Poisson Boundary Theory

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Abstract

Notes for the lecture "Introduction to Poisson boundary theory" given in the Probability student seminar in Weizmann institute. In this lecture we will present basics from "stationary dynamical system theory" and describe Furstenberg's construction for the Poisson boundary.

Contents

Part I. Informal discussion

1 What is a Poisson boundary

Let (G,μ) be a group with a measure. μ defines a random walk on G: start from e pick a random element g_1 distributed by μ. Now pick another one g_2 and the position at time 2 is g_1g_2, and so on. It is well known that such a walk can be either recurrent or transient. We are interesting in the study of transient walks. One can ask about the speed: how fast the walk escape to infinity.

We will be interesting in the different ways the walk can escape to infinity.

The Poisson boundary is a description of all the behaviors at infinity of the random walk. It can thought of as a compact measure space (Π,ν) that can be attached topologically to G in such a way that almost every $w\in(\Omega,\mathbb{P})=(G,\mu)^{\mathbb{N}}$ converges to some element of Π. The value $\nu(E)$ for $E\subset\Pi$ reflects the probability that the random walk will hit Π at E.

Example 1. $\left(\mathbb{Z}^3,\frac{1}{6}\left(\delta_{(1,0,0)}+\delta_{(-1,0,0)}+\delta_{(0,1,0)}+\delta_{(0,-1,0)}+\delta_{(0,0,1)}+\delta_{(0,0,-1)}\right)\right)$.

Example 2.$\left(F_2,\frac{1}{4}\left(\delta_a+\delta_{a^{-1}}+\delta_b+\delta_{b^{-1}}\right)\right)$.

Example 3. Lamplighter: $\mathbb{Z}/2\mathbb{Z}\wr\Gamma$ when $\Gamma=\mathbb{Z}^d$.

Question: Anyone has an idea of what are the boundaries these examples?

2 Equivalence relation point of view

The description above suggest that we need to have a measurable function $\text{bnd}:\Omega\to\Pi$ that reflects the converges to the boundary. Now this boundary function defines an equivalence relation on Ω: $w\sim w'$ if $\text{bnd}(w)=\text{bnd}(w')$.

The essential problem is that for a general group, there is no candidate space that can serve as a boundary, and the theory needs an intrinsic description: although we don't know where a walk goes to, we can say that two walks are going to the same place, using the boundary function. After having such an equivalence relation, a point in Π can be thought as an equivalence class.

In some sense, it is similar to the Cauchy sequences in calculus, and "completing a space" using them.

This theory started when Furstenberg studied during the 60' a non-commuting multiplicative version of the law of large numbers: instead of $X_i\sim\mu$ and $X_1+\cdots+X_n\to n\cdot\int\mu$, take $\mu\in\mathcal{P}(SL_2(\mathbb{R}))$, $X_i\sim\mu$ and ask what can be said on the limit of the product $X_1\cdot X_2\cdots X_n$. He found out an interesting phenomena: for large class of measures, the angels between the columns of the random product tends to zero.

In the additive version, the X_i are random real numbers and so do the limit. In the non-commuting case - the limit, or the limiting behavior, is no longer a matrix but a different object.

But what kind of object?

3 Study case - the open disc in $\mathbb{C}$

We are looking for some relation between a space and its boundary, and we start by considering an example where the boundary is known: The open disc $D \subset \mathbb{C}$ and its boundary $\partial D = S^1$.

Abstract characterization. In that case we know the boundary - we can defined it by the metric. Recall that we wish to construct a boundary for groups as F_2 where considering the metric won't yields any convergence. So we are looking for different characterization for the relation between D and S^1.

Harmonic functions. Harmonic function $h : \mathbb{R}^2 \to \mathbb{R}$ can be defined in several equivalent ways. The most convenient to our purpose is a function that admits a mean value property: the value at x can be calculated by the values on a small ball $B_\epsilon(x)$: $h(x) = \frac{1}{C} \cdot \int_{\partial B_\epsilon(x)} h(y)\, dy$.

Consider an harmonic bounded function on D. Clearly it defines a measurable function on S^1, by the limiting values.

The other direction was solved by Dirichlet (known as the Dirichlet problem): Given a measurable function on S^1 find a harmonic function on D such that its limiting values are exactly the given function.

Uniqueness. Note that by the maximum principle (harmonic function gets its extreme values on the boundary), if such harmonic function does exist, it unique (the difference between two solutions will be zero on the boundaries).

Existences. The unique solution is given by the Poisson kernel, and has the following description: Let f on S^1. We need to define h on D. From a fixed $x \in D$ start a Brownian motion until it hits S^1. Denote the hitting distribution by $\nu_x \in \mathcal{P}(S^1)$. The desired harmonic function is $h(x) = \int_{S^1} f d\nu_x$.

