1 Today’s topics

• Integer linear programs and relaxations
• Integrality gap
• Half-integrality of LP
• Total unimodular LP
• Examples: matching and vertex-cover, max-flow and min-cut, independent set
• Strengthening the LP (e.g. adding clique constraints)

2 Homework

1. Prove that the vertex-arc incidence matrix of a directed graph is totally unimodular.

2. Show that the vertex-cover LP always has an optimal solution that is half integral (i.e. all variables have values $\in \{0, \frac{1}{2}, 1\}$). (One approach: Replace every variable $x_j$ by $\frac{1}{2}(x^+_j - x^-_j)$, using two new variables $0 \leq x^+_j \leq 1$ and $-1 \leq x^-_j \leq 0$.) Is it true every bfs of the vertex-cover LP is half-integral?

3. Prove that the LP below is a relaxation of the minimum spanning tree problem (in an undirected graph $G = (V, E)$ and edge weights $w_{ij} \geq 0$), and that every integral feasible solution of it is a spanning tree. Then show for this LP the largest integrality gap you can.

$$\begin{align*}
\text{minimize} & \quad \sum_{ij \in E} w_{ij} x_{ij} \\
\text{subject to} & \quad \sum_{ij \in \delta(S)} x_{ij} \geq 1 \text{ for every subset of vertices } S \neq \emptyset, V \\
& \quad x_{ij} \geq 0 \text{ for all } ij \in E.
\end{align*}$$

Here, $\delta(S)$ is the set of edges with exactly one endpoint in $S$, i.e. edges in the cut $(S, \bar{S})$, and we use only one variable $x_{ij} = x_{ji}$ for every edge $ij \in E$.