Today’s topics

- Approximation algorithm for Sparsest-Cut via embedding into $\ell_1$ (continued from last time).
- Application of Sparsest-Cut to Minimum Bisection
- Distortion lower bounds for embedding a cube into $\ell_2$
- Distortion lower bounds for embedding expanders into $\ell_2$

Homework

1. Let $G(V, E)$ be an $r$-regular graph with edge-expansion $\alpha > 0$, i.e. $e(S, \bar{S}) \geq \alpha |S|$ for all $S \subseteq V$ with $0 < |S| \leq |V|/2$. Show that embedding the shortest-path metric of $G$ into $\ell_1$ requires distortion $\Omega(c \cdot \log |V|)$ where $c = c(r, \alpha) > 0$ is independent of $n$.
   
   Hint: Show a gap between the value of the optimal sparsest-cut and that of the LP relaxation.

2. In the Multicut problem the input is a graph $G$ and $k$ demand-pairs $\{s_1, t_1\}, \ldots, \{s_k, t_k\}$ (similarly to Sparsest-Cut). The goal is to find a partition of the vertices $V = V_1 \cup V_2 \cup \cdots$ where every $s_i$ must be separated from its corresponding $t_i$ (separated means they belong to different parts $V_j$), so as to minimize the number of edges cut (an edge is cut if its endpoints belong to different parts). Give an algorithm that approximates the Multicut problem.
   
   Hint: Use (repeatedly) an algorithm that approximates Sparsest-Cut.

3. Let the graph $H = (V, E)$ be the discrete hypercube $\{0, 1\}^m$ with an edge $(x, y)$ whenever $x$ differs from $y$ in exactly one coordinate. Prove that for every $f : V \to \ell_2$,
   
   $$\mathbb{E}_{(x,y) \in V \times V} \|f(x) - f(y)\|^2 \leq O(m) \mathbb{E}_{(x,y) \in E} \|f(x) - f(y)\|^2.$$

   Note: The lefthand side has uniform distribution over all pairs, the righthand side has uniform distribution over edges.
   
   Hint: Use the inequality for “diagonals” shown in class.