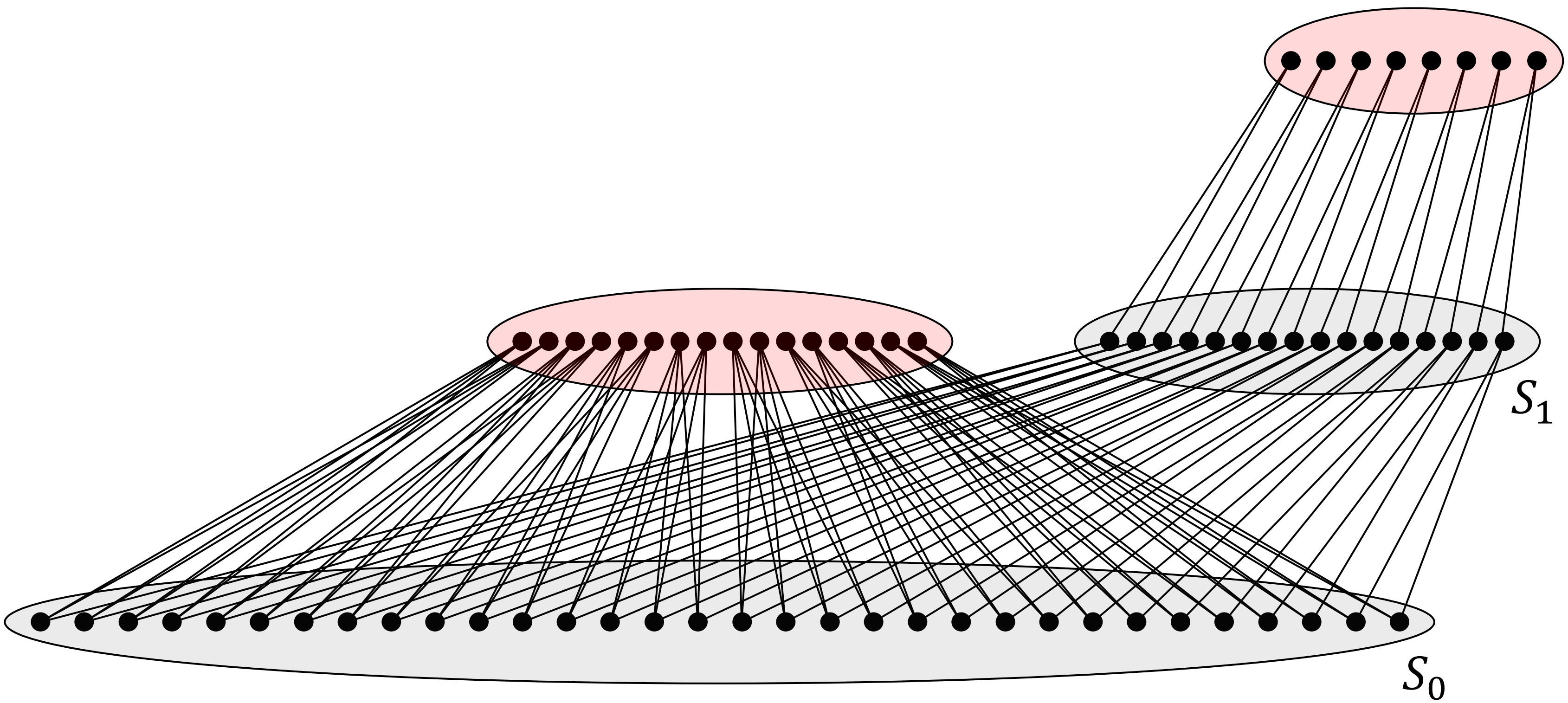

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