Advanced Algorithms – Handout 8

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1 Today's topics

- Integer linear programs and relaxations
- Integrality gap
- Half-integrality of LP
- Total unimodular LP
- Examples: matching and vertex-cover, max-flow and min-cut, independent set
- Strengthening the LP (e.g. adding clique constraints)

2 Homework

1. Prove that the vertex-arc incidence matrix of a directed graph is totally unimodular.
2. Show that the vertex-cover LP always has an optimal solution that is half integral (i.e. all variables have values $\in \{0, \frac{1}{2}, 1\}$). (One approach: Replace every variable x_j by $\frac{1}{2}(x_j^+ - x_j^-)$, using two new variables $0 \leq x_j^+ \leq 1$ and $-1 \leq x_j^- \leq 0$.) Is it true every bfs of the vertex-cover LP is half-integral?
3. Prove that the LP below is a relaxation of the minimum spanning tree problem (in an undirected graph $G = (V, E)$ and edge weights $w_{ij} \geq 0$), and that every integral feasible solution of it is a spanning tree. Then show for this LP the largest integrality gap you can.

   minimize $\sum_{ij \in E} w_{ij} x_{ij}$

   subject to $\sum_{ij \in \delta(S)} x_{ij} \geq 1$ for every subset of vertices $S \neq \emptyset, V$

   $x_{ij} \geq 0$ for all $ij \in E$.

   Here, $\delta(S)$ is the set of edges with exactly one endpoint in S, i.e. edges in the cut $(S, \bar{S})$, and we use only one variable $x_{ij} = x_{ji}$ for every edge $ij \in E$.