We discussed in class an algorithm for \((1 + \varepsilon)\)-approximate Nearest Neighbor Search, by relying on the doubling dimension of the dataset.

1. Consider the metric space \( M = \mathbb{R}^k \) with distances according to the \( \ell_\infty \)-norm, i.e., \( d(x, y) = \|x - y\|_\infty \). Prove that \( \text{ddim}(M) \leq O(k) \).

2. Let \( k = \text{ddim}(M) \) and define \( k' \) similarly using diameter instead of radius (covering by sets of half the diameter). Prove that \( k' \leq O(k) \).

3. Design a variant of the query algorithm, where instead of keeping a set of points \( Z_i \), we keep only one point \( z_i \), which is computed (iteratively) as the point closest to \( q \) inside the list \( L_{z_{i+1}, i+1} \). Show that your variant finds an \( O(1) \)-approximate NNS.

For simplicity (avoiding handling special cases) you may assume that \( OPT = d(q, S) \in [10, \text{diam}(S)/10] \). If needed, change the constant in the definition of \( L_{y, i} \).