ADIC FLOWS, TRANSVERSAL FLOWS, AND HOROCYCLE FLOWS

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ABSTRACT. We give a new construction of Shunji Ito's "transversal flow for a subshift of finite type" **[I]**, along the lines of Vershik's construction of an adic transformation. We then show how these flows arise naturally in the symbolic coding of horocycle flows on non-compact hyperbolic surfaces with finite area.

1. INTRODUCTION

This paper grew out of the study of the following classical constructions in dynamical systems.

The horocycle flow [He], [Ho]. In their study of the geodesic flow g^s on the unit tangent bundle T^1M of a hyperbolic surface M, Hedlund and Hopf used a continuous flow h^t with the following two properties:

1.1. Orbit property: $\{h^t(\omega) : t \in \mathbb{R}\} = \{\omega' \in T^1M : d(g^s(\omega'), g^s(\omega)) \xrightarrow[s\to\infty]{} 0\}$;
1.2. Renormalization Property: $g^{-s} \circ h^t \circ g^s = h^{te^s}$.

The flow h^t is called the (stable) *horocycle flow*. When lifted to the universal cover $T^1\mathbb{D}$, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, its orbits trace horocycles (circles tangent to $\partial\mathbb{D}$).

The adding machine. Let $\Sigma_2^+ := \{0,1\}^{\mathbb{N}\cup\{0\}}$, equipped with the metric $d(\underline{y}, \underline{z}) := \exp[-\min\{i \geq 0 : y_i \neq z_i\}]$, and let $\sigma : \Sigma_2^+ \to \Sigma_2^+$ denote the left shift map $\sigma(x_0, x_1, \ldots) = (x_1, x_2, \ldots)$. The *adding machine* (apparently first constructed by von Neumann) is the map $\tau : \Sigma_2^+ \to \Sigma_2^+$ given by

$$\tau : (\underbrace{1\cdots 1}_{n}, 0, x_{n+1}, x_{n+2}, \cdots) \mapsto (\underbrace{0\cdots 0}_{n}, 1, x_{n+1}, x_{n+2}, \cdots)\ ,\ \tau : 1^\infty \mapsto 0^\infty.$$

(Addition of one with "carry to the right.") Again, we have

2.1. Orbit property: $\{\tau^n(\underline{x}) : n \in \mathbb{Z}\} = \{\underline{y} : d(\sigma^n(\underline{x}), \sigma^n(\underline{y})) \xrightarrow[n\to\infty]{} 0\}$ for all $\underline{x}$ not eventually constant;
2.2. Renormalization property: $\tau^m \circ \sigma^n = \sigma^n \circ \tau^{2^n m}$ $(m, n \in \mathbb{Z})$.

The adic transformation [V]. Vershik generalized von Neumann's construction to more general Markov compacta. We focus on the special case of subshifts of finite type ("stationary adic transformations"). We review the necessary definitions.

Subshifts of finite type. Let $N > 1$, $S := \{1, \ldots, N\}$, and suppose $A := (t_{ij})_{N\times N}$ is a matrix of zeroes and ones without rows or columns made entirely of zeroes.

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The one–sided *subshift of finite type* with *states* S and *transition matrix* A is the left shift map $\sigma : \Sigma^+ \to \Sigma^+$ where

$$\Sigma^+ = \{(x_0, x_1, \dots) \in S^{\mathbb{N}} : t_{x_i x_{i+1}} = 1 \text{ for all } i\}.$$

We equip Σ^+ with the metric $d(\underline{y}, \underline{z}) := \exp[-\min\{i \geq 0 : y_i \neq z_i\}]$. It is well known that $h_{top}(\sigma) = \ln \lambda$ where λ is the leading eigenvalue of A (see e.g. **[PP]**).

Stable tail relation. One sees that $\{\underline{y} \in \Sigma^+ : d(\sigma^n \underline{x}, \sigma^n \underline{y}) \xrightarrow[n\to\infty]{} 0\} = \{\underline{y} : \underline{y} \sim \underline{x}\}$, where $\sim$ is the *stable tail relation*,

$$(\underline{x} \sim \underline{y}) \quad \Leftrightarrow \quad \exists\, p\ \big(\forall\, i \geq p\ (x_i = y_i)\big). \tag{1.1}$$

Reverse lexicographic orders. We would like to define a total order on every equivalence class of $\sim$. To do this, fix for every $a \in S$ a total order $\underset{a}{<}$ on S, and let

$$\underline{x} \prec \underline{y} \quad \text{if and only if} \quad \exists\, p\ \big(x_p \underset{x_{p+1}}{<} y_p \text{ and } \forall\, i \geq p+1\ (x_i = y_i)\big). \tag{1.2}$$

We write $\preceq$ for "$\prec$ or $=$". The symbols $\prec$ and $\precneqq$ mean the same thing.

If $\underset{a}{<}$ does not depend on a and equals the natural order on S, then we get the *standard reverse lexicographic order*. If we define $\underset{a}{<}$ to be the cyclic order on S with a as the minimal element, then we get the *cyclic reverse lexicographic order*.

Eventually extremal sequences. No matter what total orders $\underset{a}{<}$ we choose, there are only finitely many $\prec$–maximal or $\prec$–minimal sequences, all periodic. To see this let $P_{\max}, P_{\min} : S \to S$ denote the functions

$$\begin{aligned} P_{\max}(a) &:= \max\{b \in S : t_{ba} = 1\} \\ P_{\min}(a) &:= \min\{b \in S : t_{ba} = 1\} \end{aligned} \qquad \text{calculated using } \underset{a}{<}.$$

Every $\prec$–maximal (resp. $\prec$–minimal) point $\underline{x}$ must satisfy $x_i = P_{\max}(x_{i+1})$ (resp. $x_i = P_{\min}(x_{i+1})$) for all i. Since S is finite, every iteration of $P_{\max}$ and $P_{\min}$ ends at a cycle. It follows that the extremal points for $\prec$ are periodic and finite in number.

Sequences which are tail equivalent to a maximal or minimal sequence are called *eventually extremal*. Let

$$\Sigma_0^+ := \Sigma^+ \setminus \{\text{ eventually extremal sequences }\}.$$

Vershik's adic transformation. This is $\tau : \Sigma_0^+ \to \Sigma_0^+$ given by

$$\tau(\underline{x}) := \min\{\underline{y} \sim \underline{x} : \underline{y} \succneqq \underline{x}\}. \tag{1.3}$$

The definition depends on the choice of $\prec$. The *standard Vershik adic transformation* is obtained from the standard reverse lexicographic order. The *cyclic Vershik adic transformation* is obtained from the cyclic lexicographic order. In any case, we have

3.1. Orbit property: $\{\tau^n(\underline{x}) : x \in \mathbb{Z}\} = \{\underline{y} : d(\sigma^n(\underline{x}), \sigma^n(\underline{y})) \xrightarrow[n\to\infty]{} 0\}$ for all $\underline{x}$ non–eventually extremal;

3.2. Renormalization property: $\exists p_{m,n} : \Sigma_0^+ \to \mathbb{Z}$ s.t. $(\tau^m \circ \sigma^n)(\underline{x}) = (\sigma^n \circ \tau^{p_{m,n}(\underline{x})})(\underline{x})$ and $p_{m,n}(\underline{x}) = e^{n \cdot h_{top}(\sigma)}[m + O(1)]$ uniformly on Σ^+.

Ito's transversal flow and this paper [I]. Ito found a way to realize the natural extension of $\sigma : \Sigma^+ \to \Sigma^+$ as a map T of $[0,1) \times [0,1)$, and constructed a flow τ^t on $[0,1) \times [0,1)$ with the property $T^n \circ \tau^t = \tau^{\lambda^{-n}t} \circ T^n$, $\lambda = e^{h_{top}(\sigma)}$.

In this paper we give an alternative construction for Ito's flow, by working directly on the two–sided subshift of finite type. The result is a symbolic version of Ito's flow, which we call the "adic flow" because of its striking similarity to Vershik's adic transformation: compare (1.3) to (2.3) below.

The symbolic description of Ito's flows makes it is relatively easy to see how they arise in the coding of the horocycle flow on a non-compact hyperbolic surface with finite area. Recent work of Bufetov [**Bu**] suggests a similar application to translation flows.