Actually we are getting that way an isometry $\mathcal{H}^\infty(D) \approx L^\infty(S^1)$.

Recovering the topology by measures. Now we can say when a sequence in D converges to a point in S^1: Let d_n and consider the sequence of measures ν_{d_n}. If it converges to the Dirac measure $\nu_{\delta_b} \in \mathcal{P}(S^1)$ then we have that $d_n \to b$.

Corollary. We will have a notion of haminicity on groups and then we can demand the same relation: that $\mathcal{H}(G, \mu) \approx L^\infty(\Pi, \nu)$.

4 What kind of a space (Π, ν) should be?

Let's assume that we have a compact probability space (Π, ν), that we attach to G. That is we have a topology on $G \cup \Pi$ so the space is compact and almost every μ-random walk in G converges to Π. Moreover, ν reflects the hitting of the random walk on Π.

Π is a G-sapce. In other words, we have a (measurable) "boundary function": bnd:$G^{\mathbb{N}} \to \Pi$.

The G-action on $\Omega = G^{\mathbb{N}}$, which is left concatenation induces a G-action on Π: $gx = \lim gw_1 \cdots w_n$ where $x = \lim w_1 \ldots w_n$.

(Π, ν) is a stationary space.

Denote $X_n(w)$ are G-valued random variables $X_n(w) = w_n$. X_n is the increments in time n. Set $U_1^n(w) = X_1(w) \cdots X_n(w)$.

$\lim_{n\to\infty} U_1^n(w)$ converges a.s. so do $\lim_{n\to\infty} gU_1^n(w)$: The last limit is the product of g and the limit of the former.

Define $Z_1 : \Omega \to \Pi$ by $Z_1(w) = \lim_{n\to\infty} U_1^n(w)$.

Now consider the sequence $\lim_{n\to\infty} U_2^n(w)$ (just ignore the first coordinate). This sequence also converge. Denote $Z_2 : \Omega \to \Pi$ the limit, and $Z_k(w) = \lim_{n\to\infty} U_k^n(w)$.

Note that (1) all Z_k share the same distribution and (2) $Z_k(w) = X_k(w) Z_{k+1}(w)$.

If we denote the distribution of Z_k by ν we get that $\nu = \mu * \nu$.

Boundary

Furthermore, we claim that $\lim_{n\to\infty} U_1^n(w)\nu$ (which is a random measure on Π) is a.s. Dirac measure:

Let $f \in \mathcal{C}(\Pi)$, we need to show that $\int_\Pi f(\xi)\, dU_1^n(w)\nu(\xi) \to f(Z_1(w))$.

Write $Z_1(w) = U_1^n(w) \cdot Z_{n+1}(w)$ so we have

$$\begin{aligned}
\mathbb{E}(f(Z_1)|X_1,\ldots,X_n) &= \mathbb{E}(f(U_1^n Z_{n+1})|X_1,\ldots,X_n) \\
&= \int_\Pi f(X_1 \ldots X_n \xi)\, d\nu(\xi) \\
&= \int_\Pi f(\xi)\, dU_1^n(w)\nu(\xi)
\end{aligned}$$

But the first term converges to $f(Z_1)$ so we are done.

5 The boundaries of the examples

Free group. The Poisson boundary for simple random walk on F_2 is the space of all infinite canceled words, with the measure $\nu([g_1,\ldots,g_n]) = \frac{1}{4} \cdot \frac{1}{3^{n-1}}$.

Simple random walk on $\mathbb{Z}^3$ have a trivial boundary. The is due to the fact that the only bounded harmonic functions on $\mathbb{Z}^3$ are constant. So $\dim \mathcal{H} = 1$, so $\dim L^\infty(\Pi,\nu) = 1$ implies that $\Pi = \{pt\}$

One may suspect that S^2 will be the space of directions. It is well known that typically, the random walk goes on spheres so the probability that a walk will escape to infinity through a specific direction is zero.

Lamplighter. On $\mathbb{Z}^3$ is the set of all configurations, with some measure that says what is the probability that a random lamplighter will leave the space in that configuration. Informally, since the walk will eventually leave every point for ever, the last status at this point is the status of the same point in the limiting configuration.

If the underline group is recurrent, say $\mathbb{Z}$, then the Poisson boundary is trivial.

Part II. Stationary dynamical system

General settings

About G, μ.

G is a second countable group (the topology admits a countable basis), and $\mu \in \mathcal{P}(G)$ a Borel measure such that the semigroup generated by $\operatorname{supp}(\mu)$ is the whole G, and that there exists n such μ^n is non-singular w.r.t. the Haar measure of G.