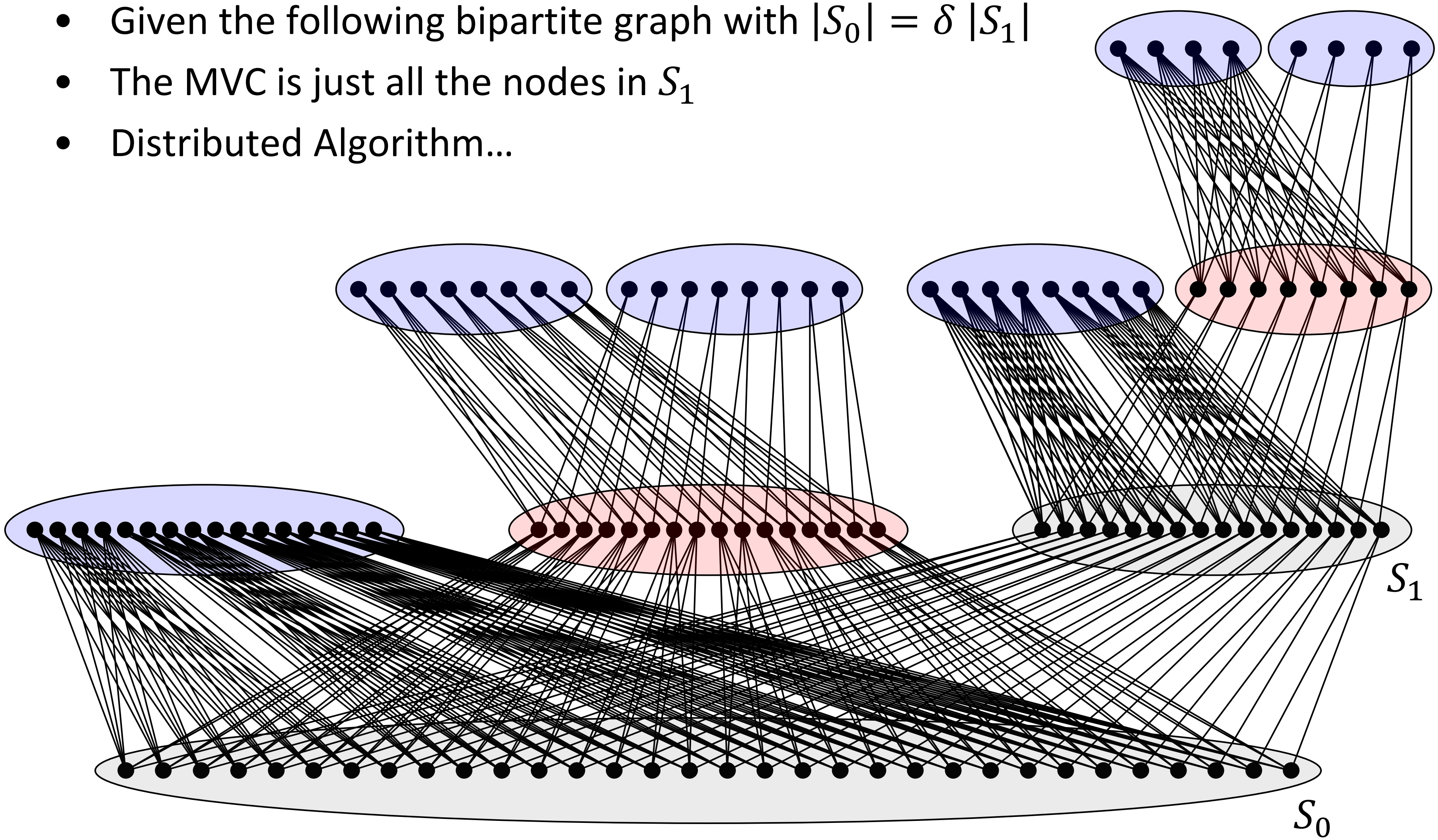
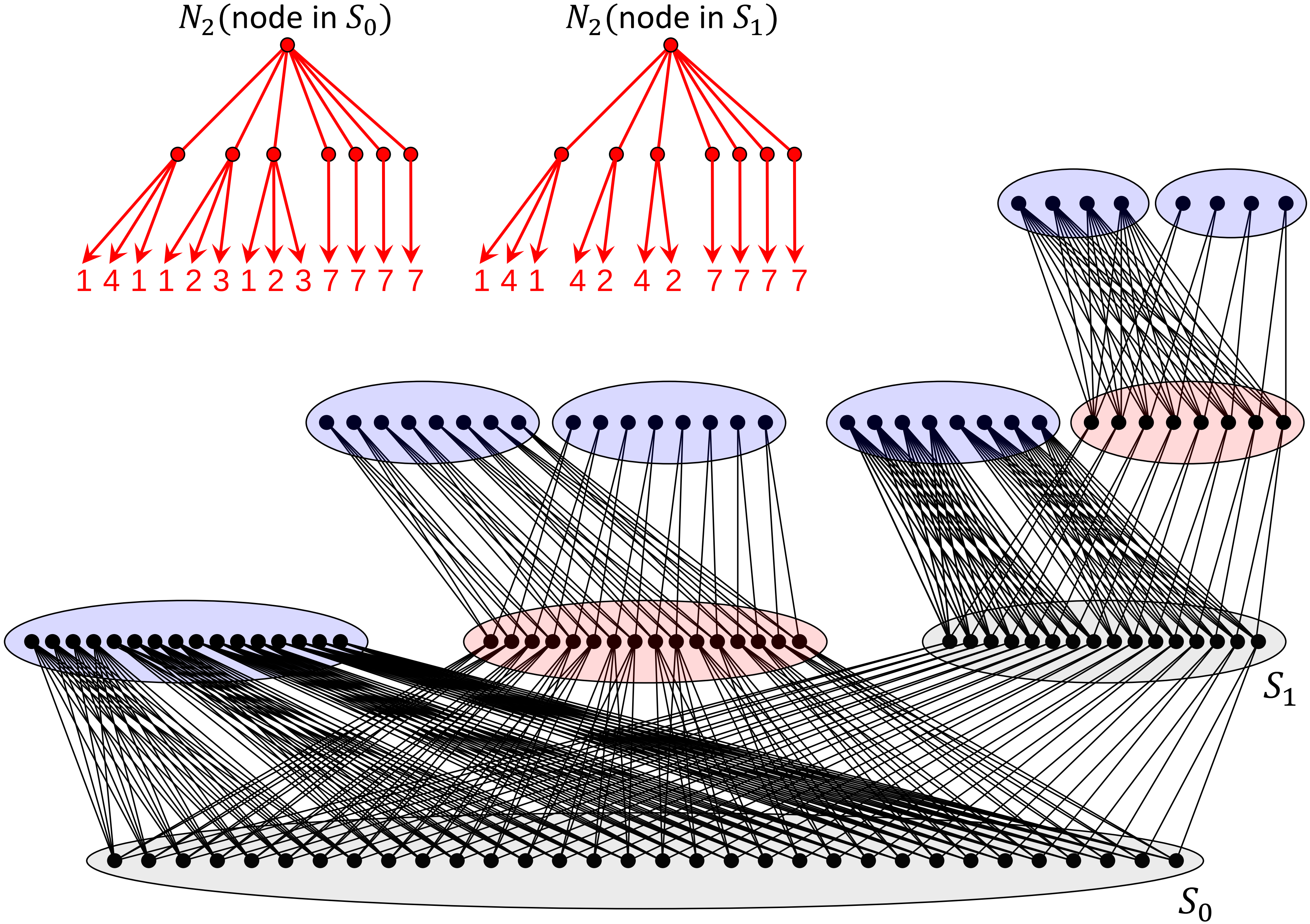
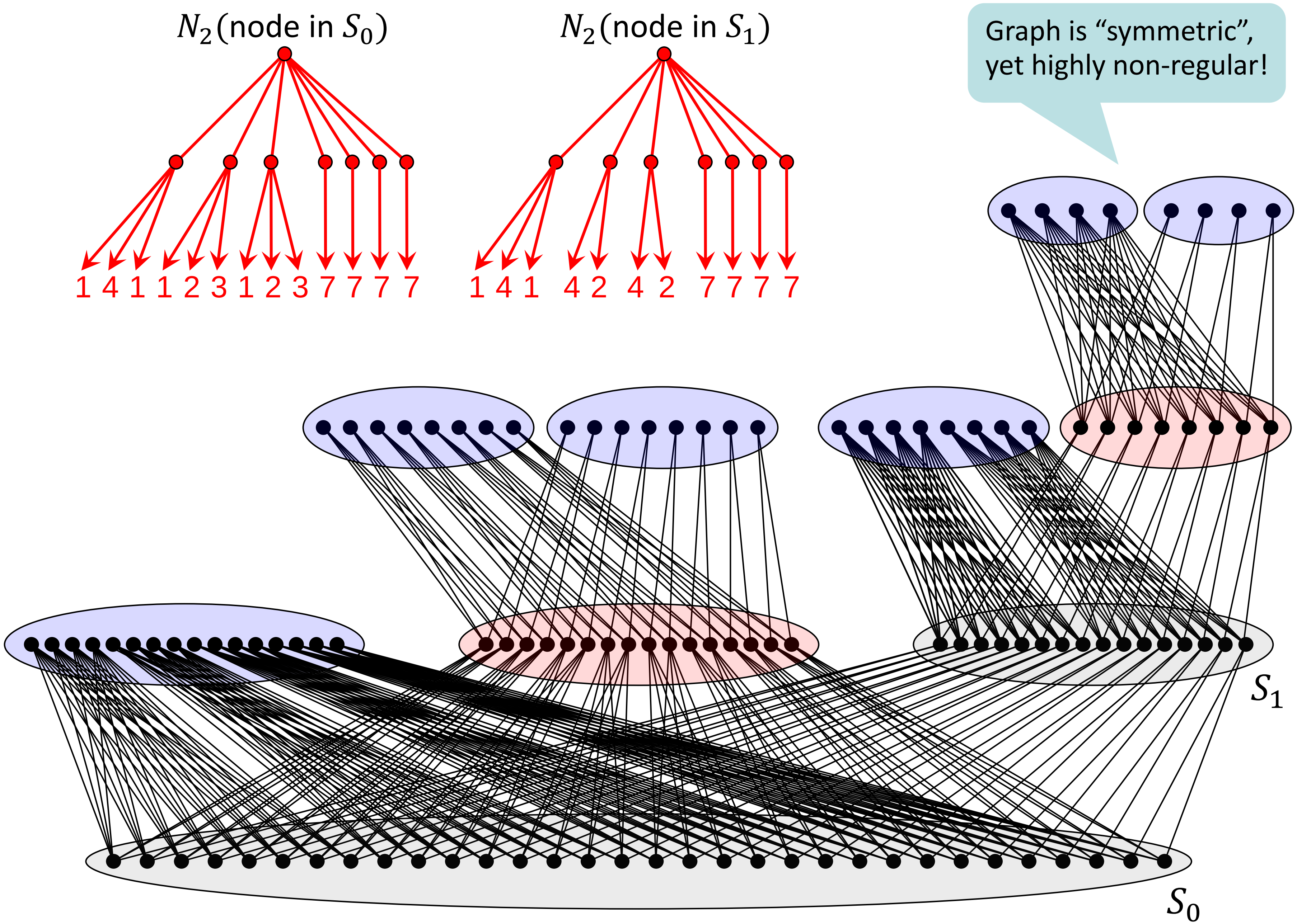


