Proximity Oblivious Testing

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Proximity Oblivious Testing?

• A tester with additional constraints
  – Must work in repeated basic rounds
  – Sees a constant number of locations each round
  – Each round is executed with random locations
  – Rejects if one of the rounds rejects
  – Accepts if all rounds accept
  – **Forgets everything between rounds**
Formal Definition

- Let $\Pi = \bigcup_{n \in \mathbb{N}} \Pi_n$ ($\Pi_n$ - instances if size $n$)
- Let $\rho: (0,1] \to (0,1]$ (success given $\epsilon$)
- $T$ is a Proximity-Oblivious tester with detection probability $\rho$ when:
  - $f \in \Pi_n \Rightarrow \Pr[T^f(n) = 1] = 1$
  - $f \notin \Pi_n \Rightarrow \Pr[T^f(n) = 0] \geq \rho \left( \frac{\delta_{\Pi_n}(f)}{\epsilon} \right)$
- We will assume $T$ has constant query complexity
Properties we’d look at

• Query Complexity
  – Note that it will always be a constant

• The detection probability $\rho$

• Comparison to regular testers
Testing graph properties under the adjacency matrix model

- The tested function $f: [N] \times [N] \rightarrow \{0,1\}$ is the adjacency matrix
- Being $\epsilon$-far means having to change $\epsilon$ fraction of the table entries
- Best used when the graph is dense
Example – Clique Collection

• Detecting if the graph is a single clique
  – Requires a single query
  – $\rho(\epsilon) = \epsilon$

• Detecting if the graph is complete bipartite
  – Requires three queries
  – $\rho(\epsilon) = \epsilon$

• Generalizing $CC^{\leq c}$ for $c \geq 3$
  – Using $\binom{c + 1}{2}$ queries
  – $\rho(\epsilon) = \epsilon^{c+1+o(1)}$
  – Can’t do better than $\rho(\epsilon) = \omega\left(\epsilon^{c/2}\right)$
  – Regular testers are superior!
Is everything obliviously testable?

• No!
• Testing bi-partitiveness is not obliviously testable!
• But bi-partitiveness is very easy to test in the old fashioned way!
• Proof on board…
What is obliviously testable?

- Theorem: $\Pi$ has a proximity oblivious tester if and only if $\exists$ constant $c$ and an infinite sequence $\mathcal{F} = \{\mathcal{F}_N\}_{N \in \mathbb{N}}$ such that
  - Each $\mathcal{F}_N$ contains graphs of size at most $c$
  - $\Pi_N$ equals the set of $N$-vertex $\mathcal{F}_N$-free graphs

- Characterizes proximity-oblivious testers for the adjacency matrix model
Proof idea

• The proof uses results from 2 other papers
• Is in fact very intuitive
• A proximity oblivious tester actually decides everything based on possible constant views
• Identical to looking for forbidden subgraphs
• Looking for forbidden sub-graphs of constant size is clearly proximity-oblivious
• Specifies a minimal detection probability!
A special case – graph freeness

• Consider the special case in which $\mathcal{F}_n = \mathcal{F}_{n+1}$
• Can achieve better complexity
  – Using $\binom{c}{2}$ queries
  – Detection Probability $\rho_{\mathcal{F}}$
• Any other tester has detection probability $\Omega(\rho_{\mathcal{F}})$
Testing graph properties under the bounded degree model

- All degrees are bounded by the constant $d$
- The tested function $f: [N] \times [d] \rightarrow \{0, \ldots, N\}$ is the adjacency list
- Being $\epsilon$-far means having to change $\epsilon$ of the entries (similar to the previous case)
- Good for sparse graphs
Is there a difference?

• Surprisingly, there is!
• The bounded degree model achieves a very interesting characterisation
• Provides more power than the dense model
Characterisation of the bounded degree model

• A marked graph is a graph in which every vertex is marked with either full, semi-full or partial

• Instead of looking for forbidden sub-graphs – look for a forbidden embedding

• A sub-graph (induced or not) is a special case of embedding
What is an embedding?

- Full
- Semi-Full
- Partial
Characterisation of the bounded degree model (cont.)

• The property $\Pi$ is called **local** if $\exists s$ and an infinite sequence $\mathcal{F} = \{\mathcal{F}_N\}_{N \in \mathbb{N}}$ such that $\forall N$
  
  $\quad - \mathcal{F}_N$ is a set of marked graphs of size at most $s$
  
  $\quad - \Pi_N$ cannot have an $F \in \mathcal{F}_N$ embedded into it

• In this case, we say $\Pi$ is $\mathcal{F}$-local

• Being $\mathcal{F}_N$-free is also local!

• So, is being local the characterization of the bounded degree model?
Characterisation of the bounded degree model (cont.)

• Proof requires an additional property to exist
• The property is **non-propagation** (shall be defined soon)
• Still remains an open problem whether locality implies “non-propagation”
Non-propagating condition

- For a graph $G = ([N], E)$, we say $B \subseteq [N]$ covers $\mathcal{F}_N$ in $G$ if $\forall F \in \mathcal{F}_N$ for every embedding of $F$ in $G$, at least one vertex of $F$ is mapped to a vertex in $B$

- Can think of $B$ as the “inescapable set”
Non-propagating condition (cont.)

• We say that $\overline{\mathcal{F}}$ is **non-propagating** if there exists a non-decreasing function $\tau: (0,1] \rightarrow (0,1]$ such that:
  
  − $\forall \varepsilon > 0 \; \exists \beta$ such that $\tau(\beta) < \varepsilon$
  
  − For every graph $G = ([N], E)$ and every $B \subset [N]$ that covers $G$, either $G$ is $\tau(\frac{|B|}{N})$-close to being $\mathcal{F}_N$ free, or there are no $N$-vertex graphs that are $\mathcal{F}_N$-free.
Interesting observations

• For every bounded degree $d \geq 3$ we can find an $\bar{F}$ that is not non-propagating
  – Proof on the board... (if time permits)

• Induced subgraph freeness is non-propagating
  – Proof on the board... (if time permits)

• Can find non-hereditary properties that are non-propagating
  – Proof on board... (if time permits)
Main Theorem

- Theorem: A graph property $\Pi$ has a constant query proximity oblivious tester if and only if $\Pi$ is local and non-propagating.
Proof idea

• $\Rightarrow$ Being $\tau(\beta)$-far implies $\text{poly}(\beta)$ fraction of the possible choices makes the tester reject.
• $\Leftarrow$ A proximity-oblivious tester can be converted to test constant surroundings that are equivalent to a non propagating sequence.
Sum things up...

- In the adjacency matrix model we saw:
  - A proximity-oblivious tester might be inferior
  - A proximity-oblivious tester may not even exist (while still being testable the ordinary way)
  - Characterisation of being proximity-oblivious testable

- In the bounded degree model we saw:
  - Characterisation of being proximity-testable is very special
  - The bounded degree model contains the adjacency matrix model
  - Can test non-hereditary properties
  - Poses an interesting open question
Thank you!