All the measure spaces are standard and compact measured spaces are Borel (so the space of continuous functions is dense in the space of measurable functions).

Topological and measurable settings. The theory can be developed either in measurable settings or measurable-topological settings. We will prove the existence and uniqueness of the Poisson boundary, in the measurable settings. Thus, people prefer to consider this object as measurable object. On the other hand, without the topology - we lose the meaning of compactification.

There are several proofs for the existence of the Poisson boundary. Each of them use at certain point some "an existence theorem". Hence, we know that this space exists, but there is no direct way to construct it. "The identification problem of the Poisson boundary" is to give an explicit topological model (definition will be provided later) to the abstract measurable space, in a similar way we did to the examples we considered.

1 Group actions and G-space

Definition. (Group action) A measure space (X, ν) is a G-space if G acts measurably on X, that is,

There exists a measurable function $G \times X \to X$ s.t. (1) $ex = x$, $g(hx) = (gh)x$ (where $(g, x) \mapsto gx$) and (2) ν is quasi-invariant ($\nu(E) = 0 \iff \nu(g^{-1}E) = 0$ for all $g \in G$.)

If X is also a topological space, then a must be continuous.

Let $\pi : X \to Y$, then also $\pi = \pi_* : \mathcal{P}(X) \to \mathcal{P}(Y)$. Explicitly, for $\nu \in \mathcal{P}(X)$, $\pi\nu(E) = \nu(\pi^{-1}(E))$.

In particular, it holds for G action: Whenever X is a G-space, $\mathcal{P}(X)$ is a G-space as well:

For $\nu \in \mathcal{P}(X)$, $g\nu$ defined by $\int f(x)\, dg\nu(m) = \int f(gm)\, d\nu(m)$.

Definition. If $(X_1, \nu_1), (X_2, \nu_2)$ are G-spaces, (X_2, ν_2) is said to be G-factor of (X_1, ν_1) if there exists a measurable function $\pi : X_1 \to X_2$ such that $\pi\nu_1 = \nu_2$ and $\pi g = g\pi\ \forall g \in G$.

(X_1, ν_1) and (X_2, ν_2) are G-isomorphic if in addition π is a measure theoretical isomorphism (bijection m.p. between co-null sets such the inverse is measure preserving).

Theorem. *Any G-space (X, ν) admits a compact model, that is, there exists a compact metric spaceK with a probability measure η, such that G acts on K continuously and we have that (X, ν) and (K, η) are G-isomorphic.*

Example. Let (a_k) be a sequence of wights, that is$\sum a_k = 1$ and $a_k \geq 0$. Let (X, ν) be the set of points and there probabilities. Let G be a subgroup of permutations of the integers. We will build 2 compact models:

$K_1 = \left\{\frac{1}{2^n}\right\}$ and $\nu_1\left(\frac{1}{2^n}\right) = a_n$.

$K_2 = \left\{\frac{1}{2^n}\right\} \cup \left\{1 - \frac{1}{2^n}\right\}$ and $\nu_2\left(\frac{1}{2^n}\right) = a_{2n+1}, \nu_2\left(1 - \frac{1}{2^n}\right) = a_{2n}$.

K_1 and K_2 are different in the topological sense but still they serve as models for the same abstract space (X, ν).

This illustrates the disadvantage of working in the topological setup.

In particular, consider the measurable Poisson boundary. In general, there are many (topological) different compact models which realize the Poisson boundary.

2 Stationary spaces

Let (M, ν) be a G-space.

Denote by $\mu * \nu \in \mathcal{P}(M)$ the image of $\mu \times \nu$ by the map $G \times M \to M$.

Explicitly, for any measurable $f \in L^\infty(M, \nu)$,

$$\int_M f(m)\, d(\mu * \nu)(m) = \int_G \int_M f(gm)\, d\nu(m)\, d\mu(g).$$

Definition. A probability space (M, ν) is a (G, μ)-stationary or just (G, μ)-space, if it is G-space and $\nu = \mu * \nu$.

Lemma. *A factor of a stationary space is a stationary space.*

Proof. $\mu * \nu_2 = \mu * \pi\nu_1 = \pi(\mu * \nu_1) = \pi\nu_1 = \nu_2$. □

Weak topology. Consider the space of probability measures $\mathcal{P}(M)$, equipped with the weak topology, that is $\nu_n \to \nu$ if for any continuous function f, $\int f d\nu_n \to \int f d\nu$.

When K is a compact space, $\mathcal{P}(K)$ is compact in this weak topology.

Theorem. *Every compact G-space admits a stationary measure.*

Proof. Markov-Kakutani fixed point theorem for the map $\theta \mapsto \mu * \theta$.

