The probable value of the Lovász-Schrijver relaxations for maximum independent set*

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August 21, 2002

Abstract

Lovász and Schrijver [LS91] devised a lift-and-project method that produces a sequence of convex relaxations for the problem of finding in a graph an independent set(or a clique) of maximum size. Each relaxation in the sequence is tighter than the one before it, while the first relaxation is already at least as strong as the Lovász theta function [Lov79]. We show that on a random graph $G_{n,1/2}$, the value of the rth relaxation in the sequence is roughly $\sqrt{n/2^r}$, almost surely. It follows that for those relaxations known to be efficiently computable, namely for $r = O(1)$, the value of the relaxation is comparable to the theta function. Furthermore, a perfectly tight relaxation is almost surely obtained only at the $r = \Theta(\log n)$ relaxation in the sequence.

1 Introduction

Let $G(V, E)$ be a graph on n vertices. An *independent set* (a.k.a. *stable set*) in G is a subset of the vertices no two of which are connected by an edge. The *maximum independent set* problem requires to find an independent set of maximum size in an input graph G. The *independence number* (a.k.a. *stability number*) of G, denoted $\alpha(G)$, is the maximum size of an independent set in G.

A *clique* in G is a subset of the vertices every two of which are connected by an edge. The *maximum clique* problem requires to find a clique of maximum size in an input graph G. The *clique number* of G, denoted $\omega(G)$, is the maximum size of a clique in G. A clique in G forms an independent set in the edge complement graph $\overline{G}$, so $\omega(G) = \alpha(\overline{G})$. It follows that the maximum clique problem and the maximum independent set problem are equivalent in many respects, including in our context. For consistency with related literature, we refer to one problem in some parts and to the other problem in others.

The maximum independent set problem is fundamental in the area of combinatorial optimization, and is closely related, in addition to the maximum clique problem, also to the *vertex cover*

*This is a preprint of an article accepted for publication in SIAM Journal on Computing© (copyright owned as specified in the journal).

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problem (the vertex complement of an independent set) and the *chromatic number* problem (minimum cover by independent sets). The maximum independent set problem (or even finding $\alpha(G)$) is one of the first problems shown to be NP-hard in [Kar72].

A common way to cope with NP-hardness of a problem is to devise algorithms that give approximate solutions. An *efficient* (i.e. polynomial time) algorithm is said to have an *approximation ratio* $\rho > 1$ for the maximum independent set problem if for every input graph, the ratio between $\alpha(G)$ and the size of the independent set returned by the algorithm is at most $\rho = \rho(n)$. It is known through work culminating in [Hås96] that for any fixed $\epsilon > 0$ it is impossible to approximate the independence number $\alpha(G)$ within a ratio of $n^{1-\epsilon}$, unless NP has randomized polynomial time algorithms (NP=ZPP). The best approximation algorithm that is known for $\alpha(G)$, due to [BH92], has approximation ratio $O(n/\log^2 n)$.

The intractability of the maximum independent set problem in the worst case suggests studying the performance of algorithms on average instances. A possible rigorous description of average instances is by probabilistic models, see e.g. [FM97] for a survey on average-case analysis of graph algorithms on random graphs.

The problem of finding a maximum independent set on a random graph appears to be difficult. Let $G_{n,1/2}$ denote the random graph on n labeled vertices obtained by connecting each pair of vertices by an edge independently with probability 1/2. It is known that the independence number of $G_{n,1/2}$ is roughly $2\log_2 n$, *almost surely*, i.e. with probability that approaches 1 as n tends to infinity, see e.g. [AS92]. Several simple and natural algorithms (e.g. the greedy one) find an independent set of size roughly $\log_2 n$, almost surely. However, no algorithm is known to find efficiently an independent set of size significantly larger than $\log_2 n$, see e.g. [Kar76, FM97]. Finding independent sets of size $\frac{3}{2}\log_2 n$ in random graphs was even suggested as a hard computational problem on which to base cryptographic applications, see [JP00].

Lovász theta function. A well-known relaxation of the maximum independent set problem is the *theta* function of a graph, denoted $\vartheta(G)$, introduced by Lovász [Lov79] (see also [GLS93, Chapter 9] and Knuth's survey [Knu94]). The theta function can be formulated as a semidefinite program and thus it can be computed, up to arbitrary precision, in polynomial time, see e.g. [GLS93]. We may consider the theta function also as a relaxation of the maximum clique problem, by formally referring to $\vartheta(\bar{G})$.

In terms of approximation ratio, the theta function appears to have little to offer. The ratio between $\vartheta(G)$ and the independence number $\alpha(G)$ can be as large as $n^{1-o(1)}$, as shown in [Fei97].

Also on the average there is a large gap between the Lovász theta function $\vartheta(G)$ and the independence number $\alpha(G)$. While the independence number of a random graph $G_{n,1/2}$ is almost surely roughly $2\log_2 n$, it was shown by Juhász [Juh82] that the value of the theta function is, almost surely, $\Theta(\sqrt{n})$.

The hidden clique problem. Jerrum [Jer92] and Kučera [Kuč95] suggested independently the following *hidden clique problem*. A random graph $G_{n,1/2}$ is chosen and then a clique of size k is randomly placed in the graph, and we wish to find in this graph a maximum clique. Jerrum showed that the Metropolis process almost surely does not find the clique when $k = o(\sqrt{n})$. Kučera observed that when $k > c\sqrt{n\log n}$ for an appropriate constant c, the vertices of the planted clique would almost surely be the ones with the largest degrees in G, and hence it is easy to recognize them efficiently. Alon, Krivelevich and Sudakov [AKS98] showed an algorithm that almost surely finds the planted clique whenever $k \geq \Omega(\sqrt{n})$. Their algorithm is based on spectral properties of the graph, namely, it uses the eigenvector that corresponds to the second largest eigenvalue of the adjacency matrix of the graph. (See also [McS01]).

Feige and Krauthgamer [FK00] devised another algorithm that is based on the semidefinite programming relaxation provided by the Lovász theta function. Their algorithm works for the same planted clique size k as the algorithm of [AKS98], but it has the advantage of being more robust; it works also in a semi-random model in which an adversary can remove edges that are outside the planted clique. Another advantage of their algorithm is that it certifies, almost surely, the optimality of its solution.

The approach of [FK00] is motivated by Juhász' result [Juh82] that the theta function of a random graph $G_{n,1/2}$ is $\Theta(\sqrt{n})$, almost surely. It follows that the maximum clique relaxation $\vartheta(\bar{G})$ is also almost surely $\Theta(\sqrt{n})$ for a random graph $G_{n,1/2}$. When a clique of size $k \geq c\sqrt{n}$, for a sufficiently large constant $c > 0$, is planted in a random graph, the theta function (being a relaxation) must increase to at least k. Furthermore, it is plausible that such a noticeable increase in the theta function will allow to find the planted clique. Indeed, it is shown in [FK00] that on the hidden clique graph $G_{n,1/2,k}$, the theta function almost surely gives exactly k, the planted clique size, in which case it allows to find the planted clique (with some extra work). In contrast, when a clique of size $k = o(\sqrt{n})$ is planted in a random graph, the monotonicity properties of the theta function, see e.g. [Knu94, Sections 18-19]), guarantee its value can only increase, but not by more than k. It follows that on the hidden clique graph $G_{n,1/2,k}$, the value of the theta function is also almost surely $\Theta(\sqrt{n})$, and it is therefore possible that the planted clique has no noticeable effect on the theta function.

A possible direction for extending the approach of [FK00] to a planted clique of smaller size $k = o(\sqrt{n})$, is to use relaxations that are stronger than the Lovász theta function. In particular, it is desirable to find a relaxation whose value on a random graph $G_{n,1/2}$ is almost surely $o(\sqrt{n})$.

The general Lovász-Schrijver technique. Lovász and Schrijver [LS91] propose a general technique for obtaining stronger and stronger relaxations of 0-1 integer programming problems. Specifically, they devise several procedures called *matrix-cut operators*, that produce from a convex (e.g. linear programming) relaxation $P \subseteq [0,1]^n$ of the problem, a convex set that is an improved relaxation for the 0-1 (i.e. integral) vectors in P. That is, the resulting convex set is contained in P and contains all the 0-1 vectors in P. The matrix-cut operators follow a *lift-and-project* approach; they lift the convex relaxation P into a higher (quadratic) dimension by introducing new variables and new constraints, and then project it back into the original space.

The two main matrix-cut operators of Lovász and Schrijver [LS91] are denoted by N and N_+. The difference between the two operators is that the lifting of the latter involves, in addition, a positive semidefinite constraint. That is, if P is a linear programming relaxation, then $N(P)$ is also a linear programming relaxation, while $N_+(P)$ is a semidefinite programming relaxation.

The matrix-cut operators can be applied iteratively, say $r \geq 0$ times, and the iterated operators are denoted N^r and N_+^r. The *N-rank of a convex relaxation* P is defined as the number of iterations of the N operator, that are needed to obtain the convex hull of the 0-1 vectors of P (i.e. a perfectly tight relaxation). The *N_+-rank* is defined similarly. Lovász and Schrijver [LS91] show that the N-rank of a relaxation is always at most the dimension d (e.g. number of variables in a linear program). The N_+ operator is a strengthening of the N operator, and hence also the N_+-rank is always at most d. Goemans and Tunçel [GT00] and Cook and Dash [CD01] show independently that there exist relaxations whose N_+-rank meets the upper bound d.

Furthermore, Lovász and Schrijver [LS91] show that the N and N_+ operators have the following important algorithmic property. If it is possible to efficiently optimize (linear objective functions) over a relaxation P, then it is also possible to efficiently optimize over the relaxation obtained by applying the operator on P. It follows that for every *fixed* $r \geq 0$, the iterated operators N^r and N_+^r also satisfy this property.

Strong relaxations for maximum independent set. To obtain relaxations of the maximum independent set problem, Lovász and Schrijver [LS91] apply their general technique of matrix-cut operators on a classical linear programming relaxation FRAC of the problem. The relaxation FRAC is a linear program of polynomial size, and hence for every fixed $r \geq 0$, one can efficiently optimize over $N_+^r(\mathrm{FRAC})$. In contrast, the dimension d (i.e. number of variables) of FRAC is the number of vertices n in the graph, and so optimizing over $N^n(\mathrm{FRAC})$ is NP-hard.

Lovász and Schrijver [LS91] show that the semidefinite programming relaxation $N_+(\mathrm{FRAC})$ is at least as strong as the Lovász theta function. It follows, for example, that for any graph on which the theta function is not tight, the relaxation $N_+^r(\mathrm{FRAC})$ for $r \geq 2$ is stronger than the theta function.

The *N-rank of a graph* is defined as the N-rank of the relaxation FRAC. The *N_+-rank* is defined similarly. It follows that for graphs with bounded N_+-rank, the maximum independent set problem can be solved in polynomial time. This family includes, for example, all perfect graphs, since the above connection with the theta function implies that their N_+-rank is at most 1.

Stephen and Tunçel [ST99] study the case where the n-vertex graph G is the line graph of a graph H on h vertices. They show that the N_+-rank of G is at most $\lfloor h/2 \rfloor$, and that this bound is met if H is a complete graph on an odd number of vertices, in which case $n = \binom{h}{2}$, and so the N_+-rank of G is $\Omega(\sqrt{n})$. Note that independent sets in G correspond to matchings in H, and that a maximum weight matching can be found efficiently; it follows that there are graphs with unbounded (and rather large) N_+-rank, in which the maximum (weighted) independent set problem can be solved in polynomial time.

Our results. We examine the asymptotic behavior on the random graph $G_{n,1/2}$ of the relaxations of Lovász and Schrijver [LS91] for the maximum independent set problem. In particular, we show that the typical value of the semidefinite programming relaxation $N_+^r(\mathrm{FRAC})$ on a random graph is, roughly $\sqrt{n/2^r}$ for $r = o(\log n)$. We note that this characterization answers (up to a constant factor) a question of Knuth [Knu94, Section 37,Problem P6].