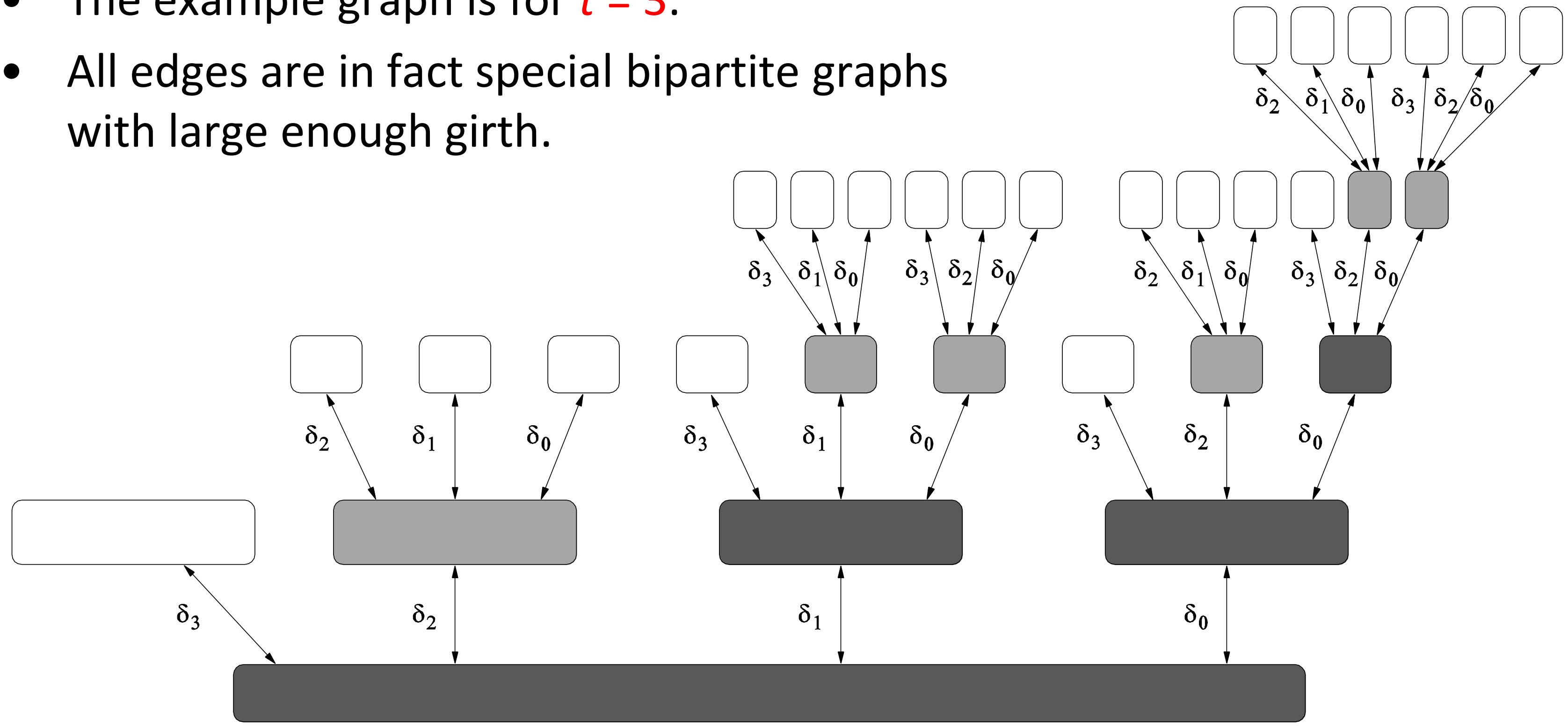

N_2(node in S_0)
1 4 1 1 2 3 1 2 3 7 7 7 7
N_2(node in S_1)
1 4 1 4 2 4 2 7 7 7 7
S_1
S_0



N_2(node in S_0)
1 4 1 1 2 3 1 2 3 7 7 7 7
N_2(node in S_1)
1 4 1 4 2 4 2 7 7 7 7
Graph is "symmetric", yet highly non-regular!
S_1
S_0

