Seminar on Algorithms and Geometry – Handout 2

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Today's topics

- Embedding finite metrics into ℓ_∞^k (continued from last time).
- Implications to embedding into ℓ_1 and ℓ_2 (non-optimal bounds).
- Approximation algorithm for Sparsest-Cut via embedding into ℓ_1
- Application of Sparsest-Cut to Minimum Bisection

Homework

Let (X, d) be a metric space. Let $B(x, r)$ denote a ball of radius $r \geq 0$ around center point $x \in X$, i.e. the set of all points whose distance from x is at most r.

A metric (X, d) is called C-growth restricted if for every center $x \in X$ and every $r > 0$ we have $|B(x, 2r)| \leq C \cdot |B(x, r)|$.

1. Let (X, d_X) be a k-dimensional grid with sidelength n (thus containing $N = n^k$ points), endowed with the ℓ_1 metric, i.e. $X = \{1, \ldots, n\}^k$ and $d_X(x, y) = \|x - y\|_1$. Prove that every such metric (X, d_X) is C-growth restricted for $C = C(k)$ (i.e. we can bound C as a function of k, regardless of n).

2. Let X be as above for $k = 2$, i.e. a two-dimensional n by n grid. Prove or disprove the following: There is an absolute constant C (in particular, C is independent of n) such that every subset $X' \subseteq X$ is C-growth restricted. (Remark: X' is endowed with the same metric.)

3. **[Corrected on April 6]** Prove that every n-point metric that is C-growth restricted embeds with distortion $O(1)$ into ℓ_∞^k for $k = O(C^4 \log^2 n)$.

   Hint: use (a variant of) the embedding in class, and show that for every $x, y \in X$, there is a coordinate f_i where (with high probability) $f_i(x) \leq d(x, y)/4$ and $f_i(y) \geq d(x, y)/2$.

4. Extra credit: Read about Bourgain's embedding of n-point metrics into ℓ_2 (recommended: Section 15.7 in [Mat02], it's actually Section 15.8 in the online version). Explain how for C-growth restricted metrics, the same embedding achieves distortion $O(C^2 \sqrt{\log n})$ (which improves over $O(\log n)$ for small C).

5. Prove that every finite metric (X,d) that embeds isometrically into ℓ_1 is a non-negative linear combination of cut psuedometrics, i.e. there are cut psuedometrics τ_i (on point set X) and coefficients $\alpha_i > 0$ such that

$$d(x,y) = \sum_i \alpha_i \tau_i(x,y), \quad \forall x,y \in X.$$

Recall: A metric (X,τ) is called a *cut metric* if there exists $g : X \to \{0,1\}$ such that $\tau(x,y) = |g(x) - g(y)|$ for all $x,y \in X$.

Hint: Consider first X which is a subset of the real line.

References

[Mat02] J. Matoušek. *Lectures on discrete geometry*, volume 212 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002.