2. ADIC FLOWS

These will be flows on two–sided subshifts of finite type

$$\Sigma := \{(x_i)_{i\in\mathbb{Z}} \in S^{\mathbb{Z}} : t_{x_i x_{i+1}} = 1 \text{ for all } i \in \mathbb{Z}\},$$

metrized by $d(\underline{x}, \underline{y}) = \exp[-\min\{|n| : x_n \neq y_n\}]$. As in the one–sided case,the left shift map $\sigma : \Sigma \to \Sigma$, $\sigma(\underline{x})_i = x_{i+1}$, has topological entropy $h_{top}(\Sigma) = \ln\lambda$ where λ is the leading eigenvalue of the transition matrix $(t_{ij})_{N\times N}$.

The left shift map is known to be topologically mixing iff there exists some m s.t. all the entries of the m–th power of $(t_{ij})_{N\times N}$ are positive (see e.g. [**B**]).

Fix once and for all a reverse lexicographic order on Σ^+ (see (1.2)), and let τ be the corresponding adic transformation. (1.2) can also be used to define a partial order on Σ, and we use the same symbol $\prec$ for the two orders. If $\pi : \Sigma \to \Sigma^+$ is the projection onto the non-negative coordinates, then $(\underline{x} \prec \underline{y}) \Rightarrow (\pi(\underline{x}) \preceq \pi(\underline{y}))$.

Let $(\underline{x}, \underline{y}) := \{\underline{z} : \underline{x} \precneqq \underline{z} \precneqq \underline{y}\}$ and define

$$\Sigma_0 := \{\underline{x} \in \Sigma : \pi(\underline{x}) \in \Sigma_0^+ \text{ and } \forall\, \underline{y} \succneqq \underline{x}\ \ [(\underline{x}, \underline{y}) \neq \varnothing]\}.$$

This excludes sequences with a right maximal tail ($x_i = P_{\max}(x_{i+1})$ for all i large); sequences with a right minimal tail ($x_i = P_{\min}(x_{i+1})$ for all i large); and sequences with a left maximal tail ($x_i = P_{\max}(x_{i+1})$ for all i sufficiently negative).

Sequences with left minimal tails ($x_i = P_{\min}(x_{i+1})$ for all i sufficiently negative) are not excluded.

Theorem 1. *Suppose (Σ, σ) is topologically mixing. There exists a measurable flow $\tau_s^t : \Sigma_0 \to \Sigma_0$ such that:*

(a) Orbit property: $\{\tau_s^t(\underline{x}) : t \in \mathbb{R}\} = \{\underline{y} \in \Sigma_0 : d(\sigma^n(\underline{x}), \sigma^n(\underline{y})) \xrightarrow[t\to\infty]{} 0\}$ *for every $\underline{x} \in \Sigma_0$.*

(b) Renormalization property: $\tau_s^t \circ \sigma^n = \sigma^n \circ \tau_s^{\lambda^n t}$.

(c) Order property: *if $t_1 < t_2$ then $\tau_s^{t_1}(\underline{x}) \prec \tau_s^{t_2}(\underline{x})$.*

(d) Almost Continuity: *Suppose $\tau_s^t(\underline{x})$ has no left minimal tails, then $d(\tau_s^{t_n}(\underline{x}^{(n)}), \tau_s^t(\underline{x})) \to 0$ for all sequences $(\underline{x}^{(n)}, t_n) \to (\underline{x}, t)$.*

This flow is unique up to a change of time scale.

The existence part follows from Ito's work in [**I**], but the proof we give is different.

To determine a flow one has to specify the orbits, the order of movement on the orbits, and the time it takes to flow from one point to another. We show that conditions (a),(b),(c) and (d) dictate these choices, and that these choices lead to a well defined flow.

The orbits. Since $\{\underline{y} \in \Sigma_0 : d(\sigma^n(\underline{x}), \sigma^n(\underline{y})) \to 0\} = \{\underline{y} \in \Sigma_0 : \exists p\ (\forall i > p\ y_i = x_i)\}$, property (a) forces the orbits to equal the equivalence classes of the *(stable) tail equivalence relation* on Σ_0:

$$\underline{x} \sim \underline{y} \Leftrightarrow \exists p \in \mathbb{Z} \text{ s.t. } \forall i > p\ (x_i = y_i).$$

We denote these equivalence classes by $[\underline{x}]$.

The order of movement. Property (c) says that the order on the orbits of τ_s is $\prec$. The question arises whether such an order can be realized by a flow. This is guaranteed by the following lemma.

Lemma 1. *$([\underline{x}], \prec)$ is order isomorphic to $(\mathbb{R}, <)$ for every $x \in \Sigma_0$.*

Proof. By **[C]**, it is enough to check that

(i) $[\underline{x}]$ is totally ordered by $\prec$.
(ii) $[\underline{x}]$ contains a countable set Z which is order isomorphic to $\mathbb{Z}$, and such that for every $\underline{y} \in [\underline{x}]$, there exist $\underline{z}_1, \underline{z}_2 \in Z$ s.t. $\underline{z}_1 \prec \underline{y} \prec \underline{z}_2$.
(iii) $[\underline{x}]$ contains a countable set Q with the property that for any $\underline{y}_1 \precneqq \underline{y}_2$ in $[\underline{x}]$, there exists some $\underline{z} \in Q$ s.t. $\underline{y}_1 \precneqq \underline{z} \precneqq \underline{y}_2$.
(iv) Every bounded monotonic sequence of elements in $[\underline{x}]$ has a limit in $[\underline{x}]$ (convergence w.r.t. the order topology).
(v) Every element of $[\underline{x}]$ is the limit of a non–eventually constant increasing (resp. decreasing) sequence in $[\underline{x}]$.

The first property is trivial. (ii) holds with $Z := \{(\Phi \circ \tau^n \circ \pi)(\underline{x}) : n \in \mathbb{Z}\}$, where $\pi : \Sigma \to \Sigma^+$ is the natural projection, $\tau : \Sigma_0^+ \to \Sigma_0^+$ is Vershik's adic transformation, and $\Phi : \Sigma_0^+ \to \Sigma$ is $\Phi(y_0, y_1, y_2, \ldots) := (\ldots, y_{-2}, y_{-1}, \dot{y}_0, y_1, y_2, \ldots)$ with $y_{-i} := P^i_{\min}(y_0)$ for $i \in \mathbb{N}$. The dot above y_0 signifies the position of the zeroth coordinate. (iii) holds with $Q := \{\underline{y} \in [\underline{x}] : \exists n \text{ s.t. } \forall i \le n\ \big(y_{i-1} = P_{\min}(y_i)\big)\}$.

Here is the proof of (iv). We do the increasing case, and leave the decreasing case to the reader. Let $\underline{y}^{(n)}$ be an increasing sequence in $[\underline{x}]$ and suppose $\underline{y}^{(n)} \prec \underline{y}$ for all n. Since $\underline{y}^{(1)} \sim \underline{y}$, there exists some p s.t. $y_i^{(1)} = y_i$ for all $i \ge p$. Since $\underline{y}^{(1)} \preceq \underline{y}^{(n)} \preceq \underline{y}$, (1.2) forces $y_i^{(n)} = y_i$ for all $n \in \mathbb{N}$ and $i \ge p$. It also forces $\{y_{p-1}^{(n)}\}_{n \ge 1}$ to be an increasing sequence in $(S, \underset{y_p}{<})$. Since S is finite, $\{y_{p-1}^{(n)}\}_{n\ge1}$ is eventually constant. Suppose $y_{p-1}^{(n)}$ equals c for all $n > N_1$, then $\{y_{p-2}^{(n)}\}_{n \ge N_1}$ must be an increasing sequence in $(S, \underset{c}{<})$. Since S is finite, it also eventually stabilizes. Continuing in this way, we see that $\exists \underline{z} \in \Sigma$ s.t. $d(\underline{y}^{(n)}, \underline{z}) \to 0$.

The limiting sequence belongs to $[\underline{x}]$, because $z_i = y_i$ for all $i \ge p$ and $\underline{y} \in [\underline{x}]$. If $\underline{z} \in \Sigma_0$, then we are done. Otherwise for some $q \in \mathbb{Z}$

$$z_{i-1} = P_{\max}(z_i) \text{ for all } i \le q.$$

Suppose q is maximal with this property (a maximum exists, since $\underline{z} \sim \underline{x}$ and $\underline{x} \in \Sigma_0$). Then $z_q < P_{\max}(z_{q+1})$ and $\underline{y}^{(n)}$ converges in the order topology of $[\underline{x}]$ to

$\underline{w} := (\cdots, P^2_{\min}(w_q), P_{\min}(w_q), w_q, z_{q+1}, z_{q+2}, \cdots)$, where the q-th coordinate is

$$w_q := \min\{w \in S : t_{wz_{q+1}} = 1 \text{ and } w > z_q\} \text{ w.r.t. } \underset{z_{q+1}}{<}.$$

This is an element of $[\underline{x}]$.