Lower Bound: The Argument

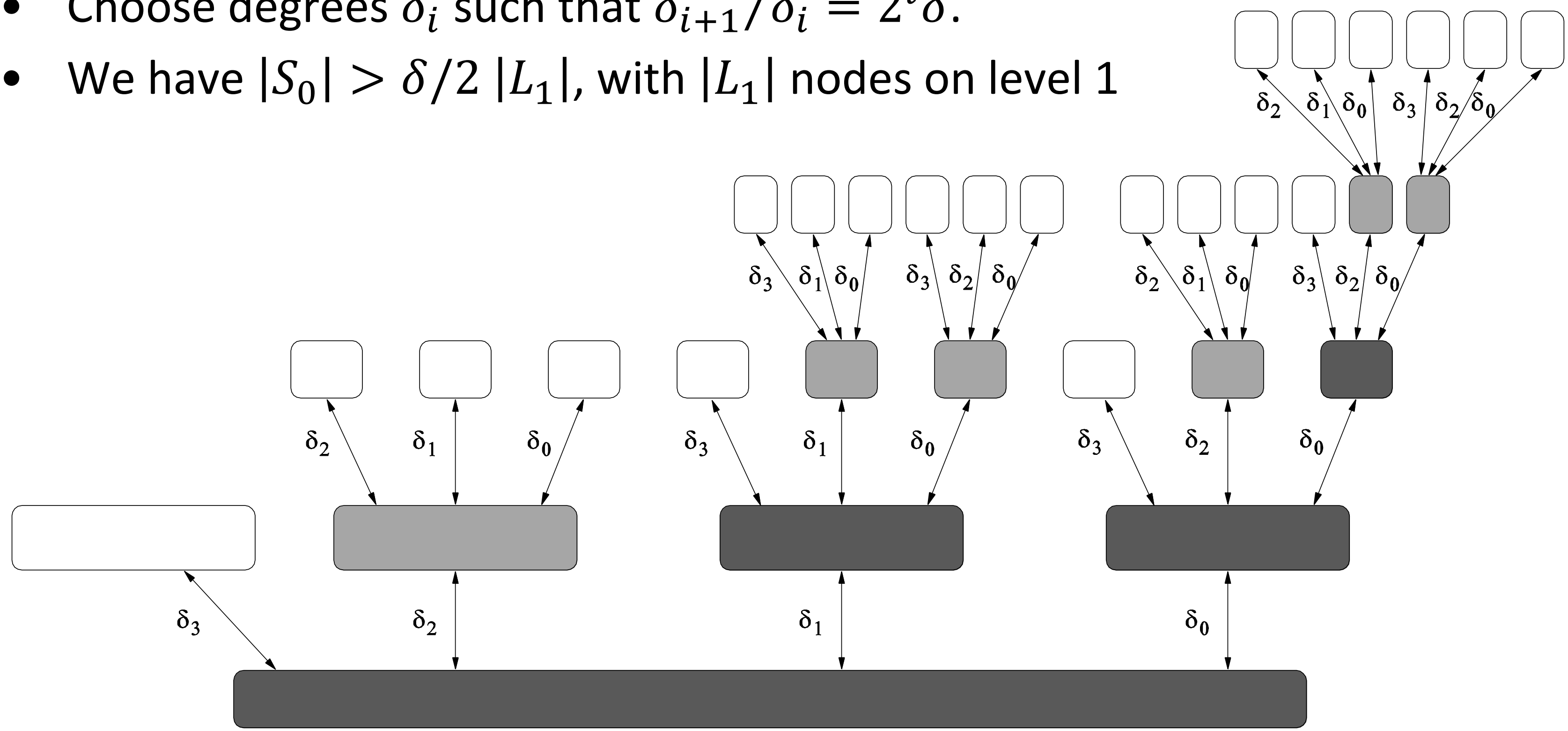
- The example graph is for $t = 3$.
- All edges are in fact special bipartite graphs with large enough girth.



- If you use the graph of recursion level t, then a distributed algorithm cannot find a good MVC approximation in time t.

Lower Bound: The Math

- Choose degrees δ_i such that $\delta_{i+1}/\delta_i = 2^i\delta$.
- We have $|S_0| > \delta/2\ |L_1|$, with $|L_1|$ nodes on level 1



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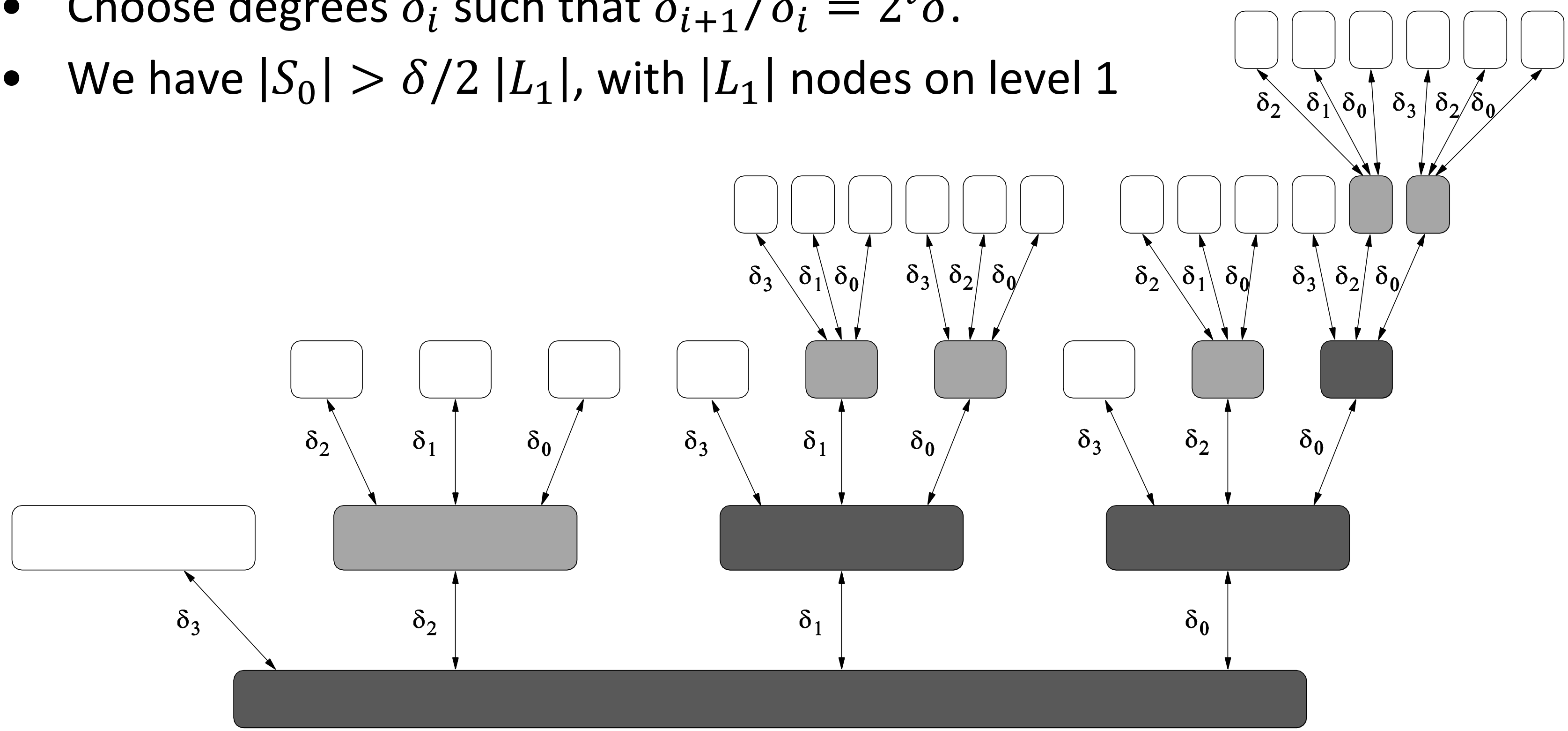

- By induction we have a $(1 - \Theta(1/\delta))$ fraction of the nodes is in S_0.
- Now δ, n, Δ are depending on the recursion level t.

Lower Bound: The Math

Graph useful for proving lower bounds in sublinear algos?

