Extra credit:

1. Design a variant of Cheeger’s inequalities, where the input graph comes with two distinguished vertices $s, t \in V$, and we only consider cuts that separate $s$ from $t$. Specifically, find a value $\lambda_{st}(G)$ that satisfies something like:

$$\frac{1}{2} \lambda_{st}(G) \leq \min_{S \colon |\{s;t\} \cap S| = 1} \phi_{G}(S) \leq \sqrt{2 \lambda_{st}(G)}$$

Remark: This $\lambda_{st}(G)$ need not be an eigenvalue, although it is derived from the (normalized) Laplacian matrix.

If helpful, you may assume that $G$ is $d$-regular.

2. Let $G = (V, E)$ be a capacitated graph (i.e., with edge capacities $c_{e} \geq 0$). Show that the algorithm below outputs a capacitated tree $T$ on the same vertex set $V$ that is flow-equivalent to $G$, in the sense that for all $s, t \in V$, the maximum $st$-flow in $G$ is equal to that in $T$. (Notice that $T$ has same vertices, but in general different edges and capacities than $G$.)

Algorithm:

(1) Construct on $V$ a complete graph $G'$, with every edge capacity $c'_{st}$ is equal to $s, t \in V$ are connected by an edge whose capacity $c'_{st}$ is defined to be the maximum $st$-flow in $G$.

(2) Compute a maximum spanning tree $T$ for $G'$ (with respect to the edge capacities $c'_{e}$).

Hint: A maximum spanning tree is analogous to a minimum spanning tree; in particular, it is just a minimum spanning tree for the capacities $-c'_{e}$. 