To prove (v), fix some $\underline{y} \in [\underline{x}]$. We construct a non eventually constant decreasing sequence in $[\underline{x}]$ with limit $\underline{y}$. Since $\underline{y} \in \Sigma_0$, it has no left maximal tails, so $\exists i_n \downarrow -\infty$ s.t. $y_{i_n} \neq P_{\max}(y_{i_n+1})$. Let

$$w_{i_n} := \min\{w \in S : t_{wy_{i_n+1}} = 1 \text{ and } w > y_{i_n}\} \text{ w.r.t } \underset{y_{i_n+1}}{<}.$$

Define $\underline{y}^{(n)} := (\cdots, P^2_{\min}(w_{i_n}), P_{\min}(w_{i_n}), w_{i_n}, y_{i_n+1}, y_{i_n+2}, \cdots)$. Then $y^{(n)}$ is a decreasing sequence of points in $\Sigma_0 \cap [\underline{x}]$ whose limit is $\underline{y}$. The sequence is not eventually constant, because the starting point of the longest left minimal tail of $\underline{y}^{(n)}$ is i_n, and i_n is not eventually constant.

Next we construct a non–eventually constant increasing sequence in $[\underline{x}]$ with limit $\underline{y}$. Suppose first that $\underline{y}$ does not have a left minimal tail. Let

$$\underline{y}^{(n)} := (\cdots, P^2_{\min}(y_{-n}), P_{\min}(y_{-n}), y_{-n}, y_{-n+1}, y_{-n+2}, \ldots).$$

This is an increasing sequence in $[\underline{x}]$ which converges to $\underline{y}$, and it is not eventually constant because there are $n_k \to -\infty$ s.t. $y_{n_k} \neq P_{\min}(y_{n_k+1})$.

Now suppose $\underline{y}$ does have a left minimal tail. There is a maximal q s.t. $y_{i-1} = P_{\min}(y_i)$ for all $i \leq q$, otherwise $\underline{y} \notin \Sigma_0$. By maximality, $y_q > P_{\min}(y_{q+1})$, therefore the following definitions are proper:

$$w_q := \max\{w \in S : t_{wy_{q+1}} = 1 \text{ and } w < y_q\} \text{ w.r.t. } \underset{y_{q+1}}{<}$$

$$\begin{aligned}\underline{y}^{(n)} := (\cdots, P^3_{\min}(P^n_{\max}(w_q)), P^2_{\min}(P^n_{\max}(w_q)), P_{\min}(P^n_{\max}(w_q)),& \\ P^n_{\max}(w_q), P^{n-1}_{\max}(w_q), \cdots, P_{\max}(w_q), w_q, y_{q+1}, y_{q+2}, \cdots).&\end{aligned}$$

The sequence $\{\underline{y}^{(n)}\}_{n\geq 1}$ is increasing sequence. It is not difficult to see using the topological mixing of (Σ, σ) that there are infinitely many n's s.t. $P^{n+1}_{\max}(w_q) \neq P_{\min}[P^n_{\max}(w_q)]$, and this implies as above that $\{\underline{y}^{(n)}\}_{n\geq 1}$ is not eventually constant. In (Σ, d), $\lim \underline{y}^{(n)} = \underline{z} := (\cdots, P^3_{\max}(w_q), P^2_{\max}(w_q), P_{\max}(w_q), w_q, y_{q+1}, y_{q+2}, \cdots)$ which lies outside of Σ_0. With respect to the order topology, $\lim \underline{y}^{(n)}$ is the smallest element of $[\underline{x}] \cap \Sigma_0$ above $\underline{z}$, which is exactly $\underline{y}$. □

Time parametrization. The time parametrization of the orbits of a flow can be encoded in terms of a measure as follows. Suppose there were a flow $\tau^t_s : \Sigma_0 \to \Sigma_0$ satisfying the requirements of the theorem. For every $\underline{x} \in \Sigma_0$ and $\underline{y} \prec \underline{z}$ in $[\underline{x}]$ let $[\underline{y}, \underline{z}) := \{\underline{w} : \underline{y} \preceq \underline{w} \precneqq \underline{z}\}$ and

$$\mu_{[\underline{x}]}\big([\underline{y}, \underline{z})\big) := \text{the unique } t \text{ s.t. } \underline{z} = \tau^t_s(\underline{y}). \tag{2.1}$$

This makes sense, since $\underline{y}, \underline{z}$ belong to the same orbit of τ_s. Since the orbit order is $\prec$, $\mu_{[\underline{x}]}\big([\underline{y}, \underline{z})\big) > 0$. Since τ^t_s is a flow, $\mu_{[\underline{x}]}$ is σ–additive on the semi–algebra of left–closed right–open intervals in $[\underline{x}]$. By Carathéodory's Extension Theorem, $\mu_{[\underline{x}]}$

extends to a measure on the Borel σ–algebra of $[\underline{x}]$ (Borel w.r.t. the order topology). We denote this measure by $\mu_{[\underline{x}]}$.

The measures $\{\mu_{[\underline{x}]} : \underline{x} \in \Sigma_0\}$ contain complete information on the speed of movement on the orbits, and they must satisfy the following properties:

(M1) $\mu_{[\underline{x}]}$ is a non–atomic measure on $[\underline{x}]$.
(M2) In $[\underline{x}]$, non–empty bounded open intervals have finite positive measure and infinite rays (sets of the form $\{\underline{z} : \underline{z} \preceq \underline{y}\}$ or $\{\underline{z} : \underline{z} \succeq \underline{y}\}$, $\underline{y} \in [\underline{x}]$) have infinite measure.
(M3) $\mu_{[\sigma(\underline{x})]} \circ \sigma = \lambda^{-1}\mu_{[\underline{x}]}$.
(M4) for every continuous function $f : \Sigma \to \mathbb{R}$, the real valued function $\underline{x} \mapsto \int_{[\underline{x}]} f(\underline{y})\delta_{x_0^\infty}(y_0^\infty)d\mu_{[\underline{x}]}(\underline{y})$ is Borel measurable. Here $\delta_{x_0^\infty}(y_0^\infty)$ is the function which is equal to one if $y_0^\infty := (y_0, y_1, \ldots) = (x_0, x_1, \ldots) =: x_0^\infty$ and equal to zero otherwise.

(M1) and (M2) are obvious. (M3) is because of the renormalization property (c). (M4) is because of the measurability of the flow: the integral can be rewritten as $\sup \int_\alpha^\beta (f \circ \tau_s^t)(\underline{x})dt$ where the supremum is taken over all rational numbers $\alpha < \beta$ s.t. $(\tau_s^\alpha(\underline{x}), \tau_s^\beta(\underline{x})) \subset \{\underline{y} \in \Sigma : y_0^\infty = x_0^\infty\}$.

Lemma 2 (Bowen & Marcus)**.** *There exists a measurable family of measures* $\{\mu_{[\underline{x}]} : \underline{x} \in \Sigma_0\}$ *satisfying (M1),(M2),(M3), and (M4).*

Proof. This is essentially in **[BM]**, albeit in different notation. We begin with a simple, but useful, description of $[\underline{x}]$. Abusing notation, we use the same symbol $\sim$ for the tail relation on Σ_0 and Σ_0^+. Then $[\underline{x}] = \{\underline{y} \in \Sigma_0 : \underline{y} \sim \underline{x}\}$ and $\pi[\underline{x}] = \{\underline{y}^+ \in \Sigma^+ : \underline{y}^+ \sim \pi(\underline{x})\}$. The last set is countable. We obtain the following countable disjoint decomposition of $[\underline{x}]$:

$$[\underline{x}] = \biguplus_{\underline{y}^+ \sim \pi(\underline{x})} A(\underline{y}^+), \text{ where } A(\underline{y}^+) := \{\underline{z} \in \Sigma_0 : z_0^\infty = y_0^\infty\}.$$