Open

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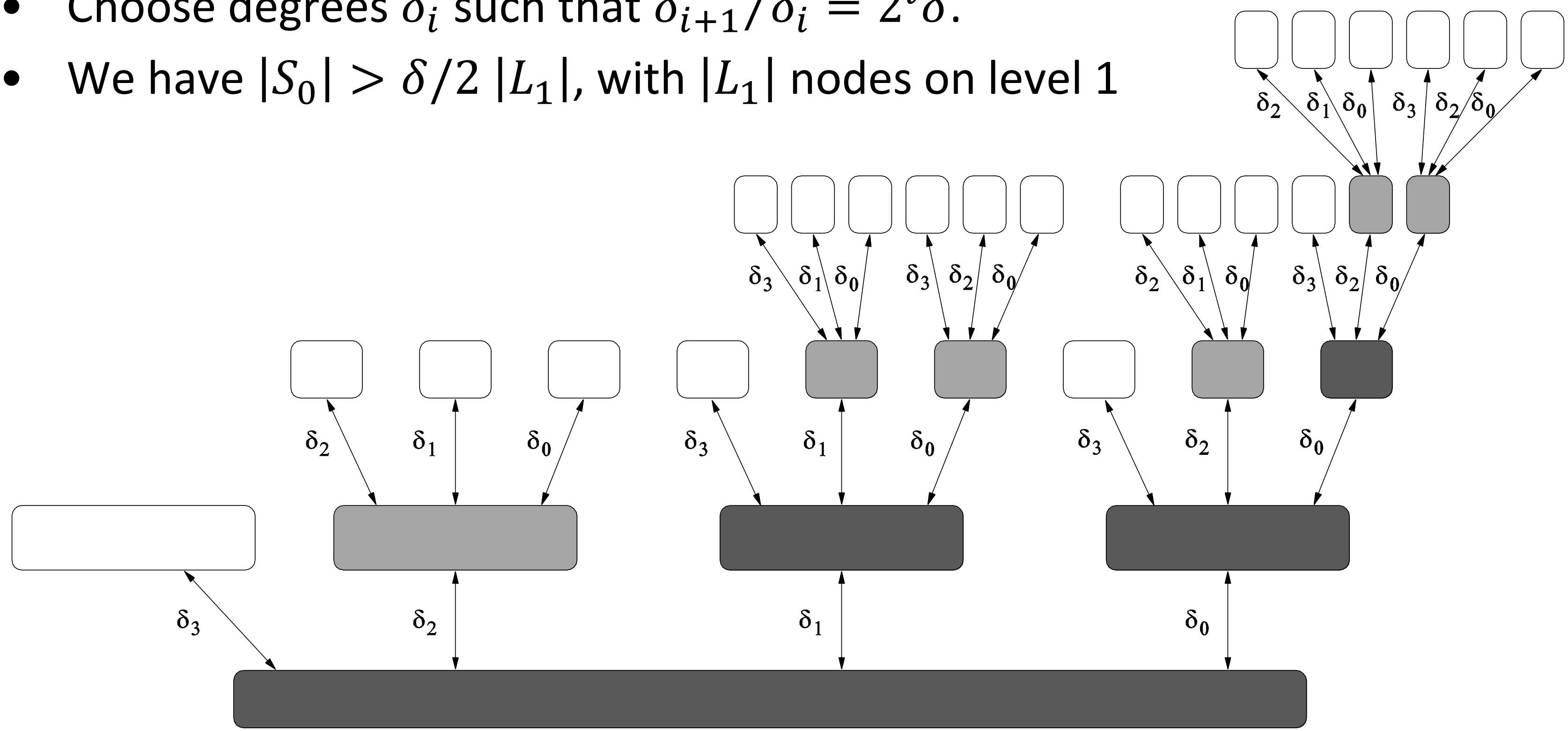

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Lower Bound: Results

- We can show that for $\epsilon > 0$, in t time, the approximation ratio is at least

$$\Omega\left(n^{\frac{1/4-\varepsilon}{t^2}}\right) \quad and \quad \Omega\left(\Delta^{\frac{1-\varepsilon}{t+1}}\right)$$

- Constant approximation needs at least $\Omega(\log \Delta)$ and $\Omega(\sqrt{\log n})$ time.
- Polylog approximation $\Omega(\log \Delta / \log \log \Delta)$ and $\Omega(\sqrt{\log n / \log \log n})$.

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tight for MVC

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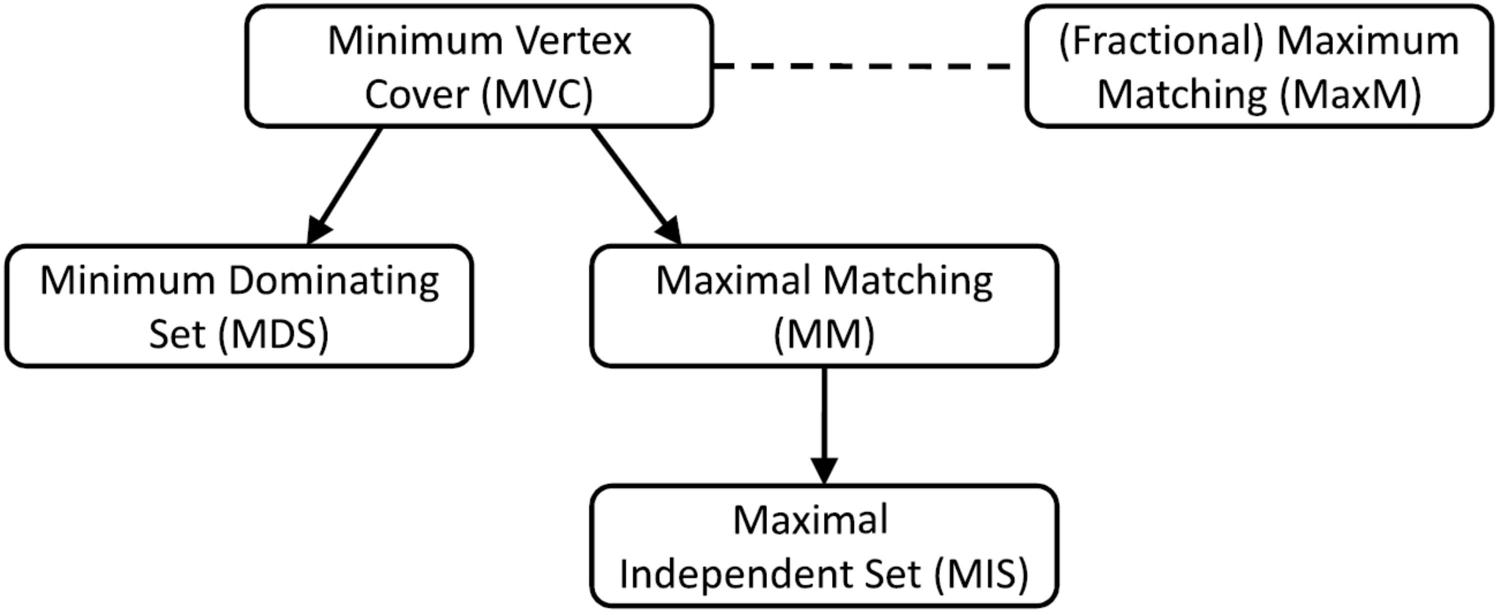
Lower Bound: Reductions

- Many "local looking" problems need non-trivial t, in other words, the bounds $\Omega(\log \Delta)$ and $\Omega(\sqrt{\log n})$ hold for a variety of classic problems.