Theorem 1.1. *For every fixed $\delta > 0$ and $r = o(\log n)$, the value of the relaxation $N_+^r(\mathrm{FRAC})$ on a random graph $G_{n,1/2}$ is at least $\sqrt{n/(2+\delta)^{r+1}}$ and at most $4\sqrt{n/(2-\delta)^{r+1}}$, almost surely.*

Recall that the strongest relaxations of Lovász and Schrijver [LS91] whose value is known to be efficiently computable are $N_+^r(\mathrm{FRAC})$ for $r = O(1)$. Theorem 1.1 shows that on a random graph, the typical value of these relaxations is smaller than that of the theta function by no more than a constant factor. In the hidden clique problem, the planted clique size k that a heuristic can handle can be improved by an arbitrarily large constant factor using a method of [AKS98], and therefore it appears that the improvement offered by these stronger relaxations can be achieved by other methods.

We use Theorem 1.1 to characterize, up to a constant factor, the typical N_+-rank of a random graph $G_{n,1/2}$.

Theorem 1.2. *The N_+-rank of a random graph $G_{n,1/2}$ is almost surely $\Theta(\log n)$.*

Our results for the N_+ operator extend to a slightly stronger variant of the matrix-cut operators of Lovász and Schrijver [LS91]. This operator, denoted $N_{\mathrm{FR}+}$, is specialized for the maximum independent set problem and retains the important algorithmic property of N_+, namely an efficient optimization over P implies an efficient optimization over $N_{\mathrm{FR}+}(P)$.

Organization. Section 2 is a technical description of the matrix-cut operators of Lovász and Schrijver [LS91] (including our variant $N_{\mathrm{FR}+}$). We present the formal definitions in Section 2.1, and state in Section 2.2 some basic useful properties (whose proof is deferred to Appendix A.1).

Section 3 describes our results on matrix-cuts in a random graph. Specifically, a lower bound on the value of the relaxation $N_+^r(\mathrm{FRAC})$ is shown in Section 3.1, and an upper bound is shown in Section 3.2.

Appendix A proves several useful properties of the matrix-cut operators. In Section A.1 we give some basic properties that are needed for our main results, and in Section A.2 we give bounds on the ranks of the different matrix-cut operators.

Preliminaries. Throughout, we omit the graph $G(V, E)$ if it is clear from the context. We let n denote the number of vertices in the graph G, and assume, without loss of generality, that $V = \{1, \ldots, n\}$. For a vertex i in the graph, let $\Gamma(i)$ denote the set of the vertices that are adjacent to i in the graph, i.e. $\Gamma(i) := \{j \in V : ij \in E\}$, and let $\Gamma(S)$ denote the set of vertices in V that are adjacent to at least one vertex of S, i.e. $\Gamma(S) := \cup_{i\in S}\Gamma(i)$.

An $n \times n$ (real) matrix Y is *positive semidefinite* if Y is symmetric and $x^T Y x \geq 0$ for all $x \in \mathbb{R}^n$. It is well-known that a symmetric matrix Y is positive semidefinite if and only if all the eigenvalues of Y are nonnegative.

A *Gram matrix representation* of an $n \times n$ matrix Y is a set of real-valued vectors $\{v_1 \ldots, v_n\}$ such that $Y_{ij} = v_i^T v_j$ for all i, j (i.e. $Y = B^T B$ for a corresponding matrix B). It is well-known that a matrix Y is positive semidefinite if and only if it has a Gram matrix representation.

2 The Lovász-Schrijver matrix-cut operators

In this section we describe the so-called matrix-cut operators that were proposed by Lovász and Schrijver [LS91]. Given a convex set (e.g. a polytope) P, the matrix-cut operators consider P as a relaxation of the convex hull of its 0-1 vectors, and produce another relaxation that is tighter than P. In other words, these operators produce a convex set that is sandwiched (in terms of containment) between P and (the convex hull of) the 0-1 vectors in P. Furthermore, the produced relaxation is strictly tighter than P, unless P is already tight. Our description and notation mostly follows that of Lovász and Schrijver [LS91](but also those of [CD01, GT00]). An alternative formulation of the matrix-cut operators is given by Lovász in [Lov94].

Section 2.1 reviews the definitions of the Lovász-Schrijver matrix-cut operators. In Section 2.2 we state some of their known properties (that we need), focusing on the application of these operators to the stable set problem. For completeness (and to aid readers who are unfamiliar with these operators), we give the proofs of these properties in Appendix A, where these and relevant known results and examples are repeated and extended to a more general setting that includes the $N_{\mathrm{FR}+}$ operator.

Throughout, let e_j be the jth unit vector, let $\mathbf{0}$ be the vector of all zeros, and let $\mathbf{1} = \sum_j e_j$ be the vector of all ones. The sizes (dimensions) of $\mathbf{0}, \mathbf{1}$ and e_j will be clear from the context. Recall that a set is called a *cone* if it is closed under multiplication by a nonnegative number. A *convex cone* is thus a set that is closed under a nonnegative linear (a.k.a. *conic*) combination. (Throughout, we will consider convex cones rather than polytopes.) A *polyhedral cone* is a cone that is also a polyhedron; equivalently, a polyhedral cone is a set that can be defined by $\{x : Ax \geq 0\}$ for some matrix A.

2.1 Definitions

Homogenization. It will be convenient to deal with homogenous systems of inequalities. We therefore embed the n-dimensional space $\mathbb{R}^n$ in $\mathbb{R}^{n+1}$ as the hyperplane $x_0 = 1$ (throughout, the 0th variable plays a special role), and work with convex cones in $\mathbb{R}^{n+1}$, as follows.

Since we deal with 0-1 programming on n variables, our basic example is a polytope P that is contained in $[0,1]^n$ (the convex hull of the n-dimensional hypercube $\{0,1\}^n$). To homogenize P using the new variable x_0, first embed P in the hyperplane $x_0 = 1$ of $\mathbb{R}^{n+1}$, and then generate from it a convex cone. That is, if

$$P = \{x \in \mathbb{R}^n : Ax \leq b,\ \mathbf{0} \leq x \leq \mathbf{1}\}, \tag{1}$$

then the convex cone obtained by homogenization is

$$K := \left\{ \begin{pmatrix} x_0 \\ x \end{pmatrix} \in \mathbb{R}^{n+1} : Ax \leq x_0 b,\ 0 \leq x \leq x_0 \mathbf{1} \right\}. \tag{2}$$

Note that such K can be described as the intersection of finitely many halfspaces defined by linear constraints $u^t x \geq 0$ (here $x \in \mathbb{R}^{n+1}$), and hence it is a polyhedral cone.

We denote by $Q \subset \mathbb{R}^{n+1}$ the convex cone that is obtained from the polytope $[0,1]^n$ via the homogenization procedure (1)-(2). Namely,

$$Q := \left\{ (x_0, x_1, \ldots, x_n)^T : 0 \leq x_i \leq x_0 \text{ for all } 1 \leq i \leq n \right\}. \tag{3}$$

Note that Q is a polyhedral cone that can be described by $2n$ linear inequalities.

Throughout, let $K \subseteq Q$ be a (closed) convex cone. We denote by K_I the convex cone that is generated by all 0-1 vectors in K. Observe that within the hyperplane $x_0 = 1$, K_I is exactly the *integral hull* (i.e. convex hull of the integral vectors) of K. For example, $Q_I = Q$.

The *polar cone* of K, denoted K^*, is the convex cone defined by

$$K^* := \{u \in \mathbb{R}^{n+1} : x^T u \geq 0 \text{ for all } x \in K\}.$$

Observe that a vector $u \in K^*$ corresponds to a linear constraint $u^T x \geq 0$ that is *valid* for K (i.e. satisfied by all vectors $x \in K$). The polar cone K^* is thus the collection of valid linear constraints for K. For example, Q is defined in (3) by $2n$ linear constraints, and hence Q^* is spanned by the vectors e_i and $f_i = e_0 - e_i$, for $i = 1, \ldots, n$.

Fractional stable sets. We will be mostly interested in the stable set problem. Let $G(V, E)$ be a graph with no isolated vertices and $|V| = n$. Then the stable sets of G correspond to the 0-1 solutions of the system of linear inequalities

$$x_i \geq 0 \quad \text{for all } i \in V \qquad \textit{(nonnegativity constraints)} \tag{4}$$

and

$$x_i + x_j \leq 1 \quad \text{for all } ij \in E \qquad \textit{(edge constraints)} \tag{5}$$

Let $\mathrm{STAB}(G) \subset \mathbb{R}^n$ denote the convex hull of the 0-1 solutions of the system (4)-(5). Let $\mathrm{FRAC}(G) \subset \mathbb{R}^n$ (for "fractional stable sets") denote the solution set of the system (4)-(5) (i.e. without integrality restriction). Clearly, $\mathrm{STAB}(G) \subseteq \mathrm{FRAC}(G)$.

Let $\mathrm{FR}(G) \subset \mathbb{R}^{n+1}$ be the polyhedral cone that is obtained from the polytope $\mathrm{FRAC}(G)$ via the homogenization procedure (1)-(2). That is, $\mathrm{FR}(G)$ is the solution set of the following homogenous system of linear inequalities for the stable set problem:

$$x_i \geq 0 \quad \text{for each } i \in V \tag{6}$$

$$x_0 - x_i - x_j \geq 0 \quad \text{for each } ij \in E \tag{7}$$

Let $\mathrm{ST}(G)$ be the polyhedral cone that is obtained from the polytope $\mathrm{STAB}(G)$ via the homogenization procedure (1)-(2). It is straightforward that $(\mathrm{FR}(G))_I = \mathrm{ST}(G)$.

Throughout, we omit the graph G when it is clear from the context, denoting $\mathrm{STAB}(G)$ by STAB etc. It can be seen that the polar cone FR^* is spanned by the vectors e_i for $i = 1, \ldots, n$ and the vectors $f_{ij} = e_0 - e_i - e_j$ for $ij \in E$. Note that $\mathrm{FR} \subseteq Q$ and hence $\mathrm{FR}^* \supseteq Q^*$.

Matrix-cut operators. Let $K_1, K_2 \subseteq Q$ be closed convex cones in $\mathbb{R}^{n+1}$ (e.g. $K_1 = \mathrm{FR}(G)$ and $K_2 = Q$). Consider the cone $K_1 \cap K_2$. For each $u \in K_1^*$ the constraint $u^T x \geq 0$ is valid for K_1, and for each $v \in K_2^*$ the constraint $v^T x \geq 0$ is valid for K_2. It follows that the quadratic inequality $(u^T x)(x^T v) \geq 0$ is valid for $K_1 \cap K_2$. Furthermore,

$$K_1 \cap K_2 = \left\{x : u^T x x^T v \geq 0 \text{ for all } u \in K_1^*, v \in K_2^*, x_0 \geq 0\right\}$$

because any original inequality, say $u^T x \geq 0$ for K_1, can be recovered by adding the two quadratic inequalities obtained by $e_i, f_i \in Q^* \subseteq K_2^*$, giving $u^T x \cdot x_0 = u^T x x^T (e_i + f_i) \geq 0$.

Furthermore, all 0-1 vectors in $K_1 \cap K_2$ satisfy $x_i^2 = x_i$. Therefore, if x is a 0-1 vector in $K_1 \cap K_2$ and with $x_0 = 1$, then setting $Y = xx^T$ we have that

(a) Y is symmetric.

(b) $Ye_0 = diag(Y)$, i.e. $Y_{ii} = Y_{i0}$ for all $1 \leq i \leq n$.

(c) $u^T Y v \geq 0$ for all $u \in K_1^*$ and $v \in K_2^*$.

(d) Y is positive semidefinite.

Note that (c) can be written as

(c') $YK_2^* \subseteq K_1$

Lovász and Schrijver [LS91] proposed the following lift-and project procedure. Given K_1, K_2, consider the derived cones:

$$M(K_1, K_2) := \{Y \in \mathbb{R}^{(n+1)\times(n+1)} : Y \text{ satisifies (a)-(c)}\}$$

$$M_+(K_1, K_2) := \{Y \in \mathbb{R}^{(n+1)\times(n+1)} : Y \text{ satisifies (a)-(d)}\}$$

and define the projections of these liftings on $\mathbb{R}^{n+1}$:

$$N(K_1, K_2) := \{Ye_0 : Y \in M(K_1, K_2)\}$$

$$N_+(K_1, K_2) := \{Ye_0 : Y \in M_+(K_1, K_2)\}.$$

It follows from the above discussion that

$$(K_1 \cap K_2)_I \subseteq N_+(K_1, K_2) \subseteq N(K_1, K_2) \subseteq K_1 \cap K_2 \tag{8}$$

Relevant variants of the operators. We shorten notation to easily handle two important special cases. When $K_2 = Q$ we omit K_2, i.e. $N(K) := N(K, Q)$ and $N_+(K) := N_+(K, Q)$. In this case, we have that (c') is equivalent to:

(c") Every column of Y is in K_1; the difference of the first column and any other column of Y is in K_1.