Each of the sets $A(y^+)$ is a bounded open interval in $[\underline{x}]$, and therefore must have finite measure w.r.t. $\mu_{[\underline{x}]}$. So instead of describing the infinite measure $\mu_{[\underline{x}]}$, we will describe the finite measures

$$\mu_{\underline{y}^+}(E) := \mu_{[\underline{x}]}(E \cap A(\underline{y}^+)) \text{ for every } \underline{y}^+ \sim \pi(\underline{x}).$$

It is enough to determine $\mu_{\underline{y}^+}$ consistently on the family of cylinders

$$C(a_{-n}, \ldots, a_{-1}; \underline{y}^+) := \{\underline{z} \in \Sigma_0 : z_{-n}^{-1} = a_{-n}^{-1}, z_0^\infty = \underline{y}^+\}, \text{ where } z_m^n := (z_m, \ldots, z_n).$$

Let $A = (t_{ij})_{S \times S}$ be the transition matrix of Σ. Construct using the Perron-Frobenius theorem a row vector $\ell = \langle \ell_a : a \in S\rangle$ with positive entries such that $\ell A = \lambda \ell$. The eigenvalue λ is bigger than one, and is the leading eigenvalue of A. We have

$$\sum_{p \in S} \ell_p t_{pa} = \lambda \cdot \ell_a, \text{ in particular } \sum_{p \in S} g_{pa} = 1 \text{ where } g_{pa} := \frac{\ell_p t_{pa}}{\lambda \ell_a}.$$

Just as in the case of Markov chains, the set function

$$\mu_{\underline{y}^+}[C(a_{-n}, \ldots, a_{-1}; \underline{y}^+)] := g_{a_{-n}a_{-n+1}} g_{a_{-n+1}a_{-n+2}} \cdots g_{a_{-2}a_{-1}} g_{a_{-1}y_0} \ell_{y_0}$$

extends to a σ–additive measure on $A(\underline{y}^+)$ (with total mass ℓ_{y_0}). This is our measure $\mu_{\underline{y}^+}$. Having $\mu_{\underline{y}^+}$, we proceed to define $\mu_{[\underline{x}]} := \sum_{\underline{y}^+\sim\pi(\underline{x})} \mu_{\underline{y}^+}$.

It is easy to check that $\mu_{[\underline{x}]}$ satisfies the following identity:

$$\mu_{[\underline{x}]}[C(a_{-n},\ldots,a_{-1};\underline{y}^+)] = \ell_{a_{-n}}\lambda^{-n}. \tag{2.2}$$

We can use this to show that $\mu_{[\underline{x}]}$ is non-atomic: For every $\underline{y} \in [\underline{x}]$,

$$\mu_{[\underline{x}]}\{\underline{y}\} \leq \mu_{[\underline{x}]}[C(y_{-n},\ldots,y_{-1};\pi(\underline{y}))] = O(\lambda^{-n}) \xrightarrow[n\to\infty]{} 0.$$

Another consequence of (2.2) is that every non-empty cylinder has finite positive measure. Since every non–empty bounded open interval contains a non–empty cylinder, and is contained in a finite union of cylinders, every non–empty bounded open interval has finite positive measure. Finally, infinite rays have infinite measure, because they contain infinitely many cylinders of the form $C(a^i_{-1};\underline{y}^+(i))$ with a^i_{-1} constant. In summary, (M1) and (M2) hold.

To verify (M3), it is enough to show that $\mu_{[\underline{x}]}\circ\sigma = \lambda^{-1}\mu_{[\underline{x}]}$ when evaluated on cylinders. This holds because

$$\frac{(\mu_{[\underline{x}]}\circ\sigma)[C(a_{-n},\ldots,a_{-1};y_0^\infty)]}{\mu_{[\underline{x}]}[C(a_{-n},\ldots,a_{-1};y_0^\infty)]} = \frac{\mu_{[\underline{x}]}[C(a_{-n},\ldots,a_{-1},y_0;y_1^\infty)]}{\mu_{[\underline{x}]}[C(a_{-n},\ldots,a_{-1};y_0^\infty)]} =$$
$$= \frac{g_{a_{-n}a_{-n+1}}\cdots g_{a_{-1}y_0}g_{y_0y_1}\ell_{y_1}}{g_{a_{-n}a_{-n+1}}\cdots g_{a_{-1}y_0}\ell_{y_0}} = \frac{g_{y_0y_1}\ell_{y_1}}{\ell_{y_0}} = \frac{1}{\lambda}.$$

(M4) is obvious. □

Lemma 3. *Let m denote the measure of maximal entropy of $\sigma : \Sigma \to \Sigma$. For any two families $\mu_{[\underline{x}]}, \nu_{[\underline{x}]}$ which satisfy (M1), (M2),(M3), and (M4) there exists a positive constant c and a set $\Omega \subset \Sigma$ of full m–measure such that $\mu_{[\underline{x}]} = c\nu_{[\underline{x}]}$ for every $x \in \Omega$.*

Proof. Define for a function $f : \Sigma^+ \to \Sigma^+$ a new function $(Lf)(\underline{x}) = \sum_{a\in S} t_{ax_0}f(a,\underline{x})$. Ruelle's Perron–Frobenius Theorem (see e.g. [**B**]) states that there exists a unique probability measure ν_0 on Σ^+ and a unique positive continuous function h_0 on Σ^+ s.t. for some $\lambda > 0$, $Lh = \lambda h$, $L^*\nu_0 = \lambda\nu_0$, $\int h_0 d\nu_0 = 1$. We have already met the function h_0: it is $h_0(\underline{x}) = \ell_{x_0}$ where ℓ is the left eigenvector of $(t_{ij})_{N\times N}$.

The measure ν_0 is a Markov measure, $h_0(x)$ only depends on x_0, and λ is the leading eigenvalue of the transition matrix A. The measure $h_0 d\nu_0$ is globally supported, invariant, and ergodic. It is the measure of maximal entropy for $\sigma : \Sigma^+ \to \Sigma^+$ [**P**]. The measure of maximal entropy for the two sided shift is its natural extension.

Suppose $\mu_{[\underline{x}]}$ is a family of measures satisfying (M1), (M2), (M3), and (M4). Fix $p \geq 1$, and define a function $\varphi^{(p)} : \Sigma_0^+ \to \mathbb{R}$ by

$$\varphi^{(p)}(x_0,x_1,\ldots) = \mu_{[\sigma^p(\underline{x}^*)]}\{\underline{y} \in \Sigma_0 : y_{-p}^{-1} = x_0^{p-1}\ ;\ y_0^\infty = x_p^\infty\}.$$

where $\underline{x}^*$ is some (any) extension of $\underline{x} = (x_0, x_1, \ldots)$ to a two sided sequence in Σ_0. The definition is proper, since $[\sigma^p(\underline{x}^*)]$ is the same for all possible extensions.

We claim that $L\varphi^{(p)} = \lambda\varphi^{(p)}$ on Σ_0^+. For every $\underline{x} \in \Sigma$

$$(L\varphi^{(p)})(\underline{x}) = \sum_{a\in S} t_{ax_0}\varphi^{(p)}(a,\underline{x}) =$$

$$
\begin{aligned}
&= \sum_{a\in S} t_{ax_0}\mu_{[\sigma^p((a,\underline{x})^*)]}\{\underline{y}\in\Sigma_0 : y_{-p}^{-1} = (a\underline{x})_0^{p-1}\ ;\ y_0^\infty = (a\underline{x})_p^\infty\}\\
&= \sum_{a\in S} t_{ax_0}\mu_{[\sigma^{p-1}(\underline{x}^*)]}\{\underline{y}\in\Sigma_0 : y_{-p} = a, y_{-p+1}^{-1} = x_0^{p-2}\ ;\ y_0 = x_{p-1}, y_1^\infty = x_p^\infty\}\\
&= \lambda\sum_{a\in S} t_{ax_0}\mu_{[\sigma^{p}(\underline{x}^*)]}\{\sigma(\underline{y}) : \underline{y}\in\Sigma_0, y_{-p} = a, y_{-p+1}^{-1} = x_0^{p-2}\ ;\ y_0 = x_{p-1}, y_1^\infty = x_p^\infty\}\\
&= \lambda\sum_{a\in S} t_{ax_0}\mu_{[\sigma^{p}(\underline{x}^*)]}\{\underline{z}\in\Sigma_0 : z_{-p-1} = a, z_{-p}^{-2} = x_0^{p-2}\ ;\ z_{-1} = x_{p-1}, z_0^\infty = x_p^\infty\}\\
&= \lambda\mu_{[\sigma^{p}(\underline{x}^*)]}\{\underline{z}\in\Sigma_0 : z_{-p}^{-1} = x_0^{p-1}\ ;\ z_0^\infty = x_p^\infty\} = \varphi^{(p)}(\underline{x}).
\end{aligned}
$$

Since $\varphi^{(p)}$ is positive and measurable, $\varphi^{(p)}d\nu_0$ is a well defined σ–finite measure equivalent to ν_0. This measure is σ–invariant, because for every positive bounded measurable function $f:\Sigma^+\to\mathbb{R}$, the identity $L^*\nu=\lambda\nu$ implies that

$$\int f\circ\sigma\,\varphi^{(p)}d\nu_0 = \lambda^{-1}\int L[f\circ\sigma\cdot\varphi^{(p)}]d\nu_0 = \lambda^{-1}\int fL\varphi^{(p)}d\nu_0 = \int f\,\varphi^{(p)}d\nu_0.$$

We see that $\varphi^{(p)}$ is an invariant density for ν_0. So is $h_0(\underline{x}) = \ell_{x_0}$. Since $h_0 d\nu_0$ is an ergodic probability measure, $\exists c_p>0$ s.t. $\varphi^{(p)} = c_p h_0$ ν_0–almost surely.