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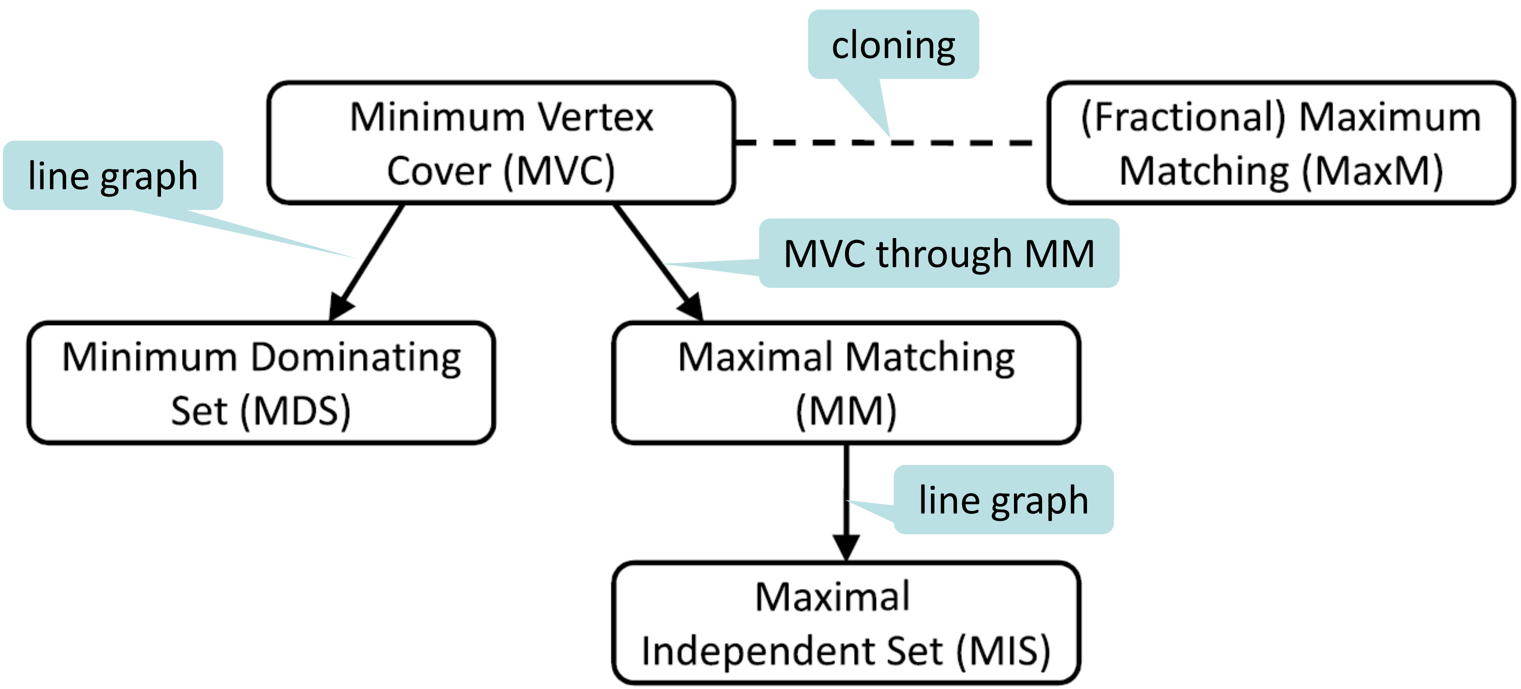
