Handout on tree width and graph minors

December 29, 2011, and January 5, 2012

We shall touch upon the theory of *Graph Minors* by Robertson and Seymour. As a motivating example, we show that maximum weight independent set on trees can be solved in polynomial time, using dynamic programming. (Note however that bandwidth is NP-hard on trees.) We then introduce the notion of a tree decomposition of a graph.

Definition: A *tree decomposition* of a graph $G(V, E)$ is a tree T, where each node i of T is labeled by a subset (*bag*) $B_i \subset V$ of vertices of G, each edge of G is in a subgraph induced by at least one of the B_i, and the nodes of T labeled by any vertex $v \in V$ are connected in T. The *tree width* of G is the minimum integer p such that there exists a tree decomposition G with all subsets of cardinality at most $p + 1$.

Theorem: For every graph $G(V, E)$ of treewidth p and $X \subset V$, there is a vertex separator S, $|S| \leq p + 1$, that partitions X into two subsets X_1 and X_2 of size at most $2|X|/3$ with all paths between X_1 and X_2 going through S.

We shall show that a tree has tree width 1, a series-parallel graph has tree-width 2, a k-clique has tree width $k - 1$, and an n by n grid has tree width $\Theta(n)$.

Definition: H is a *minor* of G if it can be obtained from G by a sequence of operations of taking subgraphs and edge *contractions* (merging endpoints together).

Kuratowski showed that non-planar graphs must contain either K_5 or $K_{3,3}$ as minors.

Theorem (planar minor): Let H be a planar graph. If G has no H-minor, then the tree width of G is bounded by some function of H, independent of G. (Not proved in class.)

Theorem: For graphs that have bounded tree width, a corresponding tree decomposition can be found in polynomial (in fact, linear) time.

We shall use this and dynamic programming to design a polynomial time algorithm for k-coloring graphs of bounded tree width.

Robertson and Seymour show that every family of graphs that is closed w.r.t. taking minors has a finite *obstruction set* of minors (such as K_5 or $K_{3,3}$ for planarity). They further show that for every fixed H, testing whether G contains H as a minor can be done in time $O(n^3)$. A consequence of their theory is that any property of graphs that is inherited by minors (such as being embeddable in 3-dimensional space without linked cycles) can be *decided* in polynomial time. (Their theory does not necessarily produce an algorithm, and if it does, the hidden constants in the running time are often huge.)

Homework:

1. Recall that graphs of tree width at most 1 are those without a K_3 minor (trees) and graphs of tree width at most 2 are those without a K_4 minor (series parallel graphs). Prove that for every p there is a finite list of forbidden minors (that depends on p) such that graphs of tree width at most p are those without any of these subgraphs as a minor.

2. Prove the following approximate min-max relation between treewidth and grid minors – for every graph, its treewidth (which is a minimization problem) is "approximately" related to the size of its largest grid minor.

   (a) If a graph has a k by k grid as a minor, then it has treewidth at least $\Omega(k)$.

   (b) If a graph has treewidth more than p then if has an $f(p)$ by $f(p)$ grid as a minor, for some nonnegative function $f(p)$ that tends to infinity as p grows. (Hint: show that for every planar graph H there is a large enough grid for which H is a minor, and use Theorem "planar minor". You need not specify f explicitly, and in fact it is still not known what the best f can be in this relation.)