Note that we have from (8) that

$$K_I \subseteq N_+(K) \subseteq N(K) \subseteq K \tag{9}$$

For the stable set problem, we may take $K_2 = \mathrm{FR}$, denoting it in the subscript, i.e. $N_{\mathrm{FR}}(K) := N(K, FR)$ and $N_{\mathrm{FR}+}(K) := N_+(K, FR)$. In this case, we have that (c') is equivalent to:

(c"') $Ye_i \in K_1$ for all $i \geq 1$, and $Yf_{ij} \in K_1$ for all $ij \in E$.

We assume throughout that $K \subseteq \mathrm{FR}$, and then we have from (8) that

$$K_I \subseteq N_{\mathrm{FR}+}(K) \subseteq N_{\mathrm{FR}}(K) \subseteq K \tag{10}$$

It follows from the definition that using $K_2 = \mathrm{FR}$ is at least as strong as using $K_2 = Q$ in the same operator, i.e. $N_{\mathrm{FR}}(K) \subseteq N(K)$ and $N_{\mathrm{FR}_+}(K) \subseteq N_+(K)$. We therefore have that

$$K_I \subseteq N_{\mathrm{FR}+}(K) \subseteq N_{\mathrm{FR}}(K) \subseteq N(K) \subseteq K \tag{11}$$

$$K_I \subseteq N_{\mathrm{FR}+}(K) \subseteq N_+(K) \subseteq N(K) \subseteq K \tag{12}$$

It can also be seen that $N_{\mathrm{FR}}(K) \not\subseteq N_+(K)$ (e.g., when G is a clique on 5 vertices and taking $K = \mathrm{FR}$, see Appendix A.2), but it is not clear (to us) whether $N_+(K) \subseteq N_{\mathrm{FR}}(K)$. The strength of these operators is further discussed in Appendix A.2.

Iterated operators. Define the iterated operator $N^r(K)$ recursively by $N^0(K) = K$ and $N^r(K) = N(N^{r-1}(K))$ for $r \geq 1$. For other operators, the iterated operator is defined similarly.

The following Theorem of Lovász and Schrijver [LS91] proves that even without the positive semidefiniteness constraint (d), it suffices to apply n iterations in order to get from a convex cone $K \subseteq Q$ the cone K_I. It follows that applying the N operator on $K \neq K_I$ produces a relaxation of K_I that is strictly tighter than K.

Theorem 2.1 (Lovász and Schrijver [LS91]). *Let $K \subseteq Q$ be a convex cone in $\mathbb{R}^{n+1}$. Then $N^n(K) = K_I$.*

It is often easier to work in the original n-dimensional space (without homogenization), so in the case that K is the cone obtained from a polytope (or a convex set) P in $[0,1]^n$ via the homogenization procedure (1)-(2), define

$$N(P) := \left\{ x \in \mathbb{R}^n : \begin{pmatrix} 1 \\ x \end{pmatrix} \in N(K) \right\}$$

and similarly for the other operators (including the iterated ones).

For the stable set problem, K will be one of the cones obtained from $\mathrm{FR}(G)$ by an iterated operator, e.g. $N^r(\mathrm{FR}(G))$. Going back to the original n-dimensional space we shall abbreviate $N^r(G) := N^r(\mathrm{FRAC}(G))$ and similarly for the other operators. We then have from Theorem 2.1 that $N^n(G) = \mathrm{STAB}(G)$.

Ranks. The *N-rank of an inequality* $u^T x \geq 0$ that is valid for K_I, is the smallest nonnegative integer r such that $u^T x \geq 0$ is valid for $N^r(K)$. (Note that the rank is relative to K). For N_+,N_{FR} and $N_{\mathrm{FR}+}$ the rank is defined similarly. Theorem 2.1 implies that the rank of any valid inequality is at most n (the dimension).

The *N-rank of a cone* K is the smallest nonnegative integer r such that $N^r(K) = K_I$, and similarly for the other operators. By Theorem 2.1, the N-rank of K is at most n (the dimension).

The N-rank of a graph G, is the N-rank of $\mathrm{FR}(G)$, and similarly for the other operators. For example, for a bipartite graph STAB = FRAC and hence the N-rank of a bipartite graph is 0. We discuss bounds on the rank in Appendix A.2.

2.2 Useful properties

Algorithmic aspects. Lovász and Schrijver [LS91] give sufficient conditions for efficient *weak* (i.e. up to arbitrary precision) optimization (of linear objective functions) over $N(K)$, $N_+(K)$, $N_{\mathrm{FR}}(K)$ and $N_{\mathrm{FR}+}(K)$. Technically, the matrix-cut operators have the following algorithmic property.

Theorem 2.2 (Lovász and Schrijver [LS91]). *A polynomial time weak separation oracle for K gives a polynomial time weak separation oracle for $N^r(K), N^r_+(K), N^r_{\mathrm{FR}}(K)$ and $N^r_{\mathrm{FR}+}(K)$ for any fixed constant r.*

By the equivalence between weak (i.e. up to arbitrary precision) optimization and weak separation (see [GLS93]), Theorem 2.2 implies a weak optimization of any linear objective function over these relaxations of K_I.

Lovász and Schrijver [LS91] suspect that Theorem 2.2 does not extend to $N(K,K)$. They remark, however, that it if K is given by an explicit system of polynomially many linear inequalities, then Theorem 2.2 does extend to $N(K,K)$.

For the stable set problem, the cone $K = \mathrm{FR}$ is given by an explicit linear program of polynomial size, so one can solve the separation problem for it in polynomial time. We thus obtain the following theorem.

Theorem 2.3. *For every fixed $r \geq 0$, the weak optimization problem for $N^r(G)$ can be solved in polynomial time, and similarly for $N^r_+, N^r_{\mathrm{FR}}, N^r_{\mathrm{FR}+}$.*

Down-monotonicity. A non-empty convex set $P \subseteq [0,1]^n$ is called *down-monotone* (in $[0,1]^n$) if for every $x \in P$, every $y \in [0,1]^n$ with $y \leq x$ is also in P (see e.g. [GLS93, page 11]). Similarly, a convex cone $\{\mathbf{0}\} \neq K \subseteq Q$ is called *down-monotone* if for every $x \in K$, every $y \in Q$ with $y \leq x$ and $y_0 = x_0$ is also in K.

The next lemma shows that the relaxations of the stable set problem that are produced by iterated matrix-cut operators are down-monotone. Its proof appears in Appendix A.1.

Lemma 2.1. *$N^r(G)$ is down-monotone for every $r \geq 0$, and similarly for N^r_+,N^r_{FR},$N^r_{\mathrm{FR}+}$.*

Removing vertices from the graph. Recall that $V = \{1, \ldots, n\}$. For a vector $x \in \mathbb{R}^n$ and a subset $W \subset V$, we denote by x_W the restriction of x to the coordinates of W.

The next lemma characterizes the relaxations of the stable set problem that are produced by iterated matrix-cut operators when one of the coordinates is fixed (i.e. $x_i = 0$ or $x_i = x_0$). Its proof appears in Appendix A.1.

Lemma 2.2. *Let $x \in \mathbb{R}^n$ and assume that i satisfies $x_i = 1$ and $x_j = 0$ for all $j \in \Gamma(i)$. Then for all $r \geq 0$, $x \in N^r(G)$ if and only if $x_{V-\Gamma(i)-i} \in N^r(G - \Gamma(i) - i)$, and similarly for N_+^r,N_{FR}^r and $N_{\mathrm{FR}+}^r$.*

Vertex deletion and contraction. Let $a^T x \leq b$ be an inequality valid for STAB(G). For a subset $W \subset V$, we denote by a_W the restriction of a to the coordinates of W. For every $i \in V$, if $a^T x \leq b$ is valid for STAB(G), then $a_{V-i}^T x \leq b$ is valid for STAB$(G - i)$ and $a_{V-\Gamma(v)-i}^T x \leq b - a_i$ is valid for STAB$(G - \Gamma(i) - i)$. Following the terminology of Lovász and Schrijver [LS91], we say that these inequalities arise from $a^T x \leq b$ by the *deletion* and *contraction* of vertex i, respectively. Note that if $a^T x \leq b$ is an inequality such that for some i, both the deletion and the contraction of i yield inequalities valid for the corresponding graphs, then $a^T x \leq b$ is valid for G.

Upper bounds on the N_+-rank.

Lemma 2.3 (Lovász and Schrijver [LS91]). *If $a^T x \leq b$ is an inequality valid for* STAB(G) *such that for all $i \in V$ with $a_i > 0$ the contraction of i gives an inequality with N_+-rank at most r, then $a^T x \leq b$ has N_+-rank at most $r + 1$.*

Lemma 2.4 (Lovász and Schrijver [LS91]). *The N_+-rank of a graph G is at most its stability number $\alpha(G)$.*

3 The Lovász-Schrijver relaxations in a random graph

In this section we show that the N_+-rank of a random graph $G_{n,1/2}$ is almost surely $\Theta(\log n)$. In particular, we analyze the asymptotic behavior of $\max\{\mathbf{1}^T x : x \in N_+^r(G)\}$ for $r = o(\log n)$. Loosely speaking, we show that the value of this relaxation is almost surely roughly $\sqrt{n/2^r}$. The precise formulations of our lower bound and upper bound on $\max\{\mathbf{1}^T x : x \in N_+^r(G)\}$ appear below. Our analysis extends the proof of Juhász [Juh82] that shows that the theta function of a random graph is almost surely $\Theta(\sqrt{n})$.

Theorem 3.1. *For any $c > \sqrt{2}$ there exists an $\epsilon' > 0$, such that if $0 \leq r \leq \epsilon' \log n$, then almost surely* $\max\{\mathbf{1}^T x : x \in N_+^r(G_{n,1/2})\} \geq \sqrt{n}/c^{r+1}$*, and similarly for $N_{\mathrm{FR}+}^r$.*

The proof of Theorem 3.1 appears in Section 3.1. Technically, we show that $N_+^r(G_{n,1/2})$ almost surely contains the "uniform" solution $(1/c^{r+1}\sqrt{n})\mathbf{1}$, and hence obtain a lower bound on the probable value of the relaxation.

To show that the above lower bound is nearly tight, we next give an upper bound on the value of the relaxation. Its proof appears in Section 3.2.

Theorem 3.2. *For any $d < \sqrt{2}$ there exists an $\epsilon' > 0$, such that if $1 \leq r \leq \epsilon' \log n$, then almost surely* $\max\{\mathbf{1}^T x : x \in N_+^r(G_{n,1/2})\} \leq 4\sqrt{n}/d^{r+1}$*, and similarly for $N_{\mathrm{FR}+}^r$.*

It is straightforward that Theorem 1.1 follows from Theorems 3.1 and 3.2 by taking $c = \sqrt{2+\delta}$ and $d = \sqrt{2-\delta}$.

The N_+-rank of a random graph $G_{n,1/2}$. Using Theorem 3.1 and Lemma 2.4 we can now show that the N_+-rank of a random graph is almost surely $\Theta(\log n)$, proving Theorem 1.2. For comparison, it follows from Corollary A.24 that the N-rank of a random graph is almost surely at least $\Omega(n/\log n)$.

Proof of Theorem 1.2. Let G be a random graph from the distribution $G_{n,1/2}$, and let us first show a lower bound on the N_+-rank. It is well known that, almost surely, the maximum size of a stable set in G is roughly $2\log_2 n$, i.e.

$$\max\{\mathbf{1}^T x : x \in \text{STAB}\} \leq O(\log n)$$

We have from Theorem 3.1 with $r = \epsilon' \log n$ that, almost surely,

$$\max\{\mathbf{1}^T x : x \in N_+^r(\text{FRAC})\} \geq n^{\Omega(1)}$$

It follows that $N_+^r(\text{FRAC}) \neq \text{STAB}$, and hence the N_+-rank of FRAC (and therefore of G), is larger than $r = \epsilon' \log n = \Omega(\log n)$.