We now interpret this in terms of the two–sided shift. Suppose $\underline{x}\in\Sigma_0$, then there is a set of full measure Ω of $\underline{x}\in\Sigma_0$ such that for every p and every word $(a_{-p},\ldots,a_{-1})$ s.t. $t_{a_{-p},a_{-p+1}}\cdots t_{a_{-1},x_0}=1$,

$$\mu_{[\underline{x}]}\Big(C(a_{-p},\ldots,a_{-1};x_0^\infty)\Big) = \varphi^{(p)}(\underline{a},x_0^\infty) = c_p h_0(\underline{a},x_0^\infty) = c_p\ell_{a_{-p}}.$$

Keeping $(a_{-p+1},\ldots,a_{-1},x_0^\infty)$ constant and summing the left hand side over all a_{-p} s.t. $t_{a_{-p}a_{-p+1}}=1$, we see that $c_p\sum_{b\in S} t_{ba_{-p+1}}\ell_b = c_{p-1}\ell_{a_{-p+1}}$, whence since $\ell A=\lambda\ell$, $c_p=\lambda^{-1}c_{p-1}$. The conclusion of all this is that for all $\underline{x}\in\Omega$,

$$\mu_{[\underline{x}]}\Big(C(a_{-p},\ldots,a_{-1};x_0^\infty)\Big) = c_1\Big(\lambda^{-p}\ell_{a_{-p}}\Big).$$

Comparing this with (2.2), we see that $\mu_{[\underline{x}]}$ is c_1 times the Bowen–Marcus measure we constructed in the previous step.

We now repeat this argument for the second family of measures $\nu_{[\underline{x}]}$ and obtain a second constant c_1' and a second set Ω' of full measure of $\underline{x}$ such that $\nu_{[\underline{x}]}$ is c_1' times the Bowen–Marcus measure from the previous step. The lemma follows. $\square$

Proof of Theorem 1: Existence. The idea is to define for $t>0$,

$$
\begin{aligned}
\tau_s^t(\underline{x}) &:= \min\{\underline{z}\sim\underline{x} : \underline{z}\succ\underline{x}\ ,\ \mu_{[\underline{x}]}\big([\underline{x},\underline{z})\big)\geq t\},\\
\tau_s^{-t}(\underline{x}) &:= \max\{\underline{y}\sim\underline{x} : \underline{y}\prec\underline{x}\ ,\ \mu_{[\underline{x}]}\big([\underline{y},\underline{x})\big)\geq t\}.
\end{aligned}
\tag{2.3}
$$

To see that the definition is proper, and that it defines a flow, we have to show that for every $t>0$ there are unique $\underline{y},\underline{z}\in\Sigma_0$ such that $\underline{y}\prec\underline{x}\prec\underline{z}$ and

$$\mu_{[\underline{x}]}\big([\underline{y},\underline{x})\big) = t = \mu_{[\underline{x}]}\big([\underline{x},\underline{z})\big). \tag{2.4}$$

Here is the proof. By (M2), there exist $\underline{y}' \prec \underline{x} \prec \underline{z}'$ s.t. $\mu_{[\underline{x}]}\big([\underline{x},\underline{z}')\big) > t$ and $\mu_{[\underline{x}]}\big([\underline{y}',\underline{x})\big) > t$. Set

$$\underline{y} := \sup\{\underline{y}' \in \Sigma_0 : \underline{y}' \prec \underline{x} \text{ and } \mu_{[\underline{x}]}\big([\underline{y}',\underline{x})\big) > t\},$$
$$\underline{z} := \inf\{\underline{z}' \in \Sigma_0 : \underline{z}' \succ \underline{x} \text{ and } \mu_{[\underline{x}]}\big([\underline{x},\underline{z}')\big) > t\}.$$

Since $([\underline{x}],\prec)$ is order isomorphic to $(\mathbb{R},<)$, the infimum and supremum exist, they belong to $[\underline{x}]$, and one can find sequences $\underline{y}^{(n)} \uparrow \underline{y}$ and $\underline{z}^{(n)} \downarrow \underline{z}$ such that

$$\mu_{[\underline{x}]}\big([\underline{y}^{(n)},\underline{x})\big) > t \text{ and } \mu_{[\underline{x}]}\big([\underline{x},\underline{z}^{(n)})\big) > t.$$

It follows that $\mu_{[\underline{x}]}\big([\underline{y},\underline{x})\big) \geq t$ and $\mu_{[\underline{x}]}\big([\underline{x},\underline{z}]\big) \geq t$.

Similarly, one can find sequences $\underline{x} \succ \widehat{\underline{y}}^{(n)} \downarrow \underline{y}$ and $\underline{x} \prec \widehat{\underline{z}}^{(n)} \uparrow \underline{z}$. These must satisfy $\mu_{[\underline{x}]}\big([\widehat{\underline{y}}^{(n)},\underline{x})\big) \leq t$ and $\mu_{[\underline{x}]}\big([\underline{x},\widehat{\underline{z}}^{(n)})\big) \leq t$. Consequently

$$\mu_{[\underline{x}]}\big((\underline{y},\underline{x})\big) \leq t \text{ and } \mu_{[\underline{x}]}\big([\underline{x},\underline{z})\big) \leq t.$$

Since $\mu_{[\underline{x}]}$ is non–atomic, we must have $\mu_{[\underline{x}]}\big([\underline{y},\underline{x})\big) = t$ and $\mu_{[\underline{x}]}\big([\underline{x},\underline{z})\big) = t$.

This shows that $\underline{y}$ and $\underline{z}$ exist. They are unique, because if there were other solutions $\widehat{\underline{y}},\widehat{\underline{z}}$ to (2.4), then the intervals with endpoints $\underline{y},\widehat{\underline{y}}$ or $\underline{z},\widehat{\underline{z}}$ would be non-empty bounded intervals with zero measure, in contradiction to (M2).

We claim that the flow thus defined satisfies the statement of the theorem. All properties are clear, except for almost continuity. Fix $\underline{x}, t$ s.t. $\tau_s^t(\underline{x})$ has no left minimal tail. Fix N. We construct $N', \epsilon > 0$ s.t. for every $\underline{x}' \in \Sigma_0$

$$\left.\begin{array}{r} t' \in (t-\epsilon, t+\epsilon) \\ x_i' = x_i \text{ for all } |i| \leq N' \end{array}\right\} \Rightarrow \tau_s^{t'}(\underline{x}')_i = \tau_s^t(\underline{x})_i \text{ for all } |i| \leq N. \tag{2.5}$$

By assumption $\tau_s^t(\underline{x})$ has no minimal left tails, therefore there exists some $N_1 > N$ s.t. $\tau_s^t(\underline{x})_{-N_1} \neq P_{\min}(\tau_s^t(\underline{x})_{-N_1+1})$. Since $\tau_s^t(\underline{x}) \in \Sigma_0$, it has no maximal left tails, therefore there exists some $N_2 > N_1$ s.t. $\tau_s^t(\underline{x})_{-N_2} \neq P_{\max}(\tau_s^t(\underline{x})_{-N_2+1})$.

Let $\underline{y}$ denote the element of Σ_0 s.t. $y_i = \tau_s^t(\underline{x})_i$ for $i \geq -N_1+1$ and $y_i = P_{\min}(y_{i+1})$ for $i \leq -N_1$. Note that $\underline{y} \prec \tau_s^t(\underline{x})$.