Another construction: Take some $\theta \in \mathcal{P}(K)$ and consider the sequence $\nu_n = \frac{1}{n}(\theta + \mu * \theta + \cdots + \mu^n * \theta)$. The compactness assets that there exists a limiting measure ν (for a subsequence).

$$\begin{aligned} \mu * \nu_n - \nu_n &= \frac{1}{n}\left(\mu * \theta + \cdots + \mu^{n+1} * \theta\right) - \frac{1}{n}(\theta + \mu * \theta + \cdots + \mu^n * \theta) \\ &= \frac{1}{n}\left(\mu^{n+1} * \theta - \theta\right) \to 0 \end{aligned}$$

□

Corollary. *Unlike G-invariant measures, every G-space admits a stationary measure.*

Remark. This illustrates the advantage for considering topological settings: we have more structure so we can use more tools. Thinking of compact model as representation of an abstract space, it is interesting to look for properties that holds for all the topological models. We will see such a property later.

Remark. **General comment about stationary actions.** The study of stationary spaces is very interesting and can be a topic for other lecture. The classical ergodic theory studies the action of a measure preserving transformation on a probability space.

Consider the following generalization: instead of single transformation, one may consider many transformations - represented by group elements - and suppose that we are given a probability measure on the group. Each of the transformations may not preserve the measure, but they preserve the measure on average (w.r.t. the given measure on the group). This is the meaning of stationary space, and one can ask what results from ergodic theory holds for stationary spaces.

Stationary dynamical system is consist of (G,μ) and a stationary space (M,ν).

It is know that there is an ergodic theorem and multiple recurrence phenomena, in this stationary dynamical systems. Also there is kind-of structure theorem.

3 Harmonic functions

Definition. A G-function $h : G \to \mathbb{R}$ is μ-harmonic if $h(g) = \int_G h(g\gamma)\, d\mu(\gamma)$.

Denote the Banach space of all bounded harmonic functions by $\mathcal{H}^\infty(G,\mu)$. Note that always exist functions in $\mathcal{H}^\infty$ - the constant functions.

This is a vector space and we equipped it with the sup norm. Note that pointwise multiplication of harmonic functions is not harmonic.

Definition. A function $f : G \to \mathbb{R}$ is left uniform continuous (l.u.c.) if for any $\epsilon > 0$, there exists open neighborhood U of e such that for any $u \in U$ and $g \in G$, $|f(ug) - f(g)| < \epsilon$.

Denote by $B_{luc}(G)$ the space of all bounded l.u.c. functions, and by $\mathcal{H}^\infty_{luc}(G,\mu)$ the Banach space of all harmonic l.u.c. functions.

Lemma. *Every $h \in \mathcal{H}^\infty$ is a pointwise limit of a sequence in $\mathcal{H}^\infty_{luc}$, or, $\mathcal{H}^\infty_{luc}$ is dense in $\mathcal{H}^\infty$.*

$\mathcal{H}^\infty$ will be used for abstract measurable settings, and $\mathcal{H}^\infty_{luc}$ for topological settings.

Why harmonic functions are so important?

Denote by $(\Omega,\mathbb{P}) = (G,\mu)^{\mathbb{N}}$ - the space of increments of a walk.

Lemma. *The limit* $\lim_{n\to\infty} h(w_1 w_2 \cdots w_n)$ *exists for a.e.* $w \in \Omega$.

Moreover, denote $\tilde{h}(g,w) = \lim_{n\to\infty} h(g w_1 \cdots w_n)$, *then* $h(g) = \mathbb{E}_\Omega\left(\tilde{h}(g,w)\right) = \int_\Omega \tilde{h}(g,w)\, d\mathbb{P}(w)$.

Proof. Define $M_n : \Omega \to \mathbb{R}$ by $M_n(w) = h(w_1 \cdots w_n)$. Then $\{M_n\}$ forms a bounded martingale:

$$\mathbb{E}(M_{n+1}|M_n) = \int_G h(w_1 \cdots w_n g)\, d\mu(g) = h(w_1 \cdots w_n) = M_n$$

so it converges a.s., that is for a.e. $w \in \Omega$. □

Remark. Recall that we want to consider the space Ω and decide whether two walk go to the "same place" in an intrinsic way. The topology on Ω is meaningless. We do have convergence in $\mathbb{R}$. So we can think of harmonic function h as an opinion on Ω that maps a walk to a value. Then we can say that two walks converge to the same place if we cannot distinguish them by harmonic functions.

One may ask: "harmonic functions can serve as opinions, but are there other opinion-functions?". We will answer this question below and will see that to distinguish by harmonic functions yields the same equivalence relation as distinguish using all the opinion-functions.