The upper bound on N_+-rank of G follows from Lemma 2.4. Indeed, the stability number of a random graph $G_{n,1/2}$ is, almost surely, roughly $2\log_2 n$, and hence the N_+-rank of G is, almost surely, $O(\log n)$, as claimed. □

3.1 Lower bound on the value of $N_+^r(G_{n,1/2})$

We prove Theorem 3.1 by showing that $N_+^r(G_{n,1/2})$ almost surely contains the "uniform" solution $(1/c^{r+1}\sqrt{n})\mathbf{1}$. First we exhibit in Lemma 3.1 certain conditions that are sufficient for such a uniform solution to be feasible in $N_+^r(G)$. We then show in Lemma 3.2 that these conditions are almost surely satisfied by a random graph $G_{n,1/2}$.

We will say that two vertices are *non-adjacent* if they are not adjacent and they are not equal (i.e. they are adjacent in the complement graph). We make no attempt to optimize constants.

Lemma 3.1. *Let G be a graph on n vertices, let $c = \sqrt{2}(1+\epsilon)^{10}$ for $0 < \epsilon < 1/5$ and let $r \geq 0$. Assume that for every $S \subset V$ with $|S| \leq r$, the graph $G' = G - S - \Gamma(S)$ satisfies (let n' denote the number of vertices in G'):*

(i) All eigenvalues of the adjacency matrix of $\overline{G'}$ are at least $-(1+\epsilon)\sqrt{n'}$.

(ii) The degree of every vertex in $\overline{G'}$ is between $\frac{1}{1+\epsilon}\frac{n'}{2}$ and $(1+\epsilon)\frac{n'}{2}$.

If $c^{r+1} \leq \epsilon\sqrt{n}$ then $(1/c^{r+1}\sqrt{n})\mathbf{1} \in N_+^r(G)$, and similarly for $N_{\text{FR}+}^r(G)$.

Proof. Proceed by induction on r. For the base case $r = 0$, observe that $(1/c^{r+1}\sqrt{n})\mathbf{1}$ (and even $(1/2)\mathbf{1}$) satisfies the nonnegativity and edge constraints and therefore is in $\text{FR}(G)$ by definition.

For the inductive step, assume it holds for $r \geq 0$, and let us show that it holds for $r+1$. Let G be a graph with (i),(ii) holding for any $|S| \leq r+1$, and $c^{r+2} \leq \epsilon\sqrt{n}$. We can choose, in particular, $|S| = 0$ and have that (i),(ii) hold for the graph G itself. To ease notation, define

$$\mu := (1+\epsilon)^5 (c^{r+1}/\sqrt{2})\sqrt{n} \tag{13}$$

Let A be the $n \times n$ adjacency matrix of $\overline{G}$, i.e. $A_{ij} = 0$ whenever $(i,j) \in E$ or $i = j$ and $A_{ij} = 1$ otherwise. We know from (i) that all eigenvalues of A are at least $-(1+\epsilon)\sqrt{n} \geq -\mu$. Hence, the matrix $B = A + \mu I$ is positive semidefinite, and there exist vectors $z_1, \ldots, z_n$ such that $B_{ij} = z_i^T z_j$. Therefore

$$\|z_i\|^2 = B_{ii} = \mu, \quad \forall i \geq 1. \tag{14}$$

Let $z_0 = \sum_{i=1}^n z_i$. Then

$$\|z_0\|^2 = (\sum_{i>0} z_i)^T (\sum_{j>0} z_j) = \sum_{i,j>0} B_{ij} = \sum_{i>0}\sum_{j>0} B_{ij}.$$

To estimate $\sum_{j>0} B_{ij} = \sum_{j>0} A_{ij} + \mu$ for $i > 0$, observe that we have from (ii) that

$$\frac{1}{1+\epsilon}\frac{n}{2} \leq \sum_{j>0} A_{ij} \leq (1+\epsilon)\frac{n}{2}$$

while $\mu \leq (c^{r+2}/2)\sqrt{n} \leq \epsilon n/2$. Hence,

$$\frac{1}{1+\epsilon}\frac{n}{2} \leq \sum_{j>0} B_{ij} \leq (1+\epsilon)^2\frac{n}{2}, \tag{15}$$

and we conclude that

$$\frac{1}{1+\epsilon}\frac{n^2}{2} \leq \|z_0\|^2 \leq (1+\epsilon)^2\frac{n^2}{2} \tag{16}$$

For every $i \geq 0$ let v_i be the unit length vectors in the direction of the vector z_i, i.e. $v_i = z_i/\|z_i\|$, and let $x_i = (v_i^T v_0)^2$. Observe that $x_0 = (v_0^T v_0)^2 = 1$.

We claim that $x = (x_1, \ldots, x_n)^T$ is in $N_+^{r+1}(G)$. Let us first show how the proof of Lemma 3.1 follows from this claim. Indeed, from (ii) we have that

$$v_i^T v_0 = (\frac{z_i}{\|z_i\|})^T(\frac{\sum_{j>0} z_j}{\|z_0\|}) = \frac{\sum_{j>0} B_{ij}}{\sqrt{\mu}\|z_0\|}$$

Together with (15) and (16) we can estimate $x_i = (v_i^T v_0)^2$ by

$$\frac{1}{(1+\epsilon)^4} \cdot \frac{1}{2\mu} \leq x_i \leq (1+\epsilon)^5 \frac{1}{2\mu} \tag{17}$$

and from (13) we have that

$$x_i \geq \frac{1}{2(1+\epsilon)^4} \cdot \frac{\sqrt{2}}{(1+\epsilon)^5 c^{r+1}\sqrt{n}} \geq \frac{1}{c^{r+2}\sqrt{n}}$$

and thus $(1/c^{r+2}\sqrt{n})\mathbf{1} \leq x \in N_+^{r+1}(G)$. By the monotonicity guaranteed in Lemma 2.1 we have $(1/c^{r+2}\sqrt{n})\mathbf{1} \in N_+^{r+1}(G)$, which indeed proves the inductive step.

We now prove the claim $x \in N_+^{r+1}(G)$, by presenting a matrix $Y \in M_+(N_+^r(G))$ whose 0th column corresponds to x. Indeed, let Y be the $(n+1)\times(n+1)$ matrix defined by $Y_{ij} = (v_i^T v_j)\sqrt{x_i x_j}$ for all $i, j \geq 0$. By definition, $Y_{i0} = (v_i^T v_0)\sqrt{x_i} = x_i$ for $i \geq 0$, and in particular $Y_{00} = x_0 = 1$. We will show that Y satisfies (a),(b),(c") and (d). Three of them are straightforward:

(a) Y is symmetric by definition.

(b) $Y_{ii} = \|v_i\|^2 x_i = x_i$ and hence $Y_{ii} = x_i = Y_{i0}$.

(d) Y is positive semidefinite because it can be represented by the vectors $\{\sqrt{x_i}v_i\}$, i.e. $Y_{ij} = (\sqrt{x_i}v_i)^T(\sqrt{x_j}v_j)$ for all $i, j \geq 0$.

Before proving (c"), observe that for $i, j > 0$ we have

$$Y_{ij} = (\frac{z_i}{\|z_i\|})^T(\frac{z_j}{\|z_j\|})\sqrt{x_i x_j} = (1/\mu)B_{ij}\sqrt{x_i x_j}$$

and B_{ij} is either μ, 0 or 1. So for $i, j > 0$ we have

$$Y_{ij} = \begin{cases} x_i & \text{if } i = j \\ 0 & \text{if } i \neq j \text{ and } ij \in E \\ (1/\mu)\sqrt{x_i x_j} & \text{if } i \neq j \text{ and } ij \notin E \end{cases}$$

and the estimate of (17) gives that $x_i \sim 1/2\mu$ and $\sqrt{x_i x_j} \sim 1/2\mu$. Hence,

$$Y = \left[\begin{array}{cccc} 1 & x_1 & \cdots & x_n \\ x_1 & x_1 & 0 \Big| \frac{\sqrt{x_i x_j}}{\mu} & \\ \vdots & & \ddots & \\ x_n & 0 \Big| \frac{\sqrt{x_i x_j}}{\mu} & & x_n \end{array} \right] \sim \left[\begin{array}{cccc} 1 & \frac{1}{2\mu} & \cdots & \frac{1}{2\mu} \\ \frac{1}{2\mu} & \frac{1}{2\mu} & 0 \Big| \frac{1}{2\mu^2} & \\ \vdots & & \ddots & \\ \frac{1}{2\mu} & 0 \Big| \frac{1}{2\mu^2} & & \frac{1}{2\mu} \end{array} \right]$$

Consider Ye_i, the ith column of Y, for $i > 0$, and scale it by a factor of $1/x_i$ so that its 0th entry will be 1. We get a fractional solution where vertex i has value 1, its adjacent vertices have value 0, and its non-adjacent vertices j have value $(1/\mu)\sqrt{x_j/x_i} \sim 1/\mu$. Let G' be the subgraph of G induced on the latter vertices (i.e. those non-adjacent to i), and let n' denote the number of vertices in G'. Then by Lemma 2.2, we have that the fractional solution Ye_i is in $N_+^r(G)$ if and only if its restriction to G' is in $N_+^r(G')$. Each coordinate in the fractional solution restricted to G' is bounded by

$$\frac{1}{\mu}\sqrt{\frac{x_j}{x_i}} \leq \frac{1}{\mu}(1+\epsilon)^{9/2} \leq \frac{\sqrt{2}}{c^{r+1}\sqrt{n(1+\epsilon)}} \leq \frac{1}{c^{r+1}\sqrt{n'}}$$

where the first inequality is due to (17), the second is due to (13), and the third follows from $n' \leq (1+\epsilon)\frac{n}{2}$ which we have from (ii). The fractional solution restricted to G' is thus dominated by the uniform solution $(1/c^{r+1}\sqrt{n'})\mathbf{1}$, which belongs to $N_+^r(G')$ by applying the induction hypothesis to G'. (Note that G' satisfies (i),(ii) for any $0 \leq |S| \leq r$ by definition, and that we have $c^{r+1} \leq \epsilon\sqrt{n}/c \leq \epsilon\sqrt{n'}$.) From the monotonicity guaranteed by Lemma 2.1, we conclude that also the fractional solution restricted to G' is in $N_+^r(G')$, and therefore $Ye_i \in N_+^r(G)$.

Consider Yf_i, the difference between column 0 and column i of Y, for $i > 0$. Its 0th entry is $1 - x_i \sim 1 - 1/2\mu$, its ith entry is 0, and any other jth entry is at most roughly $1/2\mu$. Observe that

$$x_i \leq \frac{(1+\epsilon)^5}{2\mu} \leq \frac{1}{\sqrt{2n}} \leq 1 - \frac{1}{\sqrt{2}} \tag{18}$$

where the first inequality is due to (17), the second is due to (13) and the third is due to $\sqrt{n} \geq 5\epsilon\sqrt{n} \geq 5c^{r+2} > 10$. Scaling the vector Yf_i by a factor $1/(1-x_i)$ so that its 0th entry is 1, we obtain a fractional solution in which the value of the jth entry is at most

$$\frac{x_j}{1-x_i} \leq \frac{(1+\epsilon)^5/2\mu}{1/\sqrt{2}} = \frac{1}{c^{r+1}\sqrt{n}}.$$

The fractional solution is thus dominated by $(1/c^{r+1}\sqrt{n})\mathbf{1}$, which by the induction hypothesis belongs to $N_+^r(G)$. (Note that G satisfies the requirements for r). From the monotonicity guaranteed by Lemma 2.1, (as all entries of Yf_i are nonnegative) we conclude that $Yf_i \in N_+^r(G)$.

We therefore have that (c") holds, which completes the proof of the inductive step and of Lemma 3.1.