Let $\underline{z}$ denote the element of Σ_0 s.t. $z_i = \tau_s^t(\underline{x})_i$ for $i \geq -N_2+1$, $z_{-N_2} = P_{\max}(z_{-N_2+1})$, and $z_i = P_{\min}(z_{i+1})$ for all $i < -N_2$. Note that $\underline{z} \succ \tau_s^t(\underline{x})$.

Now pick $\epsilon > 0$ smaller than $\mu_{[\underline{x}]}\big([\tau_s^t(\underline{x}),\underline{z})\big)$ and $\mu_{[\underline{x}]}\big([\underline{y},\tau_s^t(\underline{x})\big)$. We claim that (2.5) holds with ϵ and $N' := N_2$. Pick some $\underline{x}' \in \Sigma_0$ such that $x_i' = x_i$ for all $|i| < N'$, and define $\underline{y}', \underline{z}'$ by

$$y_i' := \begin{cases} y_i & i \leq N' \\ \tau_s^t(\underline{x}')_i & i > N' \end{cases} \quad \text{and} \quad z_i' := \begin{cases} z_i & i \leq N' \\ \tau_s^t(\underline{x}')_i & i > N', \end{cases}$$

then $\underline{y}' \prec \tau_s^t(\underline{x}') \prec \underline{z}'$.

Since $\underline{y}' \prec \tau_s^t(\underline{x}') \prec \underline{z}'$ and $y_i' = \tau_s^t(\underline{x})_i = z_i'$ for all $|i| \leq N_1 - 1$, we also have that $\tau_s^t(\underline{x})_i = \tau_s^t(\underline{x}')_i$ for all $|i| \leq N_1 - 1$. It follows from the structure of the measure constructed in claim 1 of part 3 that

$$\mu_{[\underline{x}']}\big((\underline{y}',\tau_s^t(\underline{x}')]\big) = \mu_{[\underline{x}]}\big((\underline{y},\tau_s^t(\underline{x})]\big)\ ,\ \mu_{[\underline{x}']}\big([\tau_s^t(\underline{x}'),\underline{z}')\big) = \mu_{[\underline{x}]}\big([\tau_s^t(\underline{x}),\underline{z})\big).$$

By the choice of ϵ, $\mu_{[\underline{x}']}\big((\underline{y}', \tau_s^t(\underline{x}')]\big) > \epsilon$ and $\mu_{[\underline{x}']}\big([\tau_s^t(\underline{x}'), \underline{z}')\big) > \epsilon$. It follows that for every t' s.t. $|t-t'| < \epsilon$, $\underline{y}' \prec \tau_s^{t'}(\underline{x}') \prec \underline{z}'$. Since $y_i' = \tau_s^t(\underline{x})_i = x_i'$ for all $i > -N_1$, $\tau_s^{t'}(\underline{x}')_i = \tau_s^t(\underline{x})_i$ for all $|i| < N_1$. Since $N_1 > N$, (2.5) follows. □

Proof of Theorem 1: Uniqueness. Suppose there are two flows τ_s, θ_s satisfying the statement of the theorem. As we saw above, τ_s and θ_s have the same (unparameterized) orbits, and they move on the these orbits in the same order. Let $\mu_{[\underline{x}]}$ and $\nu_{[\underline{x}]}$ denote the time parameterization measures, given by

$$\mu_{[\underline{x}]}[\underline{x}, \tau_s^t(\underline{x})) = t \ , \nu_{[\underline{x}]}[\underline{x}, \theta_s^t(\underline{x})) = t.$$

We saw that they must satisfy (M1),(M2),(M3), and (M4).

Let m denote the measure of maximal entropy of $\sigma : \Sigma \to \Sigma$. By Lemma 3, there exist a positive number $c > 0$ and a set Ω s.t. $m(\Omega) = 1$ and $\mu_{[\underline{x}]} = c\nu_{[\underline{x}]}$ for all $\underline{x} \in \Omega$. Since m has global support, there exists a dense set of $\underline{x} \in \Sigma_0$ s.t. $\tau_s^t(\underline{x}) = \theta_s^{ct}(\underline{x})$ for all $t \in \mathbb{R}$. By the almost continuity property $\tau_s^t(\underline{x}) = \theta_s^{ct}(\underline{x})$ whenever $\tau_s^t(\underline{x}), \theta_s^{ct}(\underline{x})$ do not have left minimal tails.

If $\tau_s^t(\underline{x})$ or $\theta_s^{ct}(\underline{x})$ have a left minimal tail, then construct $t_n \downarrow t$ s.t. $\tau_s^{t_n}(\underline{x}), \theta_s^{ct_n}(\underline{x})$ do not have left minimal tails. All but countably many parameters are like that. Since $\tau_s^t(\underline{x}) = \inf_n \tau_s^{t_n}(\underline{x}) = \inf_n \theta_s^{ct_n}(\underline{x}) = \theta_s^{ct}(\underline{x})$, we get that $\tau_s^t(\underline{x}) = \theta_s^{ct}(\underline{x})$. □

2.1. **Ergodic properties of adic flows.** The following theorem is due to Ito **[I]**:

Theorem 2. *Suppose (Σ, σ) is topologically mixing, then the adic flow on Σ_0 is uniquely ergodic and of zero entropy. The invariant probability measure is the measure of maximal entropy of $\sigma : \Sigma_0 \to \Sigma_0$ (Parry's measure).*

The key to the proof is that $\Lambda := \{\underline{x} \in \Sigma_0 : x_{i-1} = P_{\min}(x_i) \text{ for all } i \le 0\}$ is a Poincaré section for the adic flow, and the Poincaré map $T : \Lambda \to \Lambda$ satisfies $T \circ \pi = \pi \circ \tau$ where $\pi : \underline{x} \mapsto x_0^\infty$ and $\tau : \Sigma_0^+ \to \Sigma_0^+$ is Vershik's adic transformation. The height function is (in the notation of Lemma 3) $H := h_0 \circ \pi$.

Since $\tau : \Sigma_0^+ \to \Sigma_0^+$ is uniquely ergodic with zero entropy **[LV]**, $\tau_s^t : \Sigma_0 \to \Sigma_0$ is uniquely ergodic with zero entropy.

Ergodicity cannot be replaced by weak mixing: take for example $\Sigma = \{0,1\}^{\mathbb{Z}}$. In this case the function h_0 is constant, and the adic flow is conjugate to the constant suspension of the adding machine. The function $\varphi(\underline{x}) = (-1)^{x_0}$ is an eigenfunction.

3. Relation to horocycle flows

We recall some facts on the symbolic dynamics of *geodesic* flows on hyperbolic surfaces with cusps. These will be used at the end of the section to relate the horocycle flow to time changes of adic flows.

Suspension flows. Let $f : Y \to Y$ be an invertible map, and $\rho : Y \to \mathbb{R}^+$ a function such that $\sum_{n \ge 0} \rho \circ f^n = \sum_{n \le 0} \rho \circ f^n = \infty$ everywhere on Y. Let $\stackrel{r}{=}$ denote the equivalence relation on $Y \times \mathbb{R}$ generated by the equivalences $(y, \xi) \stackrel{r}{=} (f(y), \xi - \rho(y))$. Let Y_r denote the set of equivalence classes of $\stackrel{r}{=}$, denoted by $\langle y, \xi \rangle$. Let $\pi_\rho : Y \times \mathbb{R} \to Y_\rho$ be the equivalence class map $\pi_\rho : (y, \rho) \mapsto \langle y, \rho \rangle$. The *suspension flow over f with roof function ρ* is $F^t : Y_\rho \to Y_\rho$ given by $F^t\langle y, \xi \rangle = \langle y, \xi + t \rangle$.

Hyperbolic surfaces. The *Poincaré disc* is $\mathbb{D} := \{x + iy \in \mathbb{C} : x^2 + y^2 < 1\}$ with the Riemannian metric $2\sqrt{dx^2 + dy^2}/(1 - x^2 - y^2)$. The group of orientation preserving isometries of $\mathbb{D}$ is Möb$(\mathbb{D})$, the group of Möbius transformations which preserve the unit disc, see e.g. [**Ka**].