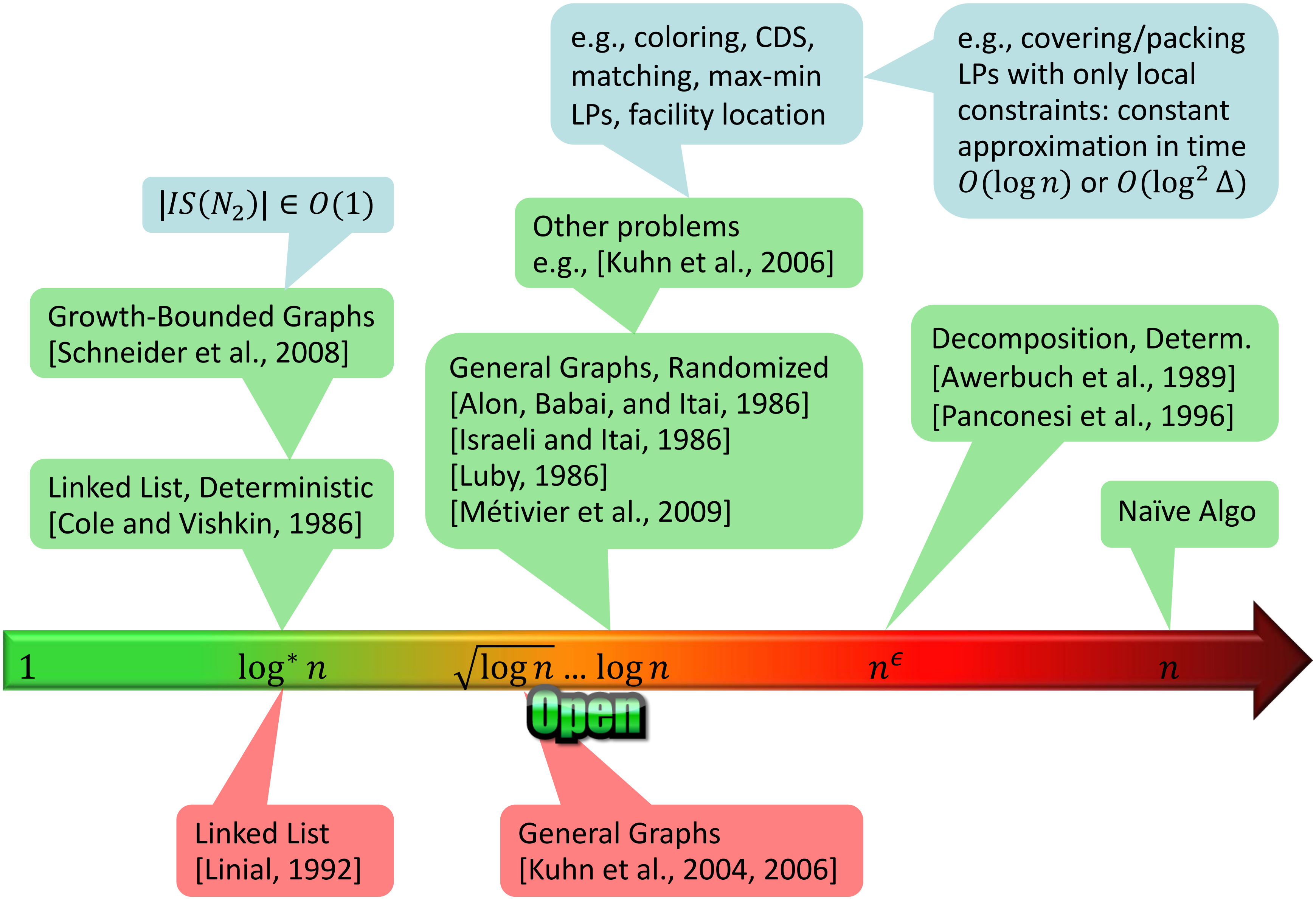
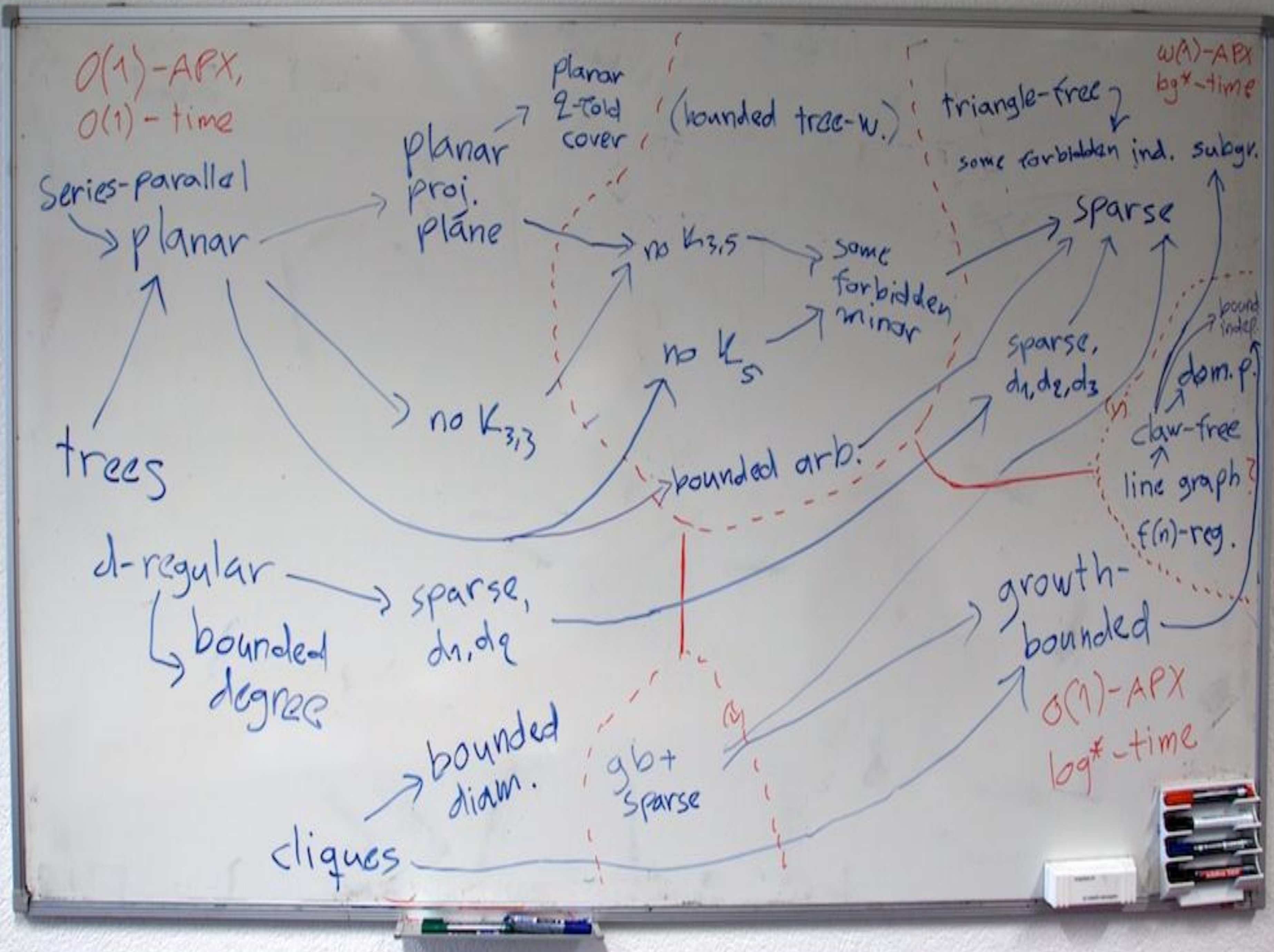
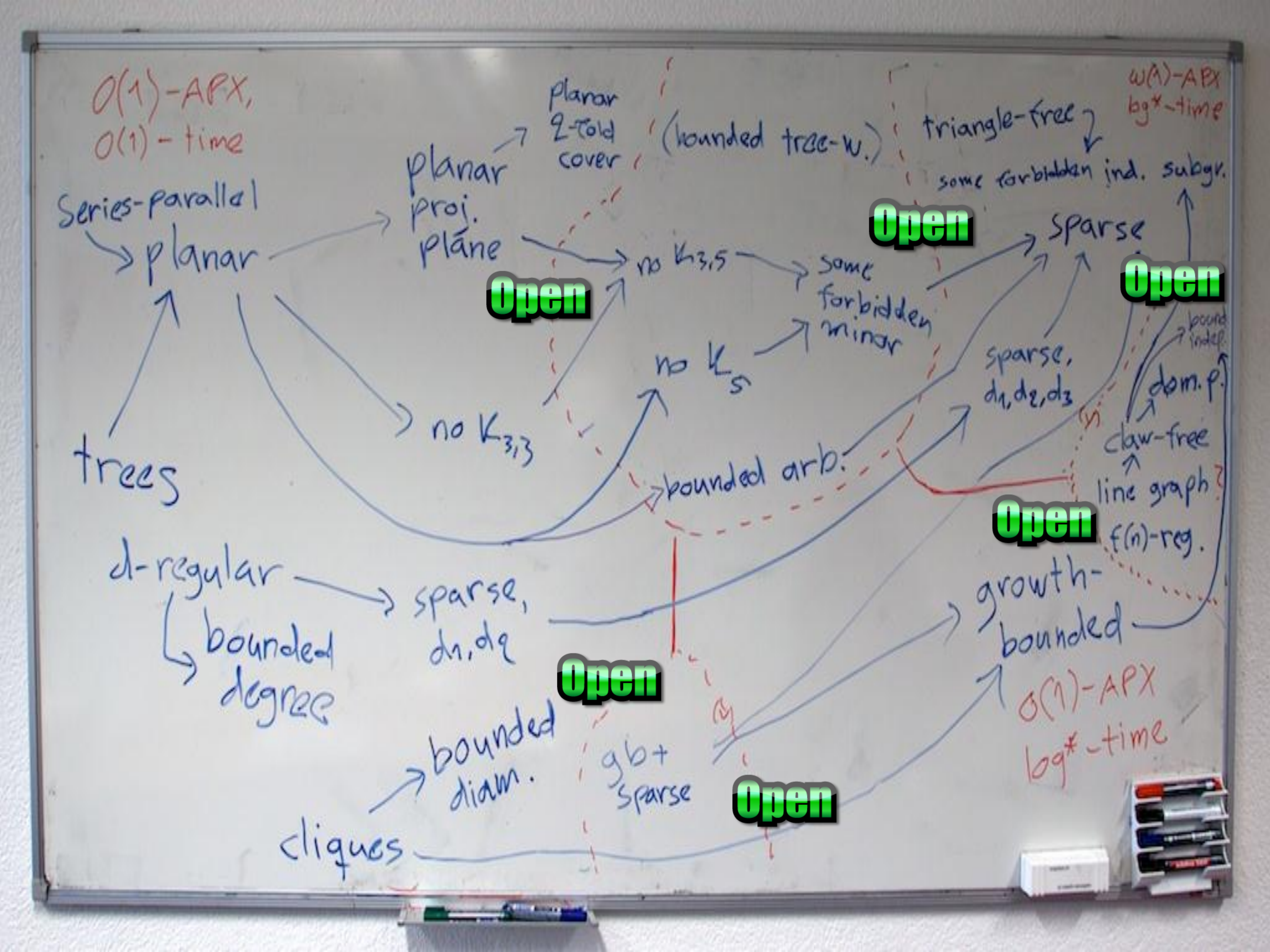