Finally, let us show that the proof extends also to $N_{\mathrm{FR}+}^r(G)$. We need to consider also Yf_{ij} for $ij \in E$. The 0th entry of this vector is $1 - x_i - x_j \sim 1 - 2/2\mu$, the ith and jth entries are 0, and any other kth entry is either roughly $1/2\mu$ if k is adjacent to both i, j, or roughly $1/2\mu - 2/2\mu^2 \sim 1/2\mu$ if k is non-adjacent to both i, j, or roughly $1/2\mu - 1/2\mu^2 \sim 1/2\mu$ if k is adjacent to exactly one of i, j. Similar to (18) we have that

$$x_i + x_j \leq 2 \cdot \frac{1}{\sqrt{2n}} \leq 1 - \frac{1}{\sqrt{2}}.$$

Scaling this vector (by a small factor) so that the 0th entry is 1, we obtain a fractional solution in which the value of the kth entry is at most

$$\frac{x_k}{1-x_i-x_j} \le \frac{(1+\epsilon)^5/2\mu}{1/\sqrt{2}} = \frac{1}{c^{r+1}\sqrt{n}}.$$

The fractional solution is thus dominated by $(1/c^{r+1}\sqrt{n})\mathbf{1}$, which by the induction hypothesis belongs to $N_+^r(G)$. From the monotonicity guaranteed by Lemma 2.1, (as all entries of Yf_{ij} are nonnegative) we conclude that $Yf_{ij} \in N_+^r(G)$.

□

Lemma 3.2. *Let $\epsilon > 0$. Then there exists an $\epsilon' > 0$ that depends only on ϵ, such that for any $r \le \epsilon' \log n$, a random graph $G_{n,1/2}$ almost surely satisfies all the requirements of Lemma 3.1.*

Proof. Observe that a sufficiently small $\epsilon' > 0$ that depends on ϵ guarantees that $c^{r+1} \le \epsilon\sqrt{n}$ (we can assume, without loss of generality, that $\epsilon < 1/5$).

Consider a particular choice of S of size $s \le r$, and its corresponding graph $G'(V', E')$ (the subgraph of G induced on the vertices that are non-adjacent to all the vertices of S). The number of vertices in G', which we denote by $n' = |V'|$, has binomial distribution $B(n-s, 1/2^s)$. Since $s \le \log n \le n/4$, we have by Chernoff bound that

$$\mathbb{P}\left[n' \le n/2^{s+1}\right] \le 2^{-\delta_1 n/2^s} \tag{19}$$

for some fixed $\delta_1 > 0$.

G' is a random graph (with edge probability 1/2) on n' vertices. Therefore, the adjacency matrix of $\overline{G'}$ is a random symmetric matrix and we can use results on the concentration of its eigenvalues. In particular, we have from Krivelevich and Vu [KV00] (which improve the concentration shown by Füredi and Komlós [FK81], see also [AKV01]) that

$$\mathbb{P}\left[G' \text{ does not satisfy (i)}\right] \le 2^{-\delta_2 n'} \tag{20}$$

for some $\delta_2 > 0$ that depends on ϵ.

Since G' is a random graph, the degree of a particular vertex in $\overline{G'}$ has binomial distribution $B(n'-1, 1/2)$. By Chernoff bound and the union bound on the n' vertices we have that

$$\mathbb{P}\left[G' \text{ does not satisfy (ii)}\right] \le n' 2^{-\delta_3 n'} \tag{21}$$

for some fixed $\delta_3 > 0$ that depends on ϵ.

Using the union bound on the events of (20) and (21) we can bound the probability that G' does not satisfy (i) or (ii). In order to obtain a bound in terms of n (rather than n'), we add to the union bound also the event of (19) and have that for some fixed $\delta > 0$ that depends on ϵ,

$$\mathbb{P}\left[G' \text{ does not satisfy (i) or (ii)}\right] \le n 2^{-\delta n/2^s}$$

Taking the union bound on all possible sets S of size at most r, the probability that the requirements of Lemma 3.1 do not hold is at most

$$\sum_{s=0}^{r} \binom{n}{s} n 2^{-\delta n/2^s} \le r n^{r+1} 2^{-\delta n/2^r} \le n^{r+2} 2^{-\delta n/2^r} \ll 1$$

when $r \le \epsilon' \log n$ for a sufficiently small $\epsilon' > 0$ that depends on ϵ, and hence these requirements hold almost surely. □

The proof of Theorem 3.1 follows from Lemma 3.1 and Lemma 3.2.

3.2 Upper bound on the value of $N_+^r(G_{n,1/2})$

We prove Theorem 3.2 by showing that the inequality $\mathbf{1}^T x \le 4\sqrt{n}/d^{r+1}$ is almost surely valid for $N_+^r(G)$. First we exhibit in Lemma 3.3 certain conditions that are sufficient for this inequality to be valid for $N_+^r(G)$. We then show in Lemma 3.4 that these conditions are almost surely satisfied by a random graph $G_{n,1/2}$.

The Lovász theta function of a graph is defined as $\vartheta(G) := \max\{\mathbf{1}^T x : x \in TH(G)\}$, where $TH(G)$ is the solution set of the nonnegativity constraints (4) and the so-called orthogonality constraints (see [Lov79, GLS93] for definition). Lovász and Schrijver [LS91] show that the orthogonality constraints have N_+-rank at most 1, and hence $N_+(G) \subseteq TH(G)$.

Lemma 3.3. *Let G be a graph on n vertices, let $d = \sqrt{2}(1-\epsilon)$ for $0 < \epsilon < 1$ and let $r \ge 1$. Assume that for every $S \subset V$ with $|S| \le r$, the graph $G' = G - S - \Gamma(S)$ satisfies (let n' denote the number of vertices in G'):*

(i) $\vartheta(G') \le 2(1+\epsilon)\sqrt{n'}$.

(ii) The degree of every vertex in $\overline{G'}$ is between $\frac{1}{1+\epsilon}\frac{n'}{2}$ and $(1+\epsilon)\frac{n'}{2}$.

If $d^{r+1} \le \epsilon^2\sqrt{n}$ then $\max\{\mathbf{1}^T x : x \in N_+^r(G)\} \le 4\sqrt{n}/d^{r+1}$, and similarly for $N_{\mathrm{FR}+}^r$.

Proof. Proceed by induction on r. For the base case $r = 1$, we can choose $|S| = 0$ and then (i) and (ii) hold for the graph G itself. In particular, we have that

$$\max\{\mathbf{1}^T x : x \in N_+(G)\} \le \vartheta(G) \le 2(1+\epsilon)\sqrt{n} < 4\sqrt{n}/d^2$$

For the inductive step, assume it holds for $r \ge 1$ and let us show that it holds for $r+1$. In other words, given a graph G with (i),(ii) holding for any $|S| \le r+1$, we will prove that the inequality $\mathbf{1}^T x \le 4\sqrt{n}/d^{r+2}$ is valid for $N_+^{r+1}(G)$. By Lemma 2.3 we know that it suffices to prove that for every vertex v, the inequality that arises from the contraction of v, i.e. $\mathbf{1}^T x \le 4\sqrt{n}/d^{r+2} - 1$, is valid for $N_+^r(G - \Gamma(v) - v)$.

By the induction hypothesis for $G' = G - \Gamma(v) - v$ we have that $\max\{\mathbf{1}^T x : x \in N_+^r(G')\} \le 4\sqrt{n'}/d^{r+1}$, i.e. the inequality $\mathbf{1}^T x \le 4\sqrt{n'}/d^{r+1}$ is valid for $N_+^r(G')$. Since (ii) holds also for G itself, we have that $n' \le (1+\epsilon)\frac{n}{2}$, and hence

$$\frac{4\sqrt{n'}}{d^{r+1}} \le \frac{4\sqrt{n}}{d^{r+1}}\frac{\sqrt{1+\epsilon}}{\sqrt{2}} = \frac{4\sqrt{n}}{d^{r+2}}\sqrt{1+\epsilon}(1-\epsilon) \le \frac{4\sqrt{n}(1-\epsilon^2)}{d^{r+2}} \le \frac{4\sqrt{n}}{d^{r+2}} - 1$$

where the last inequality follows from $d^{r+2} \le 4\epsilon^2\sqrt{n}$. Therefore we have that for $N_+^r(G')$ the inequality $\mathbf{1}^T x \le 4\sqrt{n'}/d^{r+1} \le 4\sqrt{n}/d^{r+2} - 1$ holds, which completes the proof of the inductive step.

Finally, the proof immediately extends to the $N_{\mathrm{FR}+}^r$ operator since $N_{\mathrm{FR}+}^r(G) \subseteq N_+^r(G)$. □

Lemma 3.4. *Let $\epsilon > 0$. Then there exists an $\epsilon' > 0$ that depends only on ϵ, such that for any $r \le \epsilon' \log n$, a random graph $G_{n,1/2}$ almost surely satisfies all the requirements of Lemma 3.3.*

Proof. The proof is similar to that of Lemma 3.2, but with the different requirement (i). Juhász [Juh82] shows that $\vartheta(G')$ is at most $(2 + o(1))\sqrt{n'}$, almost surely, by using the result of Füredi and Komlós [FK81] on the concentration of eigenvalues of random symmetric matrices. By using the stronger concentration result of Krivelevich and Vu [KV00] (see also [AKV01]), we have that the analog of (20) holds, and the proof follows. □

The proof of Theorem 3.2 follows from Lemma 3.3 and Lemma 3.4.

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A Properties of the matrix-cut operators

A.1 Basic properties

We collect some properties of the matrix-cut operators defined in Section 2.1. In particular, we prove Lemmas 2.1 and 2.2 (that are used in Section 3.1).

Monotonicity. It is straightforward that the matrix-cut operators are monotone with respect to containment of K_1 and K_2, as follows.

Lemma A.1. *Let $K_1' \subseteq K_1$ and $K_2 \subseteq K_2'$. Then $N(K_1', K_2') \subseteq N(K_1, K_2)$ and similarly for N_+.*

For the stable set problem it follows that the matrix-cut operators are monotone with respect to adding/removing edges.

Corollary A.2. *Let G' be a graph that is obtained from another graph G by adding edges. Then $N^r(G') \subseteq N^r(G)$, and similarly for N^r_+,N^r_{FR},$N^r_{\mathrm{FR}+}$.*

Proof. Observe that $\mathrm{FR}(G') \subseteq \mathrm{FR}(G)$. The proof follows from Lemma A.1. $\square$

Down-monotonicity. The next lemma shows that down-monotonicity (see Section 2.2 for definition) is preserved by the matrix-cut operators. It extends a similar result that is given for $N(\cdot)$ and $N_+(\cdot)$ by Goemans and Tunçel [GT00, Theorem 5.1] (under the name lower-comprehensive) and by Cook and Dash [CD01, Lemma 2.6] (under the name anti-blocking type).

Lemma A.3. *Let $K_1, K_2 \subseteq Q$ be down-monotone convex cones. Then $N(K_1, K_2)$ is down-monotone, and similarly for N_+.*

Proof. Let $x \in N(K_1, K_2)$ and $0 \le x' \le x$ with $x'_0 = x_0$. It suffices to prove that $x' \in N(K_1, K_2)$ when x, x' differ only in a single coordinate, say $i = 1$, since we can repeat the same argument for each coordinate. Furthermore, for a single coordinate $i = 1$ it suffices to prove the case $x'_1 = 0$, since $N(K_1, K_2)$ is convex, and so convex combinations of x' and x give any desired value in coordinate $i = 1$.

Since $x \in N(K_1, K_2)$, there exists a matrix $Y \in M(K_1, K_2)$ with $x = Ye_0$. Define the matrix Y' by

$$Y'_{ij} = \begin{cases} 0 & \text{if } i = 1 \text{ or } j = 1; \\ Y_{ij} & \text{otherwise.} \end{cases}$$

We claim that $Y' \in M(K_1, K_2)$. Indeed, Y' clearly satisfies (a) and (b). To prove (c), let $u \in K_1^*, v \in K_2^*$, and from Proposition A.4 below we have that $u - u_1 x_1 \in K_1^*$ and $v - v_1 x_1 \in K_2^*$ and hence

$$u^T Y' v = (u - u_1 x_1)^T Y (v - v_1 x_1) \ge 0$$

Observe that $x' = Y' e_0$, and therefore $x' \in N(K_1, K_2)$, as required.