A *hyperbolic surface* is a Riemannian surface M such that every $p \in M$ has a neighborhood which is isometric to $\{z \in \mathbb{D} : |z| < \epsilon\}$ for some $\epsilon > 0$. It is known that every complete connected orientable hyperbolic surface is isometric to an orbit space $\Gamma \setminus \mathbb{D} = \{\Gamma z := \{g(z) : g \in \Gamma\} : z \in \mathbb{D}\}$ where Γ is some discrete subgroup of Möb$(\mathbb{D})$. Henceforth we take all our hyperbolic surfaces to be complete, connected, and orientable, whence of the form above.

Cutting sequences [A],[Mo],[Ko]. Let $M = \Gamma \setminus \mathbb{D}$ be a non-compact hyperbolic surface of finite area, then Γ has a fundamental domain F with the following properties:

(1) F is a geodesic polygon all of whose vertices are on $\partial\mathbb{D}$ ("cusps");
(2) F has an even number of sides $a_1, \ldots, a_n; \overline{a}_1, \ldots, \overline{a}_n$, and there are $\varphi_{a_i} \in \Gamma$ s.t. φ_{a_i} maps a_i onto $\overline{a}_i$, and $\varphi_{\overline{a}_i} = \varphi_{a_i}^{-1}$ maps $\overline{a}_i$ onto a_i.

We re-index the sides of F as $s_1, \ldots, s_{2n}$ according to the counterclockwise order of the sides as edges of F (the choice of s_1 is arbitrary). We keep the convention that $\overline{s}_i$ is the congruent edge to s_i.

Following [**BKS**] and [**BS**], we use the symbols s_i to label the sides of F from the inside ("interior labels"). The images of F by elements of Γ tile $\mathbb{D}$. Extend the labeling to the other tiles using the action of Γ. Now every edge will also have an "exterior label" (the interior label of the adjacent tile). This labeling system is the unique scheme so that (a) if the interior label of an edge is s, then its exterior label is $\overline{s}$, and (b) the counterclockwise order of the interior labels is the same in each tile. The second property is because Γ preserves orientation.

Let $g : T^1\mathbb{D} \to T^1\mathbb{D}$ denote the geodesic flow on the unit tangent bundle of $\mathbb{D}$. Let $T^1(F)$ denote collection of unit tangent vectors with base points in F. The *cutting sequence* of $\omega \in T^1(F)$ is the sequence of the interior labels of the sides of the tiles that $\{g^t(\omega)\}_{t\in\mathbb{R}}$ intersects. This is a two sided infinite sequence as long as $g^t(\omega)$ does not converge to one of the cusps as $t \to \pm\infty$. We use the convention that the zeroth coordinate of the cutting sequence $(s_{i_k})_{k\in\mathbb{Z}}$ of ω is the side s_{i_0} such that ω belongs to the geodesic segment between s_{i_0} and s_{i_1}.

The collection of all cutting sequences is equal to the subshift of finite type

$$\Sigma := \{x \in S^{\mathbb{Z}} : x_i \neq \overline{x}_{i+1} \text{ for all } i \in \mathbb{Z}\},\ S := \{s_1, \ldots, s_{2n}\}.$$

One way to see this is to consider *boundary expansions* [**BS**]. The boundary expansion of $x \in \partial\mathbb{D}$ is the one–sided cutting sequence of (any) geodesic ray which starts at an interior point of F and terminates at x. This is a one–sided infinite sequence for all x which does not belong to the Γ–orbit of the vertices of F (a countable set of exceptions). It is easy to see that the collection of all boundary expansions is equal to $\Sigma^+ := \{x \in S^{\mathbb{N}\cup\{0\}} : x_i \neq \overline{x}_{i+1} \text{ for all } i \geq 0\}$. The identification of the collection of cutting sequences with Σ easily follows.[1]

[1] The simple relationship between cutting sequences and boundary expansions is only true for geometrically finite surfaces *with cusps*. In the cocompact case, the set of cutting sequences is not necessarily a SFT [**E**], although it is always sofic [**S2**].

A Poincaré section for the geodesic flow [S1]. We say that a geodesic on M *tends to a cusp* at ∞ (resp. $-\infty$), if one (all) of its lifts to $\mathbb{D}$ tends at ∞ (resp. $-\infty$) to one of the vertices of F or its image under some element of Γ. Let

$$X := \{\omega \in T^1(M) : \text{ the geodesic of } \omega \text{ does not tend to a cusp at } \infty \text{ or } -\infty\}.$$

If $\omega \in X$, then the lift of $g^t(\omega)$ to $\mathbb{D}$ crosses infinitely many Γ–images of F, and the set of crossing times is discrete. Consequently, the following set is a Poincaré section for the geodesic flow on X:

$$\Omega := \{\Gamma\omega : \omega \in T^1(\partial F)\ ,\forall \epsilon > 0 \text{ small}\ , g^\epsilon(\omega) \in T^1(\text{int}(F))\}.$$

Let $T : \Omega \to \Omega$ denote the *section map*,

$$T(\omega) := g^{R(\omega)}(\omega), \text{ where } R(\omega) := \inf\{t > 0 : g^t(\omega) \in \Omega\}.$$

The relation to cutting sequences is as follows. Partition Ω into the sets $A_s = \{\Gamma\omega \in \Omega : \omega \text{ is based at } \Gamma s\}$ where s is one of the sides of F. For $\omega \in \Omega$, let A_{s_n} denote the unique element of this partition so that $T^n(\omega) \in A_{s_n}$. The sequence $\{A_{s_n}\}_{n\in\mathbb{Z}}$ is called the *itinerary* of ω. It is not difficult to check that $\{A_{s_n}\}_{n\in\mathbb{Z}}$ is the itinerary of $\omega \in \Omega$, iff $\{s_n\}_{n\in\mathbb{Z}}$ is the cutting sequence of the geodesic of ω, indexed in such a way that $\omega \in A_{s_0}$.

Let $\pi : \Omega \to \Sigma$ denote the map which send $\omega \in \Omega$ to the sequence $\{s_n\}_{n\in\mathbb{Z}}$ constructed above. This is a bijection, and $\pi \circ T = \sigma \circ \pi$ where $\sigma : \Sigma \to \Sigma$ is the left shift. This allows us to represent the geodesic flow on X as a suspension flow $\sigma^t : \Sigma_r \to \Sigma_r$ over $\sigma : \Sigma \to \Sigma$ with roof function $r = R \circ \pi$.

The value of r at $x \in \Sigma$ can be calculated as follows. A cutting sequence $x \in \Sigma$ determines a geodesic $\gamma = \gamma(x)$ on $\mathbb{D}$, because it contains the boundary expansions of the endpoints of this geodesic. We let $r(x)$ be the hyperbolic length of the segment of γ between side x_0 and side x_1 (this segment can be unambiguously defined to be the collection of base points of all unit tangent vectors on γ which admit the cutting sequence x.)

It follows from constructions in [**LT**] that one can find a continuous function $u : \Sigma \to \mathbb{R}$ s.t.

$$r^+ := r + u - u \circ \sigma$$

depends only on non-negative coordinates, in the sense that for every $x, y \in \Sigma$ such that $x_i = y_i$ for all $i \geq 0$, $r^+(x) = r^+(y)$. See lemma 2.2 in [**LS**] for more details.

The strong stable foliation in symbolic coordinates [BM]. The *strong stable manifold* of $\omega \in T^1M$ is by definition

$$W^{ss}(\omega) := \{\omega' \in T^1M : d(g^t(\omega'), g^t(\omega)) \xrightarrow[t\to\infty]{} 0\}.$$

We wish to describe the lifts of these sets to the symbolic suspension space Σ_r.

We treat the stable horocycle flow; the description of the unstable flow is essentially the same. Let $\mathfrak{G}$ denote the grand tail relation of σ:

$$\mathfrak{G} := \{(x, y) \in \Sigma \times \Sigma : \exists p, q \text{ s.t. } x_p^\infty = y_q^\infty\},$$

where $x_k^\infty = (x_k, x_{k+1}, \ldots)$. If two unit tangent vectors in X lie on the same strong stable manifold, then they have $\mathfrak{G}$–equivalent cutting sequences.

Suppose x, y are two $\mathfrak{G}$–equivalent cutting sequences. As shown in [**BM**], two elements $\langle x, \xi\rangle, \langle y, \eta\rangle$ in Σ_r code two line elements on the same strong stable manifold iff

$$\exists p, q \text{ s.t. } x_p^\infty = y_q^\infty \text{ and } [\xi - r_{p+n}(x)] - [\eta - r_{q+n}(y)] \xrightarrow[n\to\infty]{} 0.$$

Here and throughout, $r_m := r + r\circ\sigma + \cdots + r\circ\sigma^{m-1}$.

