General instructions: Please keep your answers short and easy to read. You can use results, calculations or notation seen in class without repeating them, unless asked explicitly to redo them.

1. Design a 1coreset (i.e., exact and not approximate) for Minimum Enclosing Ball in $\mathbb{R}^d$ under $\ell_\infty$ norm, i.e., the cost function is

$$C^\infty_P(x) = \max_{p \in P} \|p - x\|_\infty.$$ 

What is the size of your coreset (as a function of $d$)? Does this cost function satisfy the Merge and Reduce properties? And the Disjoint Union property?

2. An unweighted graph $G$ is called $k$-connected if every cut $(S, \bar{S})$ contains at least $k$ edges.

Design a streaming algorithm that determines whether a dynamic graph $G$ on vertex set $V = [n]$ (i.e., a stream of edge insertions and deletions) is 2-connected, using storage $\tilde{O}(n)$.

Hint: First verify that $G$ is connected by constructing a spanning tree $T$. Then classify all possible cuts $(S, \bar{S})$ into those that contain two or more edges of the tree $T$ and the rest, and finally use additional (independent) samples to verify whatever is still needed.

3. Analyze Algorithm 2 below for counting triangles in a graph given as a stream, and show that with constant high probability, the additive error is $|\tilde{T} - T| \leq \epsilon t$. Can this algorithm be applied also for dynamic graphs (i.e., a stream of edge insertions and deletions)? Explain how/why not.

Notation (similar to class): Assume $t > 0$ is a known lower bound for the actual number of triangles $T$, and let $x_S$ count the number of edges internal to the vertices $S \subset V$.

Algorithm 2
1. Init: pick $k = O\left(\frac{n^3}{\epsilon^2 t}\right)$ random subsets $S_1, \ldots, S_k \subset V$ each of size 3 (with replacement)
2. Update: maintain $x_{S_1}, \ldots, x_{S_k}$ (explicitly)
3. Output: compute $z = \sum_{i \in [k]} 1_{x_{S_i}=3}$ and $N = \binom{n}{3}$, and report $\tilde{T} = \frac{N}{k} \cdot z$

Hint: Use Chebyshev’s inequality.
Extra credit:

4. Show how to improve Algorithm 2 above by choosing the random sets $S_i$ only from those sets $S$ that satisfy $x_S \geq 1$. The resulting Algorithm 2’ should have space requirement $k' = O(\frac{mn}{\varepsilon^2})$ words, and work also for dynamic graphs.

Hint: Use $\ell_0$-samplers and an estimator for $N' = \|x\|_0$. 