For the proof of N_+ we need to show that (d) also holds, and indeed from the Gram matrix representation of Y we can obtain a Gram matrix representation of Y' by replacing the vector that corresponds to coordinate $i = 1$ with the all zeros vector $\mathbf{0}$. $\square$

Proposition A.4. *Let $K \subseteq Q$ be down-monotone and let $v \in K^*$. Then $v - v_i e_i \in K^*$ for all $i \ge 1$.*

Proof. By the down-monotonicity of K, for every $x \in K$ we have that $x - x_i e_i \in K$, and hence $(v - v_i e_i)^T x = \sum_{j \ne i} v_j x_j = v^T (x - x_i e_i) \ge 0$. $\square$

We can now prove Lemma 2.1, i.e. show that $N^r(G)$ is down-monotone for every $r \ge 0$, and similarly for N_+^r,N_{FR}^r,$N_{\mathrm{FR}+}^r$.

Proof of Lemma 2.1. Observe that Q is down-monotone by its definition (3), and that FRAC is down-monotone by its definition (6)-(7). By Lemma A.3 the matrix-cut operators preserve down-monotonicity and the proof follows. $\square$

Flipping and renaming coordinates. The operators $N, N_+, N_{\mathrm{FR}}, N_{\mathrm{FR}+}$ are invariant under various operations, including *renaming coordinates* (i.e. permuting the order of coordinates), and *flipping coordinates* $x_i \to (x_0 - x_i)$ for any subset of the coordinates $\{1, 2, \ldots, n\}$. More formally,

Lemma A.5 (Lovász and Schrijver [LS91]). *Let A be a linear transformation mapping Q onto itself. Then $N(AK_1, AK_2) = AN(K_1, K_2)$ and similarly for N_+. Hence $N(AK) = AN(K)$ and similarly for N_+.*

By flipping coordinates, one can extend Lemma A.3. For example, it follows that the N and N_+ operators preserve up-monotonicity, see Cook and Dash [CD01, Section 2] (as the blocking property) and Goemans and Tunçel [GT00, Section 5] (as the "convex corner" property).

Intersection with faces. A *face* of Q is the intersection of Q with hyperplanes of the form $\{x : x_i = 0\}$ or $\{x : x_i = x_0\}$. The intersection of K with a face of Q consists of all $x \in K$ with one or more of their coordinates *fixed* to 0 or x_0 (recall that x_0 corresponds to 1 in the non-homogenous case).

The following lemma proves equivalence between fixing some coordinates before applying a matrix-cut operator (e.g. in K) and afterwards (e.g. in $N(K)$). It extends a similar result that is given by Goemans and Tunçel [GT00] for $N(\cdot)$ and $N_+(\cdot)$.

Lemma A.6. *If F is a face of Q, then $N(K_1 \cap F, K_2) = N(K_1, K_2) \cap F$ and similarly for N_+.*

Proof. The direction "$\subseteq$" follows from Lemma A.1, since $N(K_1 \cap F, K_2) \subseteq N(K_1, K_2)$ and $N(K_1 \cap F, K_2) \subseteq N(F, K_2) \subseteq F$, and similarly for N_+.

For the converse direction "$\supseteq$" with the N operator, let $x \in N(K_1, K_2) \cap F$. Then there exists a matrix $Y \in M(K_1, K_2)$ with $Ye_0 = x$. Let H be any one of the hyperplanes of the form $\{x : x_i = 0\}$ or $\{x : x_i = x_0\}$ that define F. Since $e_j, f_j \in Q^* \subseteq K_2^*$ for all j, we have that $Ye_j \in K_1 \subseteq Q$ and $Yf_j \in K_1 \subseteq Q$, while their sum satisfies $Ye_j + Yf_j = Ye_0 = x \in F \subset H$. Since H defines a face of Q then by definition of a face we have that Ye_j (and also Yf_j) must belong to H.[1] But every $v \in \mathbb{R}^{n+1}$ is a linear combination of $\{e_0, e_1, \ldots, e_n\}$ and $Ye_j \in H$ for all $j \geq 0$, and so $Yv \in H$ for every v, including all $v \in K_2^*$.

For every $v \in K_2^*$ we have that Yv belongs to $K_1 \subseteq Q$, by the definition of Y. We saw above that Yv also belongs to all hyperplanes H that define F, and we conclude that Yv belongs also to F. Hence, $Yv \in K_1 \cap F$ for all $v \in K_2^*$, implying that $Y \in M(K_1 \cap F, K_2)$ and $x \in N(K_1 \cap F, K_2)$. The proof for N_+ is similar, since Y is also known to be positive semidefinite. $\square$

We remark that the above proof of Lemma A.6 extends to the case where F is a face of K_1, as shown by Cook and Dash [CD01, Lemma 2.2] for $N(\cdot)$ and $N_+(\cdot)$. For the special cases $K_2 = Q$ and $K_2 = \mathrm{FR}$ we obtain the following.

Corollary A.7. *If F is a face of Q (or a face of K), then $N(K \cap F) = N(K) \cap F$ and similarly for N_+,N_{FR},$N_{\mathrm{FR}+}$.*

Deleting fixed coordinates. Suppose that K is contained in a face of Q. Then some of the coordinates are fixed (i.e. $x_i = 0$ or $x_i = x_0$), and it may be desirable to delete these coordinates and reduce the dimension. Formally, a *deletion* operation of indices subset $I \subset \{1, \ldots, n\}$ is the function $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1-|I|}$ where $f(x)$ is the vector x restricted to the coordinates not in I, i.e. $f(x) = (x_i)_{i \notin I}$.

For the stable set problem it is straightforward that the effect of fixing and deleting a coordinate of $\mathrm{FR}(G)$ is as follows.

Lemma A.8. *Let $F = Q \cap \{x : x_i = 0\}$, and let f be the deletion operation of coordinate i. Then $f(\mathrm{FR}(G) \cap F) = \mathrm{FR}(G - i)$.*

Lemma A.9. *Let $F = Q \cap \{x : x_i = x_0\}$, and let f be the deletion operation of coordinate i. Then $f(\mathrm{FR}(G) \cap F) = \mathrm{FR}(G - i) \cap \{x : \forall j \in \Gamma(i), x_j = 0\}$.*

[1] In other words, suppose that the hyperplane H is defined by the equality $u^T x = 0$ (with $u = e_i$ or $u = f_i$) and that the inequality $u^T x \geq 0$ is valid for Q (i.e. Q is entirely contained in one side of H). We then have that $u^T(Ye_j), u^T(Yf_j) \geq 0$ while their sum is $u^T x = 0$, implying that $u^T(Ye_j) = u^T(Yf_j) = 0$.

We show below that deleting fixed coordinates of K before applying a matrix-cut operator (e.g. in K) is equivalent to deleting them afterwards (e.g. in $N(K)$). This extends similar results that are given for $N(\cdot)$ and $N_+(\cdot)$ by Cook and Dash [CD01] (see also [ST99]). Technically, they consider an embedding operation (that introduces new coordinates that are fixed to either 0 or x_0), which is just the inverse of the deletion operation.

We first handle the basic case of one coordinate that is fixed to 0 (Lemma A.10), then extend the result to an arbitrary face F and to an arbitrary K_2 (Lemma A.11), and finally specialize it to the cases $K_2 = Q$ and $K_2 = \mathrm{FR}$ (Corollary A.12).

Lemma A.10. *Let $F = Q \cap \{x : x_n = 0\}$ and let f be the deletion operation of coordinate n. If $K_1, K_2 \subseteq F$ are convex cones then $f(N(K_1, K_2)) = N(f(K_1), f(K_2))$,[2] and similarly for N_+.*

Proof. The deletion operation f is a linear transformation from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$, and thus can be described as an $n \times (n+1)$ matrix A. Note that columns 0 to $n-1$ of A form an identity matrix and column n of A is all zeros. We first claim that $AK^* = (AK)^*$ for $K = K_1$ and for $K = K_2$. Indeed, by definition, $u \in AK^*$ if there exists $r \in \mathbb{R}$ with $\begin{pmatrix} u \\ r \end{pmatrix} \in K^*$. Note that $\begin{pmatrix} u \\ r \end{pmatrix} \in K^*$ holds either for all values of r or for no value of r, since $K \subset \{x : x_n = 0\}$. Therefore,

$$AK^* = \{u : \exists r \in \mathbb{R} \text{ with } \begin{pmatrix} u \\ r \end{pmatrix} \in K^*\} = \{u : \begin{pmatrix} u \\ 0 \end{pmatrix} \in K^*\}.$$

We also have that

$$(AK)^* = \{u : u^T(Ax) \geq 0 \quad \forall x \in K\} = \{u : A^T u \in K^*\}.$$

Since $A^T u = \begin{pmatrix} u \\ 0 \end{pmatrix}$, we obtain $AK^* = (AK)^*$.

Let us now prove that $M(AK_1, AK_2) = AM(K_1, K_2)A^T$. For the direction "$\subseteq$", let $Y \in M(AK_1, AK_2)$. Then by (c), for every $u \in K_1^*, v \in K_2^*$ we have that $u^T A^T Y A v \geq 0$. We therefore have that

$$\begin{pmatrix} Y & \mathbf{0} \\ \mathbf{0}^T & 0 \end{pmatrix} = A^T Y A \in M(K_1, K_2).$$

Multiplying by A from the left and by A^T from the right, we obtain (since AA^T is the identity matrix) that $Y \in AM(K_1, K_2)A^T$.

For the converse direction "$\supseteq$", let $Y \in AM(K_1, K_2)A^T$. Since $K_1 \subseteq \{x : x_n = 0\}$, every matrix in $M(K_1, K_2)$ has only zeros in row n, and by the symmetry (a) it has only zeros also in column n. Hence,

$$A^T Y A = \begin{pmatrix} Y & \mathbf{0} \\ \mathbf{0}^T & 0 \end{pmatrix} \in M(K_1, K_2).$$

By (c), for every $u \in K_1^*, v \in K_2^*$ it holds that $u^T A^T Y A v \geq 0$, and hence $Y \in M(AK_1, AK_2)$.

Now since $A^T e_0$ is just e_0 (in a larger dimension), we conclude that

$$N(AK_1, AK_2) = AM(K_1, K_2)A^T e_0 = AM(K_1, K_2)e_0 = AN(K_1, K_2).$$

The proof for the N_+ operator is similar since Y is positive semidefinite if and only if $A^T Y A$ is (observe that Y has a Gram matrix representation if and only if $A^T Y A$ has such a representation). $\square$

[2]Note that the application of N in the righthand side is in a smaller dimension than in the lefthand side.

Lemma A.11. *Let $F = Q \cap \{x : \forall i \in I_0, x_i = 0\} \cap \{x : \forall i \in I_1, x_i = x_0\}$ and let f be the deletion operation of the coordinates $I_0 \cup I_1$. If $K_1 \subseteq F$ and $K_2 \subseteq Q$ are convex cones, then $f(N(K_1, K_2)) = N(f(K_1), f(K_2 \cap F))$, and similarly for N_+.*

Proof. K_1 and $K_2 \cap F$ are both contained in F, so we can repeatedly apply Lemma A.10 on them, and delete the coordinates of $I_0 \cup I_1$. (Note that by using Lemma A.5 we can extend Lemma A.10 also to deleting coordinates that are fixed to x_0.) It follows that $f(N(K_1, K_2 \cap F)) = N(f(K_1), f(K_2 \cap F))$.

By Lemma A.6 we have that $N(K_1, K_2 \cap F) = N(K_1, K_2) \cap F$, and since $N(K_1, K_2) \subseteq K_1 \subseteq F$, we have that $N(K_1, K_2 \cap F) = N(K_1, K_2)$. The proof follows. □

Corollary A.12. *Let $F = Q \cap \{x : \forall i \in I_0, x_i = 0\} \cap \{x : \forall i \in I_1, x_i = x_0\}$ and let f be the deletion operation of the coordinate $I_0 \cup I_1$. If $K \subseteq F$ is a convex cone then $f(N(K)) = N(f(K))$,[3] and similarly for N_+, N_{FR} and $N_{\mathrm{FR}+}$.*

Proof. For the N operator we have from Lemma A.11 that

$$f(N(K)) = N(f(K), f(Q \cap F))$$

and $f(Q \cap F)$ is just Q in the smaller dimension, so $f(N(K)) = N(f(K))$. The proof for the N_+ operator is similar.