4 IF TIME ALLOWS: Harmonic and shift invariant functions

Let $(\boldsymbol{X}, \mathcal{B}, m)$ be the space of all walks. Note that unlike Ω, for $x \in \boldsymbol{X}$, x_n is the position of the walk at time n rather then the increment. m is the Markov measure, with the initial probability μ: $m([x_1, \ldots, x_n]) = \mu(x_1) \cdot \prod_{i=1}^{n-1} \mu\left(x_{i+1}^{-1} x_i\right)$.

Let $T : \boldsymbol{X} \to \boldsymbol{X}$ be the left shift: $T(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$.

Theorem. *Let* $\mathcal{I}^\infty \subset L^\infty(\boldsymbol{X}, \mathcal{B}, m)$ *be the space of Borel shift-invariant maps. Then* $\mathcal{H}^\infty(G,\mu)$ *is isometric to* $\mathcal{I}^\infty$.

It is known theorem. The isomorphism is given by:
$h \in \mathcal{H}^\infty(G,\mu) \mapsto \tilde{h}(e,\cdot)$ and $\tilde{h} \in \mathcal{I}^\infty \mapsto h$ where $h(g) = \int \tilde{h}(g,w)\, d\mathbb{P}(w)$.
Denote by $\mathcal{I} = \left\{E \in \mathcal{B} | T^{-1}(E) = E\right\}$. It is a sub-$\sigma$-algebra of $\mathcal{B}$.
Note that $\mathcal{I}^\infty = L^\infty(\boldsymbol{X}, \mathcal{I}, m|_{\mathcal{I}})$.

Remark. We want to interpret this theorem as follows: Divide Ω into equivalence classes using harmonic functions, is the same as divide $\boldsymbol{X}$ by invariant functions.

There are two known constructions for the Poisson boundary. One is by the "spectrum" of the space of harmonic functions (Furstenberg), and the other is done by the space of shift-ergodic-components on $\boldsymbol{X}$.

5 Furstenberg transform

Let (M,ν) be a (G,μ)-stationary space.

The G-action on $L^{\infty}(M,\nu)$ is $\phi^{g}(x)=\phi(gx)$. Thus $\nu(\phi^{g})=g\nu(\phi)$.

Definition. For each $\phi\in L^{\infty}(M,\nu)$, define a G-function $F(\phi)(g)=\int\limits_{M}\phi(gm)\,d\nu(m)=g\nu(\phi)=\nu(\phi^{g})$.

Lemma. *The Furstenberg transform is a linear operator $F:L^{\infty}(M,\nu)\to\mathcal{H}^{\infty}(G,\mu)$, F commutes with the G-action, and $\|F\|=1$.*

For compact space (K,ν), the transform is a linear operator $\mathcal{C}(K)\to\mathcal{H}^{\infty}_{luc}(G,\mu)$ with the same properties.

Proof. Let $\phi\in L^{\infty}(M,\nu)$ and denote $f=F(\phi)$.

Hamonicity:

$$\begin{aligned}\int\limits_{G}f(g\gamma)\,d\mu(\gamma) &= \int\limits_{G}\left(\int\limits_{M}\phi(g\gamma m)\,d\nu(m)\right)d\mu(\gamma)=\\ &= \int\limits_{G}\left(\int\limits_{M}\phi^{g}(\gamma m)\,d\nu(m)\right)d\mu(\gamma)\\ &= \mu*\nu(\phi^{g})=\nu(\phi^{g})=g\nu(\phi)=f(g)\,.\end{aligned}$$

Linearity: The linearity of the transform is by the linearity of the integral.

Commutativity with G-action:

$$F(\phi^{\gamma})(g)=\int\limits_{M}\phi^{\gamma}(gm)\,d\nu(m)=\int\limits_{M}\phi(\gamma gm)\,d\nu(m)=F(\phi)(\gamma g)$$

□

Remark 1. This is a generalization of the Poisson transform. If we set $G=D^{1}$, and $M=S^{1}$ so given a boundary function ϕ, we define a harmonic function in g by integrating ϕ against the measure seen from g.

6 Boundaries

Lemma. *Let (K,ν) be a compact metric (G,μ)-stationary. The limit $\nu_{w}=\lim_{n\to\infty}w_{1}\cdots w_{n}\nu$ exists for a.e. $w\in\Omega$.*

Proof. Let $\phi\in\mathcal{C}(K)$, and consider the function $F(\phi)$. Since $F(\phi)$ is harmonic the limit $l(\phi,w)=\lim\limits_{n\to\infty}F(\phi)(w_{1}\cdots w_{n})=\lim\limits_{n\to\infty}\int\limits_{M}\phi d(w_{1}\cdots w_{n})\nu$ exists for all $w\in\Omega_{\phi}$ with $\mathbb{P}(\Omega_{\phi})=1$. Repeat this for a dense set of functions $\{\phi_{i}\}$ to get $\Omega_{0}=\bigcap\limits_{i}\Omega_{\phi_{i}}$ with $\mathbb{P}(\Omega_{0})=1$.