Let $u : \Sigma \to \mathbb{R}$ be the function mentioned at the end of the previous section so that $r^+(x) := r(x) + u(x) - u(\sigma x)$ depends only on $x_0, x_1, \ldots$, then

$$\begin{aligned} r_{p+n}(x) - r_{q+n}(y) &= r^+_{p+n}(x) - r^+_{q+n}(y) + u(\sigma^{p+n}x) - u(x) - u(\sigma^{q+n}y) + u(y) \\ &\xrightarrow[n\to\infty]{} r_p^+(x) - r_q^+(y) + u(y) - u(x). \end{aligned}$$

Define $\Psi(x,y) := r_p^+(x) - r_q^+(y) + u(y) - u(x)$, whenever $x_p^\infty = y_q^\infty$ (this is a proper definition for non–eventually periodic cutting sequences). The calculation above shows that if $(x,y) \in \mathfrak{G}$ and $\xi - \eta = \Psi(x,y)$ then $\langle x,\xi\rangle, \langle y,\eta\rangle$ lie on the same strong stable manifold. When x, y are not eventually periodic, this is an iff.

The horocycle flow on a hyperbolic surface with cusps. We finally arrive to the description of the horocycle flow in the symbolic space Σ_r. The orbits of this flow are

$$\mathrm{Hor}\langle x, \xi\rangle \quad = \quad p\{\langle y, \xi + \Psi(y,x)\rangle : \exists p, q \text{ s.t. } x_p^\infty = y_q^\infty\}$$

Here $p : \Sigma_r \to T^1(\Gamma \setminus \mathbb{H})$ is the coding map, and $\langle\cdot,\cdot\rangle$ denotes the element of Σ_r corresponding to (x, ξ). We shall describe the horocycle flow as

$$(*) \qquad H^t\langle x,\xi\rangle = \langle \tau^{te^{\xi+c(x)}}(x), \xi + \Psi(\tau^{te^{\xi+c(x)}}(x), x)\rangle,$$

and proceed to describe τ^t and $c(x)$.

The flow τ^t is a time change of an adic flow on Σ. In order to describe it we need to describe a tail partial order $\preceq$ and a family of measures $\mu_{[x]}$ on $[x] := \{y : \exists p \text{ s.t. } x_p^\infty = y_p^\infty\}$ (see section 2).

The tail order is defined as follows: Recall that the elements of S (the sides of the fundamental domain F) are ordered cyclically by $s_1, s_2, \ldots, s_{2n}, s_1$. For every $1 \le i \le 2n$, let $\underset{s_i}{<}$ be the order

$$s_i \underset{s_i}{<} s_{i+1} \underset{s_i}{<} \cdots \underset{s_i}{<} s_{2n} \underset{s_i}{<} s_1 \underset{s_i}{<} \cdots \underset{s_i}{<} s_{i-1}.$$

The tail partial order $\preceq$ is

$$x \preceq y \text{ iff } x = y \text{ or } \exists p \text{ s.t. } x_{p+1}^\infty = y_{p+1}^\infty \text{ and } x_p \underset{x_{p+1}}{<} y_p.$$

The meaning of this order is as follows: let x, y be the cutting sequences of two directed geodesics with the same terminus (=limit at $+\infty$). Then $x \preceq y$ if the geodesic with cutting sequence x is mapped by the horocycle flow to the geodesic with cutting sequence y in *positive* time.

We define the measure $\mu_{[x]}$. It is enough to construct $\mu_{[x]}$ on each of the sets $\Lambda(y_0^\infty) := \{z : z_0^\infty = y_0^\infty\}$, with $y_0^\infty \in \{\underline{y}^+ \in \Sigma^+ : \exists p \text{ s.t. } y_p^\infty = x_p^\infty\}$ (a countable set). We do this by canonically identifying $\Lambda(y_0^\infty)$ with an interval, and then pulling back Lebesgue's measure.

Let $\alpha = \alpha(y) \in \partial\mathbb{D}$ be (the unique) point on $\partial\mathbb{D}$ whose boundary expansion is y_0^∞. The set $\Lambda(y_0^\infty)$ is the set of cutting sequences of all geodesics in $\mathbb{D}$ which intersect F and terminate at α. The beginning points of these geodesics form an arc in $\partial\mathbb{D}$. The horocycle flow parameterizes this arc in a natural way, and the idea is to pull back this parameterization.

Here is how to do this. Assume w.l.o.g. that $0 \in \mathrm{int}(F)$. Let $\omega(\alpha) \in T^1\mathbb{D}$ denote the unit tangent vector based at 0 and pointing at α, let $\mathbb{H} := \{x + iy : x, y \in \mathbb{R}, y > 0\}$, and let $\omega_0 \in T^1\mathbb{H}$ denote the unit tangent vector based at i and pointing north. There exists a unique Möbius transformation $\psi_\alpha : \mathbb{D} \to \mathbb{H}$ which maps $\omega(\alpha)$ to ω_0. This map transforms all geodesics which terminate at α to vertical lines pointing at infinity. The collection of geodesics coded by $\Lambda(y_0^\infty)$ transforms to a vertical band. This band can be canonically identified with an interval on $\partial\mathbb{H}$. We define $\mu_{[x]}|_{\Lambda(y_0^\infty)}$ to be the pull back of the (non-normalized) Lebesgue measure of this interval.

The meaning of $\mu_{[x]}$ is as follows. Elements of $[x] \cap \Lambda(y_0^\infty)$ can be canonically identified with an arc on the horocycle of (Euclidean) diameter one, which is tangent to $\partial\mathbb{H}$ at the point α with boundary expansion y_0^∞. The horocycle flow parametrizes this horocycle by hyperbolic length. The map ψ_α maps the horocycle of diameter one onto a horizontal line at (Euclidean) height one. The arc becomes an interval; the length parametrization of the arc becomes Lebesgue's measure. The measure $\mu_{[x]}$ is this measure in symbolic coordinates.

It remains to determine $c(x)$ in $(*)$. Unfortunately, this function depends in an essential way on the choice of the fundamental domain, and is not at all 'canonical'. Here is how it is defined. We use the notation

$$H_\alpha(z) := \text{horocycle which passes through } z \text{ and is tangent to } \partial\mathbb{H} \text{ at } \alpha.$$

Let $z(x) :=$ the base point on ∂F of the unit tangent vector coded by $\langle x, 0\rangle$. Define

$$c(x) := \ln \mathrm{Im}\,[\psi_{\alpha(x)}(z(x))]$$

This is the (signed) hyperbolic distance between the horocycle $H_{\alpha(x)}(z(x))$ and the unit horocycle $H_{\alpha(x)}(\alpha(x) + i)$. It follows that the hyperbolic distance between the unit horocycle and $H_{\alpha(x)}(\langle x, \xi\rangle)$ is $\xi + c(x)$. Therefore, the speed along $H_{\alpha(x)}(\langle x, \xi\rangle)$ is $e^{-(\xi + c(x))}$ times the speed along the unit horocycle. It follows that the horocycle flow is given by $(*)$.

A final simplification. It is possible to remove the dependence on $c(x)$ in $(*)$ by an additional change of coordinates. Let

$$\widetilde{r}(x) := r(x) + c(x) - c(\sigma(x)),$$

and let $\Sigma_{\widetilde{r}}$ denote the set of equivalence classes of the relation $\overset{\widetilde{r}}{=}$ generated by $(x, \eta) \overset{\widetilde{r}}{=} (\sigma(x), \eta - \widetilde{r}(x))$. Denote equivalence classes w.r.t. $\overset{\widetilde{r}}{=}$ by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$. The map $\vartheta : \Sigma_r \to \Sigma_{\widetilde{r}}$, $\langle x, \xi\rangle \mapsto \langle\!\langle x, \xi + c(x)\rangle\!\rangle$ is well defined, and $\widetilde{H}^t := \vartheta H^t \vartheta^{-1}$ becomes

$$(**) \qquad \widetilde{H}^t \langle\!\langle x, \xi\rangle\!\rangle = \left\langle\!\!\left\langle \tau^{te^{\xi}}(x), \xi + \Xi(\tau^{te^{\xi}}(x), x) \right\rangle\!\!\right\rangle$$

where $\Xi(x, y) := \Psi(x, y) + c(x) - c(y)$.

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