For the N_{FR} operator we have from Lemma A.11 that

$$f(N_{\mathrm{FR}}(K)) = N(f(K), f(\mathrm{FR}(G) \cap F)),$$

and it follows from Lemmas A.8 and A.9 that $f(\mathrm{FR}(G) \cap F) = \mathrm{FR}(G - I_0 - I_1) \cap H$, where $H = \{x : x_i = 0 \ \forall i \in \Gamma(I_1) - I_0 - I_1\}$. We therefore have that

$$f(N_{\mathrm{FR}}(K)) = N(f(K), \mathrm{FR}(G - I_0 - I_1) \cap H).$$

Note that $f(K) \subset H$ since $K \subseteq F \cap \mathrm{FR}(G) \subseteq H$, and so by Lemma A.6 we have that $f(N_{\mathrm{FR}}(K)) = N_{\mathrm{FR}}(f(K))$, as required. The proof for $N_{\mathrm{FR}+}(K)$ is similar. □

Removing vertices from the graph. For the stable set problem, the properties collected so far, and in particular Corollary A.12, give a useful characterization to whether $x \in N^r(G)$ in the case that x has a fixed coordinate (i.e. $x_i = 0$ or $x_i = x_0$).

Recall that $V = \{1, \ldots, n\}$. For a vector $x \in \mathbb{R}^n$ and a subset $W \subset V$, we denote by x_W the restriction of x to the coordinates of W. We can now prove Lemma 2.2, showing that if $x \in \mathbb{R}^n$ with $x_i = 1$ and $x_j = 0$ for all $j \in \Gamma(i)$, then for all $r \geq 0$, $x \in N^r(G)$ if and only if $x_{V-\Gamma(i)-i} \in N^r(G - \Gamma(i) - i)$, and similarly for N_+^r, N_{FR}^r and $N_{\mathrm{FR}+}^r$.

Proof of Lemma 2.2. It is clear that x belongs to the face F of Q that is defined by the hyperplanes $\{x : x_i = x_0\}$ and $\{x : x_j = 0\}$ for all $j \in \Gamma(i)$. Then $x \in N^r(G)$ if and only if $x \in N^r(G) \cap F$, which is equivalent, by Corollary A.7, to $x \in N^r(\mathrm{FR}(G) \cap F)$. Let f be the deletion operation of the coordinates $\Gamma(i) \cup \{i\}$, and then we have equivalently that $f(x) \in f(N^r(\mathrm{FR}(G) \cap F))$. By Corollary A.12, the latter is equivalent to $f(x) \in N^r(f(\mathrm{FR}(G) \cap F))$. By Lemmas A.8 and A.9, we have that $f(\mathrm{FR}(G) \cap F) = \mathrm{FR}(G - \Gamma(i) - i)$, and the proof follows. The proof for N_+^r, N_{FR}^r and $N_{\mathrm{FR}+}^r$ is similar. □

Lemma A.13. *Let $x \in \mathbb{R}^n$ be a vector and assume that $x_i = 0$ for some i. Then $x \in N^r(G)$ if and only if $x_{V-i} \in N^r(G - i)$, and similarly for N_+^r, N_{FR}^r and $N_{\mathrm{FR}+}^r$.*

[3] Note that the application of N in the righthand side is in a smaller dimension than in the lefthand side.

Proof. It is clear that x belongs to the face F of Q that is defined by the hyperplane $x_i = 0$. Then $x \in N^r(G)$ if and only if $x \in N^r(G) \cap F$, which is equivalent, by Corollary A.7, to $x \in N^r(\mathrm{FR}(G) \cap F)$. Let f be the deletion operation of the coordinate i, and then we have equivalently that $f(x) \in f(N^r(\mathrm{FR}(G) \cap F))$. By Corollary A.12, the latter is equivalent to $f(x) \in N^r(f(\mathrm{FR}(G) \cap F))$. By Lemma A.8 we have that $f(\mathrm{FR}(G) \cap F) = \mathrm{FR}(G - i)$, and the proof follows. The proof for N^r_+,N^r_{FR} and $N^r_{\mathrm{FR}+}$ is similar. $\square$

A.2 Bounds on the rank

We describe some general methods to obtain upper and lower bounds on the N-rank and N_+-rank of valid inequalities, and extend them to the N_{FR}-rank. We also illustrate the use of these methods on a few valid constraints for the stable set problem (see Table 1 on page 27).

The N-rank of an inequality valid for $\mathrm{STAB}(G)$ depends only on the subgraph induced by those vertices with a nonzero coefficient, and similarly for N_+,N_{FR} and $N_{\mathrm{FR}+}$. Indeed, if a vertex i has a zero coefficient, then the inequality being valid for $N^r(G)$ is equivalent, by Lemma 2.1, to the inequality being valid for $N^r(G) \cap \{x : x_i = 0\}$, which in turn is equivalent, by Lemma A.13, to the inequality being valid for $N^r(G - i)$.

Upper bounds on the N-rank. Lovász and Schrijver [LS91] give an upper bound on $N(K)$, which allows to upper bound the N-rank of an inequality, as follows.

The *sum* of two sets $K', K'' \subseteq \mathbb{R}^{n+1}$ is defined as $K' + K'' := \{x' + x'' : x \in K', x'' \in K''\}$. Note that if K', K'' are convex cones in Q then $K' + K''$ is also a convex cone in Q. Furthermore, if K', K'' are obtained via the homogenization procedure (1)-(2) from polytopes $P', P'' \subseteq \mathbb{R}^n$, respectively, then $K' + K''$ corresponds to all convex combinations of a point from P' and a point from P'' (recall that x_0 needs to be scaled to 1).

Lemma A.14 (Lovász and Schrijver [LS91]). *For all* $1 \le i \le n$,

$$N(K) \subseteq \Big(K \cap \{x : x_i = 0\}\Big) + \Big(K \cap \{x : x_i = x_0\}\Big).$$

Proof. If $x \in N(K)$ then there exists $Y \in M(K)$ with $x = Ye_0 = Ye_i + Yf_i$ for any $i \le i \le n$. Clearly, $Ye_i \in K \cap \{x : x_i = x_0\}$ and $Yf_i \in K \cap \{x : x_i = 0\}$, and the proof follows. $\square$

Corollary A.15. *If an inequality is valid for both* $K \cap \{x : x_i = 0\}$ *and* $K \cap \{x : x_i = x_0\}$*, then it is valid for* $N(K)$*.*

Goemans and Tunçel [GT00] note that repeatedly using Lemma A.14 and Corollary A.7, gives that for all $I \subseteq \{1, \dots, n\}$ with $|I| = r$,

$$N^r(K) \subseteq \sum_{I_0 \subseteq I} \Big(K \cap \{x : \forall i \in I_0, x_i = 0\} \cap \{x : \forall i \in I \setminus I_0, x_i = x_0\}\Big).$$

In particular, this shows that the N-rank of any cone K is at most n, proving Theorem 2.1.

For the stable set problem, Corollary A.15 can be rephrased as follows (using Lemmas 2.2 and A.13).

Lemma A.16 (Lovász and Schrijver [LS91]). *Let P be a convex set with* $\mathrm{STAB} \subseteq P \subseteq \mathrm{FRAC}$. *If $a^T x \le b$ is an inequality such that for some $i \in V$, both the deletion and contraction of i give an inequality valid for P, then $a^T x \le b$ is valid for $N(P)$.*

For example, if C induces a chordless odd cycle in G, the *odd hole constraint*

$$\sum_{i \in C} x_i \leq \frac{|C|-1}{2} \tag{22}$$

has N-rank at most (and actually exactly) 1, because both the contraction and the deletion of any vertex result in an inequality that is valid for FRAC. (In fact, Lovász and Schrijver [LS91] prove that $N(\text{FRAC})$ is exactly the relaxation that is obtained by adding to FRAC all the odd hole constraints.)

Lovász and Schrijver [LS91] also give the following upper bound on the N-rank of a graph. The proof follows by applying Lemma A.16 repeatedly for $n - \alpha(G) - 1$ vertices outside a maximum stable set in the graph, since the graph induced on the other vertices must be bipartite.

Corollary A.17 (Lovász and Schrijver [LS91]). *The N-rank of a graph G with stability number $\alpha(G)$ is at most $n - \alpha(G) - 1$.*

It follows that the N-rank of any graph G is at most $n-2$. Note that the N-rank of FR is at most $n-2$, while the N-rank of a general cone K is at most (and can actually be) n.

We next analyze the N-rank of a few more examples, due to Lovász and Schrijver [LS91]. By Corollary A.17, if B is a clique in G, the *clique constraint*

$$\sum_{i \in B} x_i \leq 1 \tag{23}$$

has N-rank at most (and actually exactly) $|B|-2$. Note that the class of all clique constraints strengthens the class of all edge constraints (5).

If D induces a chordless odd cycle in $\overline{G}$ (the edge complement of G), the *odd antihole constraint*

$$\sum_{i \in D} x_i \leq 2 \tag{24}$$

has N-rank at most (and actually exactly) $(|D|-3)/2$, because the contraction of a vertex results in an inequality trivially valid for FRAC, and the deletion of a vertex results in an inequality that is the sum of two clique constraints, each of size $(|D|-1)/2$ and hence of N-rank $(|D|-5)/2$.

If W induces an odd wheel in G with center $i_0 \in W$, the *odd wheel constraint*

$$\sum_{i \in W \setminus \{i_0\}} x_i + \frac{|W|-2}{2} x_{i_0} \leq \frac{|W|-2}{2} \tag{25}$$

has N-rank at most (and actually exactly) 2, since the contraction of the center vertex results in a trivial inequality, and the deletion of the center vertex results with the odd hole constraint.

Upper bounds on the N_{FR}-rank. The methods for obtaining upper bounds on the N-rank can be extended (with modifications) to upper bounds on the N_{FR}-rank, as follows.

Lemma A.18. *For all $ij \in E$,*

$$N(K) \subseteq \Big(K \cap \{x : x_i = x_j = 0\}\Big) + \Big(K \cap \{x : x_j = x_0\}\Big) + \Big(K \cap \{x : x_i = x_0\}\Big).$$

Proof. If $x \in N_{\text{FR}}(K)$ then there exists $Y \in M(K)$ with $x = Ye_0 = Ye_i + Ye_j + Yf_{ij}$ for any $ij \in E$. Clearly, $Ye_i \in K \cap \{x : x_i = x_0\}$ and $Ye_j \in K \cap \{x : x_j = x_0\}$ and $Yf_{ij} \in K \cap \{x : x_i = x_j = 0\}$, and the proof follows. ☐

Corollary A.19. *Let $ij \in E$. If an inequality is valid for $K \cap \{x : x_i = x_0\}$, for $K \cap \{x : x_j = x_0\}$ and for $K \cap \{x : x_i = x_j = 0\}$, then it is valid for $N_{\mathrm{FR}}(K)$.*

Corollary A.19 can be rephrased as follows (using Lemmas 2.2 and A.13).

Lemma A.20. *Let P be a convex set with $\mathrm{STAB} \subseteq P \subseteq \mathrm{FRAC}$. If $a^T x \le b$ is an inequality such that for some $ij \in E$, the contraction of i, the contraction of j, and the deletion of $\{i, j\}$ give an inequality valid for P, then $a^T x \le b$ is valid for $N(P)$.*

The following upper bound on the N_{FR}-rank of a graph follows by applying Lemma A.20 repeatedly on edges, so that the removal of their endpoints results in a bipartite graph (e.g. a matching that is maximal with respect to containment).

Corollary A.21. *Suppose that a graph G contains a set of β edges, whose endpoints removal results in a bipartite graph. Then the N_{FR}-rank of G is at most β.*

It follows that the N_{FR}-rank of a graph G is at most $(n-2)/2$ if n is even and $(n-1)/2$ if n is odd; in general it is at most $\lfloor (n-1)/2 \rfloor$. In particular, the N_{FR}-rank of the clique constraint (23) is at most $\lfloor (|B|-1)/2 \rfloor$.