For a given $w_0 \in \Omega_0$, $l(\cdot, w_0)$ is a linear function $\{\phi_i\} \to \mathbb{R}$. Since $\{\phi_i\}$ is dense it extended to a (positive) linear functional on $\mathcal{C}(K)$ which correspond (by Riesz) to a measure $\nu_w \in \mathcal{P}(K)$. □

Definition. A compact (G,μ)-stationary, (K,ν) is called *compact-(G,μ)-boundary* if $\nu_w = \delta_{Z(w)}$ is a Dirac measure for a.e. $w \in \Omega$, Z is a random point in K (that is $Z : \Omega \to K$).

Remark. This is the $Z_1(w)$ form the informal discussion: $Z_1 = \text{bnd} : \Omega \to B$ by $w \mapsto z(w)$ where $\lim w_1 \cdots w_n \nu = \delta_{z(w)}$.

WLOG we assume that $\text{supp}(\nu) = Im(\text{bnd})$.

Lemma. *For an abstract measurable space (M,ν) the following equivalent:*

1. *(M,ν) have a compact model which is (G,μ)-boundary.*
2. *Every compact model of (M,ν) is a (G,μ)-boundary.*

Remark. The property "compact model" is defined using the topology, but still it is holds for any compact model.

Definition. An abstract stationary space (M,ν) is a (G,μ)-boundary if every compact model of it is a compact boundary.

Remark. **The name boundary and topology.** Why the name boundary is proper? Each compact boundary (B,ν) can be attached to G:

Give $G \cup B$ the weakest topology for which $G \to G \cup B$, $B \to G \cup B$ are homeomorphic embeddings and $\varphi : G \cup B \to \mathcal{P}(B)$ defined by $\varphi(b) = \delta_b$ and $\varphi(g) = g\nu$ is continuous. With that topology, $\lim_{n\to\infty} w_1 \cdots w_n$ exists to an element if B for a.e. $w \in \Omega$.

It it not true for abstract boundary - there convergence of the measures at all ($\mathcal{P}(M)$ does not equipped with weak topology).

Note that a trivial space, $(\{pt\}, \delta)$ is always a boundary. This is a one-point compactification. We are looking for the largest boundary.

Recall the intrinsic-description of the limit (or an equivalence relation on the set of walks) that we looked for. Now we can say that two walks w, w' are converges to the same point if $\nu_w = \nu_{w'}$.

Any boundary gives such a relation on Ω and we are looking for the "maximal boundary" which will give the finest relation.

Lemma. *A factor of a boundary is a boundary.*

Proof. Let $\left(K^{(1)}, \nu^{(1)}\right) \xrightarrow{\pi} \left(K^{(2)}, \nu^{(2)}\right)$ both are compact, where $\left(K^{(1)}, \nu^{(1)}\right)$ is a boundary.

Then

$$\nu_w^{(2)} = \lim w_1 \cdots w_n \nu^{(2)} = \lim w_1 \cdots w_n \pi \nu^{(1)} = \pi\left(\lim w_1 \cdots w_n \nu^{(1)}\right) = \pi\left(\nu_w^{(1)}\right)$$

and the image of Dirac is Dirac.

The abstract settings follow. □

7 Furstenberg transform for boundaries

Recall the Furstenberg transform, $F : L^\infty(M,\nu) \to \mathcal{H}^\infty(G,\mu)$ and $F : \mathcal{C}(K) \to \mathcal{H}^\infty_{luc}(G,\mu)$ defined by

$$F(\phi)(g) = \int_M \phi(gm)\, d\nu(m) = g\nu(\phi) = \nu(\phi^g)$$

.

Lemma 2. *(K,ν) is a compact boundary if and only if $F : \mathcal{C}(K) \to \mathcal{H}^\infty_{luc}(G,\mu)$ is an isometric embedding.*

Proof. We will show here only one direction. The other direction - IOU.

Injectivity

Assume $F(\phi)(g) = F(\psi)(g)$. Let $b \in K$ we want to show that $\phi(b) = \psi(b)$, find $w \in \Omega$ such $\mathrm{bnd}(w) = b$.

$\phi(b) = \int_K \phi d\delta_b = \lim_{n\to\infty} \int \phi d(w_1 \ldots w_n \nu) = \lim_{n\to\infty} F(\phi)(w_1 \ldots w_n) = \lim_{n\to\infty} F(\psi)(w_1 \ldots w_n) = \ldots = \psi(b)$.

