Advanced Algorithms – Final Exam

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General instructions. The exam has 2 parts. You have 3 hours. No books, notes, or other external material is allowed.

Part I (40 points)

Answer 4 of the following 5 questions. Each question is composed of a statement. Determine if the statement is TRUE or FALSE, and explain your answer in 1-3 sentences (i.e. sketch of the proof or a convincing argument why it is correct/wrong).

a. In every LP, every local optimum is also a global optimum.

   Reminder: a point is a local optimum if its value is optimal among all feasible points in a small neighborhood; it is a global optimum if its value is optimal among all feasible points. Assume it is a minimization LP, hence optimality means value is $\leq$ other points.

b. In every polytope $\{x : \ Ax \leq b\}$, for every BFS z there is a linear objective function $c^t x$ such that z is the unique optimum of $\min\{c^t x : \ Ax \leq b\}$.

c. There is a symmetric positive semidefinite matrix $A_{3\times 3}$ such that $A_{11} = 0$ and $A_{13} = 7$.

d. If A is a totally unimodular matrix and the LP $\min\{c^t x : \ Ax \leq b\}$ is bounded, then this LP has an optimal solution x^* that is integral.

e. The *complementary slackness* conditions relate between pairs of optimal basic feasible solutions of a primal-dual pair of linear programs. They connect between nonzero variables in the solution to one LP and constraints that are satisfied with equality in the other LP. For optimal solutions that are not basic feasible solutions, it might be that the complementary slackness conditions do not hold.

Part II (60 points)

Answer 3 of the following 5 questions.

1. For a given graph G, a fractional packing of spanning trees is defined as follows. Every spanning tree T_i is given a nonnegative weight $w_i \geq 0$, and for every edge, the sum of weights given to spanning trees that contain it is at most 1. The *fractional packing of spanning trees*

number FPSTN(G) of a graph G is the maximum sum of weights given to all trees in a fractional packing of spanning trees.

Give a polynomial time algorithm that on input a graph G outputs the value of FPSTN(G).

2. Is the following problem solvable in polynomial time? The input is a graph G on n vertices, and the goal is to arrange its vertices on an n-dimensional unit-sphere so as to maximize the sum of the angles made by the edges.

   Remark: angles are always in the range $[0, \pi]$.

3. For instances of MAX-SAT in which every clause has either one or three literals, show a randomized polynomial time algorithm with a constant expected approximation ratio strictly bigger than 3/4. (For maximum credit, show approximation $\rho = 113/145 \simeq 0.779$.)

4. Let G be an arbitrary graph and let $\alpha_1, \ldots, \alpha_k$ be nonnegative with $\sum_{i=1}^{k} \alpha_i = 1$. Show that one can color the edges of the graph such that for every vertex v and every color class i, the number of edges of color i incident with vertex v is between $\lfloor \alpha_i d_v \rfloor - 2$ and $\lceil \alpha_i d_v \rceil + 2$, where d_v is the degree of vertex v.

5. Let $G = (V, E)$ be a graph. Recall the standard LP relaxation for vertex-cover is:

$$
\begin{array}{llll}
\mathrm{LP}_1 = & \min & \sum_{i \in V} x_i & \\
& \text{s.t.} & x_i + x_j \geq 1 & \forall (i,j) \in E \\
& & x_i \geq 0 & \forall i \in V
\end{array}
$$

   Consider an alternative LP with two variables x_i^+, x_i^- for each $i \in V$:

$$
\begin{array}{llll}
\mathrm{LP}_2 = & \min & \sum_{i \in V} (x_i^+ - x_j^-)/2 & \\
& \text{s.t.} & x_i^+ - x_j^- \geq 1 & \forall (i,j) \in E \\
& & 0 \leq x_i^+ \leq 1 & \forall i \in V \\
& & -1 \leq x_i^- \leq 0 & \forall i \in V
\end{array}
$$

   Remark: In this LP, every edge gives two different constraints, one for the pair (i, j) and one for (j, i).

   Prove the following:

   (a) every solution for LP_1 can be converted to a solution for LP_2 with the same value;

   (b) LP_2 always has an optimal solution that is integral;

   (c) integral solutions for LP_2 yield half-integral solutions for LP_1, implying that LP_1 has an optimal solution that is half-integral.

THE END.