Results: MIS



e.g., coloring, CDS, matching, max-min LPs, facility location
e.g., covering/packing LPs with only local constraints: constant approximation in time $O(\log n)$ or $O(\log^2 \Delta)$
$|IS(N_2)| \in O(1)$
Other problems e.g., [Kuhn et al., 2006]
Growth-Bounded Graphs [Schneider et al., 2008]
General Graphs, Randomized [Alon, Babai, and Itai, 1986] [Israeli and Itai, 1986] [Luby, 1986] [Métivier et al., 2009]
Decomposition, Determ. [Awerbuch et al., 1989] [Panconesi et al., 1996]
Linked List, Deterministic [Cole and Vishkin, 1986]
Naïve Algo
1
$\log^* n$
$\sqrt{\log n}$ ... $\log n$
n^ϵ
n
Open
Linked List [Linial, 1992]
General Graphs [Kuhn et al., 2004, 2006]


O(1)-APX,
O(1)-time
planar 2-fold cover
(bounded tree-w.)
triangle-free
ω(1)-APX
log*-time
planar proj. plane
some forbidden ind. subgr.
series-parallel
planar
sparse
no K3,5
some forbidden minor
no K5
sparse, d1, d2, d3
bound. indep.
dom. p.
no K3,3
claw-free
trees
bounded arb.
line graph?
f(n)-reg.
d-regular
sparse, d1, d2
growth-bounded
bounded degree
O(1)-APX
log*-time
bounded diam.
gb+ sparse
cliques



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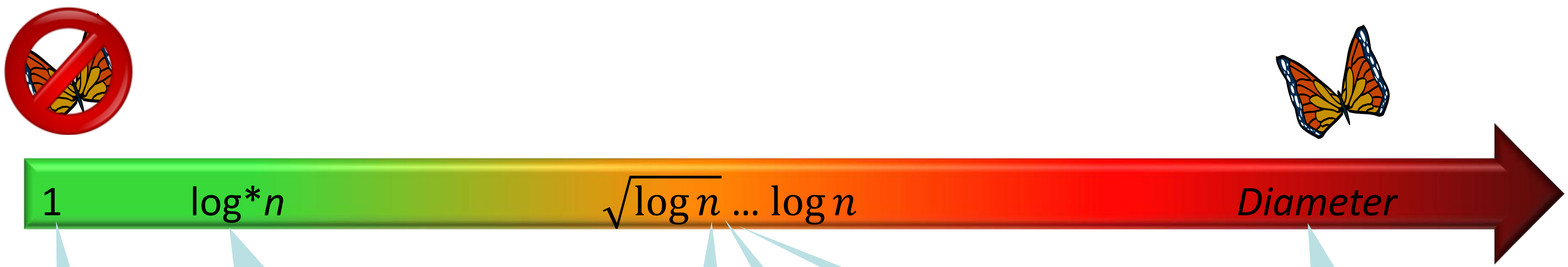
Summary



E.g., dominating set approximation in planar graphs

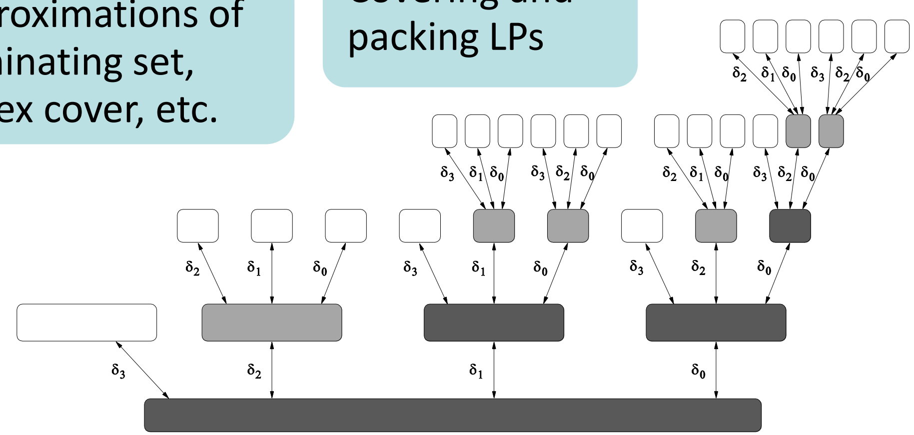
Growth-Bounded Graphs (various problems)

Approximations of dominating set, vertex cover, etc.

Covering and packing LPs

MIS, maximal matching, etc.

MST, Sum, etc.



Thank You!

Questions & Comments?

Thanks to my co-authors
Fabian Kuhn
Thomas Moscibroda
Johannes Schneider

www.disco.ethz.ch

Open Problems

- Close the gap between $\sqrt{\log n}$ and $\log n$ (for randomized algorithms)!
- Find a fast deterministic MIS algorithm (or strong det. lower bound)!
- Where are the boundaries between constant, log*, log, and diameter?
- What about algorithms that cannot even exchange messages?
- Can the lower bound graph be used in the context of sublinear algorithms?