Isometry - skip

In general, the norm may only decrease, so we need to show that $\|F_\phi\| \geq \|\phi\|$. Assume that $\|F_\phi\| > \|\phi\|$ so there exists g_0 with $F_\phi(g_0) > \sup_b \{\phi(b)\}$. Since $F_\phi(g_0) = \int F_\phi(\gamma g_0)\, d\mu(\gamma)$, there exists w_1 such $F_\phi(w_1) \geq F_\phi(g_0)$. Continue with this process to get $w \in \Omega$, such $\lim F_\phi(w_n \ldots w_1)$ not only exists but the sequence of values is monotonic. Set $b = \mathrm{bnd}(w)$ to get $\phi(b) = \lim F_\phi(w_n \ldots w_1) \geq F_\phi(g_0)$ which is a contradiction. □

Lemma. *An abstract stationary space,(B,ν) is a boundary if and only if $F : L^\infty(B,\nu) \to \mathcal{H}^\infty(G,\mu)$ is an isometric embedding.*

Proof. Assume (B,ν) is a boundary. Take (B_K,ν) a compact model. So F is linear (so continuous) isometry on the dense set $\mathcal{H}^\infty_{luc}$ so it is extended to a linear isometry $F : L^\infty(B,\nu) \to \mathcal{H}^\infty(G,\mu)$.

Assume F is isometry, and let B_K be any compact model. $F|_{\mathcal{C}(B_K)}$ is an isometry, so by the IOU, B_K is a compact boundary. So any compact model is a compact boundary thus (B,ν) is a boundary. □

Part III. The Poisson boundary

1 Statements of the main theorem

We can state the main theorem both in the topological settings and in the measurable settings.

Theorem. *Let (G,μ) be a locally compact second countable group with admissible measure. There exists a uniquely defined maximal boundary called the Poisson boundary, (Π,ν), in the sense that any other boundary is a G-factor of (Π,ν).*

We will prove here the topological version: there exists a compact boundary (Π_{top}, ν) such that any other boundary is a G-factor of (Π_{top}, ν).

2 Construction using Harmonic functions

We will find a compact space (Π_{top}, ν) which is a boundary and every other boundary is its factor.

Recall that the martingale convergence theorem suggests to use harmonic functions as indicator of the ways of escaping to infinity, but we ask what about other such functions. Let's take a look on this object, but first, recall from algebra:

2.1 Commutative C^*-algebras

A unital commutative C^*-algebra is a commutative Banach algebra (normed linear space with a multiplication and a unit element) that have nice $*$operation:

$(x+y)^* = x^* + y^*$, $(xy)^* = y^* x^*$, $(\lambda x)^* = \bar{\lambda} x^*$, $(x^*)^* = x$ and $\|x^* x\| = \|x\| \|x^*\|$.

Example for such object is $\mathcal{C}(K)$ where K is compact. This is indeed all the sources for examples: **Gelfand-Naimark theorem** says that if $\mathcal{A}$ is such a unital commutative C^*-algebra then there exists a compact space K such that $\mathcal{C}(K)$ isometrically isomorphic to $\mathcal{A}$ (or $*$-isomorphic).

Another theorem that we will use is **Stone-Banach**. Let X, Y compact spaces. If $\pi : Y \to X$ is continuous then it defines a function $\varphi : \mathcal{C}(X) \to \mathcal{C}(Y)$ by $\varphi(f)(x) = f(\pi(x))$.Stone-Banach gives the other direction: if $i : \mathcal{C}(X) \to \mathcal{C}(Y)$ is isometric and $i(\mathbf{1}_X) = \mathbf{1}_Y$ then it is induced by a continuous $\pi : Y \to X$.

In particular, if i is also isomorphism (onto) then X and Y are homeomorphic.

2.2 The construction of Π_{top}

Recall that $B_{luc}(G)$ is the set of all bounded l.u.c. function on G.

Let $\mathcal{A}$ be the algebra of all $f \in B_{luc}(G)$ such that the limit $\lim_{n\to\infty} f(gw_1w_2\cdots w_n) = \tilde{f}(g, w)$ exists for every $g \in G$ and a.e. $w \in \Omega$.

- We saw that $\mathcal{H}^{\infty}_{luc}(G, \mu) \leq \mathcal{A}$.
- This is an C^*-algebra: the pointwise multiplication preserves the existence of the limit a.e., the norm is the sup norm.
- This is also G-space: the action $f^g(\gamma) = f(g\gamma)$ preserves the condition that the limit exists for all g and a.e. w.

Now let $\mathcal{I}$ be the set of all functions in $\mathcal{A}$ that the limit exists and equal to zero for every $g \in G$ and a.e. $w \in \Omega$.

- Note that $\mathcal{I}$ is an ideal.

- The G-action preserves $\mathcal{I}$.

