SUPPLEMENTARY MATERIAL

Proof of Lemma 5.1. We only prove the lower bound for the analytic moment case (other cases are similar). We have,

$$\begin{aligned}\left|\sum_{k=3}^{\infty} \tfrac{1}{k!} m_k(\Delta) s^k\right| &\leq \tfrac{1}{2}\sum_{k=3}^{\infty} \alpha^{k-2} m_2(\Delta)^{\frac{k}{2}} s^k \\ &= \tfrac{s^2 m_2(\Delta)}{2} \sum_{k=1}^{\infty} (s\alpha\sqrt{m_2(\Delta)})^k\end{aligned}$$

For our choice of s, we have $\sum_{k=1}^{\infty}(s\alpha\sqrt{m_2(\Delta)})^k \leq \sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^k = \frac{1}{3}$. Hence,

$$\begin{aligned}\sum_{k=2}^{\infty} \tfrac{m_k(\Delta)s^k}{k!} &\geq \tfrac{s^2 m_2(\Delta)}{2}\left(1 - \sum_{k=1}^{\infty}(s\alpha\sqrt{m_2(\Delta)})^k\right) \\ &\geq \tfrac{s^2 m_2(\Delta)}{3} = \tfrac{1}{3}\tfrac{m_2(\Delta)}{\max\{16\alpha^2 m_2(\Delta),1\}}\end{aligned}$$ □

Proof of Lemma 5.2. As $s \in [0,1]$, by convexity, we have $\mathcal{L}(\theta) - \mathcal{L}(\theta^\star) \geq \mathcal{L}(\theta^\star + s\Delta) - \mathcal{L}(\theta^\star)$. We consider the analytic moment case (cumulant case is easier). By Lemma 3.5,

$$\mathcal{L}(\theta) - \mathcal{L}(\theta^\star) \geq \log(1 + \tfrac{m_2(\Delta)}{3\max\{16\alpha^2 m_2(\Delta),1\}})$$

By Jensen's inequality, we know that the 4th standardized moment (kurtosis) is greater than 1, so $\alpha^2 \geq \frac{1}{12}$ (since $\frac{4!}{2}\alpha^2 \geq 1$). Thus, $\frac{m_2(\Delta)}{3\max\{16\alpha^2 m_2(\Delta),1\}} \leq \frac{1}{48\alpha^2} \leq 1/4$ since the sum is only larger if we choose any argument in the max. Now for $0 \leq x \leq 1/4$, we have $\log(1+x) \geq 1 + \frac{3}{4}x$. Hence,

$$\log(1 + \tfrac{m_2(\Delta)}{3\max\{16\alpha^2 m_2(\Delta),1\}}) \geq \tfrac{m_2(\Delta)}{4\max\{16\alpha^2 m_2(\Delta),1\}}$$

which proves (6). For the second claim, the precondition implies that the max in (6) will be achieved at 1, which directly implies the lower bound. For the upper bound, we apply Lemma 5.1 with $s = 1$ ($s = 1$ under our precondition), which implies that $\sum_{k=2}^{\infty}\frac{m_k(\Delta)}{k!}$ is less than $\frac{2}{3}m_2(\Delta)$. The claim follows using Lemma 3.5, with $s = 1$, and the fact that $\log(1+x) \leq x$. □