We can apply these bounds to the other examples. The N_{FR}-rank of the odd hole constraint (22) is at most (and thus exactly) 1, since the N_{FR} operator is at least as strong as N. The N_{FR}-rank of the odd antihole constraint (24) is at most $\lfloor (|D|+1)/4 \rfloor$, because the contraction of a vertex results in an inequality trivially valid for FRAC, and the deletion of two vertices results in an inequality that is the sum of two clique constraints, each of size at most $(|D|-1)/2$ and hence of N_{FR}-rank $\lfloor (|D|-3)/4 \rfloor$.[4] The N_{FR}-rank of the wheel constraint (25) is at most (and thus exactly) 1, since the contraction of the center vertex results in a trivial inequality, the contraction of a non-center vertex results in an inequality that is valid for FRAC, and the deletion of these two vertices also results in an inequality that is valid for FRAC.

Lower bounds on the N-rank. Lovász and Schrijver [LS91] show that certain uniform fractional stable sets belong to $N^r(G)$, regardless of the graph G. For example, for $r = 0$ it is straightforward that $(1/2)\mathbf{1} \in \mathrm{FRAC}(G)$. The following lemma allows to extend this to larger r, with the uniform solution being smaller, depending on r.

Lemma A.22 (Lovász and Schrijver [LS91]). *Assume that P is down-monotone and contains $\mathrm{STAB}(G)$. If $(1/r)\mathbf{1} \in P$ for $r > 0$ then $1/(r+1)\mathbf{1} \in N(P)$.*

Proof. Let K be the convex cone obtained from P via the homogenization procedure (1)-(2). Define the matrix $Y \in \mathbb{R}^{(n+1)\times(n+1)}$ by

$$Y_{ij} = \begin{cases} 1 & \text{if } i = j = 0; \\ 1/(r+1) & \text{if } (i = 0, j > 0) \text{ or } (i > 0, j = 0) \text{ or } (i = j > 0); \\ 0 & \text{otherwise.} \end{cases}$$

To see that $Y \in M(K, Q)$ observe that (a),(b) clearly hold, and let us now show that (c") holds.

$$Y e_i = \frac{1}{t+1}(e_0 + e_i) \in \mathrm{ST}(G) \subseteq K$$

and

$$Y f_i = \frac{r}{r+1} e_0 + \sum_{j \ne 0,i} \frac{1}{r+1} e_j = \frac{r}{r+1} \left(e_0 + \sum_{j \ne 0,i} \frac{1}{r} e_j \right).$$

[4] In fact, direct calculations show that the N_{FR}-rank of the odd antihole constraint (24) with $|D| = 7$ is at most 1.

By the induction hypothesis we have that

$$\sum_{j\neq 0,i} \frac{1}{r} e_j \leq \sum_{j\neq 0} \frac{1}{r} e_j \in P,$$

and the down-monotonicity of P implies that $Yf_i \in K$, and thus (c") holds. We conclude that $Ye_0 \in N(K)$, i.e. $1/(r+1)\mathbf{1} \in N(P)$. □

Corollary A.23 (Lovász and Schrijver [LS91]). *$1/(r+2)\mathbf{1} \in N^r(G)$ for all $r \geq 0$.*

Proof. Proceed by induction on r. We mentioned above that the case $r = 0$ is trivial. The inductive step follows from Lemma A.22, since $N^r(\mathrm{FRAC}(G))$ clearly contains $\mathrm{STAB}(G)$ and is down-monotone by Lemma 2.1. □

Corollary A.24 (Lovász and Schrijver [LS91]). *The N-rank of a graph G with stability number $\alpha(G)$ is at least $n/\alpha(G) - 2$.*

Proof. Let r be the N-rank of G, and hence $N^r(G) = \mathrm{STAB}(G)$. By Corollary A.23 we have that $1/(r+2)\mathbf{1} \in N^r(G)$. The inequality $\mathbf{1}^T x \leq \alpha$ is valid for $\mathrm{STAB}(G) = N^r(G)$, and in particular for $1/(r+2)\mathbf{1}$, implying that $n/(r+2) \leq \alpha(G)$, and the proof follows. □

For example, the stability number of a clique B is 1, so the N-rank of B is at least, and hence exactly, $|B| - 2$. In fact, the above proof shows that the N-rank of the clique constraint (23) is at least, and hence exactly, $|B| - 2$. The stability number of an an odd antihole D is 2, so the N-rank of D is at least $|D|/2 - 2$, and since $|D|$ is odd, it must be at least $(|D| - 3)/2$. In fact, this shows that the N-rank of the odd antihole constraint (24) is at least, and hence extacly, $(|D| - 3)/2$. Corollary A.23 also yields a lower bound on the N-rank of the wheel constraint (25). Indeed, let r be the N-rank of this constraint. Then we have that this constraint is valid for $N^r(G)$ and, in particular, for $1/(r+2)\mathbf{1} \in N^r(G)$. Thus,

$$\frac{1}{r+2}\left(|W| - 1 + \frac{|W|-2}{2}\right) \leq \frac{|W|-2}{2}$$

which gives that $\frac{2(|W|-1)}{|W|-2} + 1 \leq r + 2$ and thus $r \geq 1 + \frac{2}{|W|-2}$. Since the N-rank of the wheel constraint is an integer, it must be at least, and hence exactly, 2.

Lower bounds on the N_{FR}-rank. The methods for obtaining lower bounds on the N-rank can be extended (with modifications) to lower bounds on the N_{FR}-rank, as follows.

Lemma A.25. *Assume that P be down-monotone and contains $\mathrm{STAB}(G)$. If $(1/r)\mathbf{1} \in P$ for $r > 0$ then $1/(r+2)\mathbf{1} \in N_{\mathrm{FR}}(P)$.*

Proof. Define the matrix $Y \in \mathbb{R}^{(n+1)\times(n+1)}$ by

$$Y_{ij} = \begin{cases} 1 & \text{if } i = j = 0; \\ 1/(r+2) & \text{if } (i = 0, j > 0) \text{ or } (i > 0, j = 0) \text{ or } (i = j > 0); \\ 0 & \text{otherwise.} \end{cases}$$

To see that $Y \in M(K, FR)$ observe that (a),(b) clearly hold, and let us now show that (c") holds.

$$Ye_i = \frac{1}{r+2}(e_0 + e_i) \in \mathrm{ST}(G) \subseteq K$$

and for $ij \in E$

$$Yf_{ij} = \frac{r}{r+2}e_0 + \sum_{l \neq 0,i,j} \frac{1}{r+2}e_l = \frac{r}{r+2}\left(e_0 + \sum_{l \neq 0,i,j} \frac{1}{r}e_l\right)$$

By the induction hypothesis we have that

$$\sum_{l \neq 0,i,j} \frac{1}{r}e_l \leq \sum_{l \neq 0} \frac{1}{r}e_l \in P$$

and the down-monotonicity of P implies that $Yf_{ij} \in K$, and thus (c") holds. We conclude that $Ye_0 \in N_{\mathrm{FR}}(K)$, i.e. $1/(r+2)\mathbf{1} \in N_{\mathrm{FR}}(P)$. □

Corollary A.26. *$1/(2r+2)\mathbf{1} \in N^r_{\mathrm{FR}}(G)$ for all $r \geq 0$.*

Proof. Proceed by induction on r. We mentioned above that the case $r = 0$ is trivial. The inductive step follows from Lemma A.25, since $N^r_{\mathrm{FR}}(\mathrm{FRAC}(G))$ clearly contains $\mathrm{STAB}(G)$ and is down-monotone by Lemma 2.1. □

Corollary A.27. *The N_{FR}-rank of a graph G with stability number $\alpha(G)$ is at least $n/(2\alpha(G))-1$.*

Proof. Let r be the N-rank of G, and hence $N^r(G) = \mathrm{STAB}(G)$. By Corollary A.26 we have that $1/(r+2)\mathbf{1} \in N^r(G)$. The inequality $\mathbf{1}^T x \leq \alpha(G)$ is valid for $\mathrm{STAB}(G) = N^r(G)$, and in particular for $1/(r+2)\mathbf{1}$, implying that $n/(2r+2) \leq \alpha(G)$, and the proof follows. □

For example, the N_{FR}-rank of a clique B is at least $|B|/2-1$ (since the stability number of B is 1), and it must be an integer, so we have that it is at least $\lfloor(|B|-1)/2\rfloor$. In fact, the above proof shows that the N_{FR}-rank of the clique constraint (23) is at least, and hence exactly, $\lfloor(|B|-1)/2\rfloor$. The N_{FR}-rank of an odd antihole D is at least $|D|/4-1$ (since the stability number of D is 2), and it must be an integer (while $|D|$ is odd), so we have that it is at least $\lfloor|D|/4\rfloor$. In fact, this shows that the N-rank of the odd antihole constraint (24) is at least $\lfloor|D|/4\rfloor$.

Upper bounds on the N_+-rank. Lovász and Schrijver [LS91] give also a sufficient condition for an inequality to be valid for $N_+(K)$. The following lemma considers an inequality $u^T x \geq 0$ with $u_0 \geq 0$ and $u_i \leq 0$ for $i \geq 1$. It can be extended to an arbitrary inequality $u^T x \geq 0$ by flipping the relevant coordinates according to Lemma A.5.

Lemma A.28 (Lovász and Schrijver [LS91]). *If for all i with $u_i < 0$, $u^T x \geq 0$ is valid for $K \cap \{x : x_i = x_0\}$, then $u^T x \geq 0$ is valid for $N_+(K)$.*

By applying this to the stable set problem we obtain Lemma 2.3. Indeed, considering the original n-dimensional space, the inequalities $a^T x \leq b$ (with $a \in \mathbb{R}^n$) that are valid for $\mathrm{STAB}(G)$ are non-trivial only when $b > 0$ and $a \geq 0$, and then we can use Lemma A.28.

For example, the clique, odd hole, odd wheel, and odd antihole constraints all have N_+-rank at most (and thus exactly) 1. Lovász and Schrijver [LS91] show also that the so-called orthogonality constraints (see [Lov79, GLS93] for definition) are valid for $N_+(\mathrm{FRAC})$ by definition, and hence their N_+-rank is also 1.

One simple way to derive facet-defining valid inequalities from other facet-defining inequalities is *cloning* a clique at a vertex i. That is, replacing the vertex i by a clique and replacing every edge incident to i by corresponding edges that are incident to all the clique vertices, and substituting the variable of i in the inequality with the sum of the variables of the clique vertices. In general, it is not clear how cloning influences the N_+-rank of an inequality. However, Goemans and Tunçel [GT00] note that Lemma 2.3 implies that cloning at the center vertex of an odd wheel inequality still has N_+-rank 1, and that cloning at one or several vertices of an odd wheel, odd hole, or odd antihole inequality has N_+-rank at most 2. Indeed, fixing any variable (of the corresponding subgraph) to 1, the resulting inequality can be seen to be a linear combination of clique inequalities and hence valid for $N_+(\text{FRAC})$.

Corollary A.29 (Lovász and Schrijver [LS91]). *If $G - \Gamma(i) - i$ has N_+-rank at most r for every $i \in V$, then the N_+-rank of G is at most $r + 1$.*

It follows for example, that the N_+-rank of a clique, an odd antihole or an odd wheel, is at most (and hence exactly) 1. It also follows (as stated in Lemma 2.4) that the N_+-rank of a graph G is at most its stability number $\alpha(G)$. This bound is tight for a clique.

Lower bounds on the N_+-rank. Lovász and Schrijver [LS91] give no general method to lower bound the N_+-rank. The approach taken by Stephen and Tunçel [ST99], Goemans and Tunçel [GT00], and Cook and Dash [CD01] is to obtain an analog of Corollary A.23 that holds for a specific cone K. That is, they show that $N_+^r(K)$ contains a "uniform" solution that does not belong to K_I, and thus obtain that the N_+-rank of K must be larger than r. Our analysis in Section 3 also follows this approach.

We note that Goemans and Tunçel [GT00] give a sufficient condition for $N_+(K) = N(K)$ to hold, but this condition appears to be not applicable to the stable set problem.

The ranks of the constraints exemplified above are listed in Table 1.

Constraint	N-rank	N_{FR}-rank	N_+-rank	$N_{\text{FR}+}$-rank
odd hole (22)	1	1	1	1
clique (23)	$\|B\| - 2$	$\lfloor (\|B\| - 1)/2 \rfloor$	1	1
antihole (24)	$(\|D\| - 3)/2$	$\lfloor \|D\|/4 \rfloor \le \text{rank} \le \lfloor (\|D\| + 1)/4 \rfloor$	1	1
wheel (25)	2	1	1	1

Table 1: The ranks of some example constraints