Thus, $\mathcal{Z} = \mathcal{A}/\mathcal{I}$ is a G-space with a structure of C^*-algebra.

Lemma. *$\mathcal{Z} = \mathcal{H}^{\infty}_{luc}$, or $\mathcal{A} = \mathcal{H}^{\infty}_{luc} \oplus \mathcal{I}$.*

Proof. First, $\mathcal{H} \cap \mathcal{I} = \{0\}$ since we saw that $h(g) = \mathbb{E}_{\Omega}\left(\tilde{f}(g,w)\right)$ so if $\tilde{f}(g,w) = 0$ for every g then $h(g) = 0$.

Given $f \in \mathcal{A}$ define $h_f(g) = \int \tilde{f}(g,w)\, d\mathbb{P}(w)$. So h is harmonic since

$$\begin{aligned}\int_G h_f(g\gamma)\, d\mu(\gamma) &= \int_G \int_\Omega \tilde{f}(g\gamma, w)\, d\mu(\gamma)\, d\mathbb{P}(w) \\ &= \int_\Omega \tilde{f}(g,w)\, d\mathbb{P}(w) = h_f(g)\end{aligned}$$

$f - h_f \in \mathcal{I}$:

$$\begin{aligned}\tilde{h_f}(g,w) &= \lim_{n\to\infty} h_f(gw_1 \cdots w_n) \\ &= \lim_{n\to\infty} \int_\Omega \tilde{f}(gw_1 \cdots w_n w')\, d\mathbb{P}(w') \\ &= \lim_{n\to\infty} \mathbb{E}\left(\tilde{f}(g, w_1 \cdots w_n w')\, | g, w_1, \ldots, w_n\right) \\ &= \tilde{f}(g,w)\end{aligned}$$

The last equality: $Z, (X_n)$ are random variables and Z is $\mathcal{F}(X_n)$-measurable for all n then $\lim_{n\to\infty} \mathbb{E}(Z|X_1, \ldots X_n) = Z$. □

Remark 3. Note that $\mathcal{H}^{\infty}_{luc}$ is a Banach space. We have just defined a multiplication in $\mathcal{H}^{\infty}_{luc}$: the pointwise multiplication modulo $\mathcal{I}$. That is, $h_1 \cdot h_2 = h$ where $h_1 \cdot h_2 = h + i$.

Thus, $\mathcal{H}^{\infty}_{luc}$ admits a Gelfand representation: there exists a compact space Π_{top} such that $\mathcal{H}^{\infty}_{luc} \approx \mathcal{C}(\Pi_{top})$. For $h \in \mathcal{H}^{\infty}_{luc}$, denote by $\bar{h}$ the image by Gelfand.

Π is a G-space

We have an G-action on $\mathcal{C}(\Pi_{top})$, so the G-action defines a group of automorphism on $\mathcal{C}(\Pi_{top})$. Such a group always correspond (Stone-Banach) to a group of homeomorphisms of Π_{top} so we have a G-action on Π_{top}.

2.3 The measure ν

Until now we have a G-sapce. We are looking for a measure on the space.

Define for each $g \in G$, $L_g : \mathcal{H}^{\infty}_{luc} \to \mathbb{R}$ by $L_g(h) = h(g) = \int_\Omega \tilde{h}(g,w)\, d\mathbb{P}(w)$. In words, consider that we start form g, what we will see in the future.

It induces a positive linear function $\bar{L}_g$ on $\mathcal{C}(\Pi_{top})$ and by Riesz, we have a $\nu_g \in \mathcal{P}(\Pi_{top})$ such that for $\bar{h} \in \mathcal{C}(\Pi_{top})$,

$$\nu_g(\bar{h}) = \int_{\Pi_{top}} \bar{h}(x)\, d\nu_g(x) = \bar{L}_g(\bar{h}) = L_g(h) = \int_{\Omega} \tilde{h}(g, w)\, d\mathbb{P}(w) = h(g)$$

Properties of these measures:

- **The G-action is $\nu_g = g\nu_e$**

$$g\nu_e(\bar{h}) = \nu_e(\bar{h}^g) = \nu_e(\overline{h^g}) = h^g(e) = h(g) = \nu_g(\bar{h})$$

- $\nu = \nu_e$ **is stationary.**

$$\nu(\bar{h}) = h(e) = \int_G h(g)\, d\mu(g) = \int_G \nu_g(\bar{h})\, d\mu = \mu * \nu(\bar{h})$$

- **Gelfand=Furstenberg.**

  With this measure, the Gelfand representation $h \leftrightarrow \bar{h}$ and the Furstenberg transform coincide: For $\bar{h}$ we need to show that $h = F(\bar{h})$.

$$F(\bar{h})(g) = g\nu(\bar{h}) = \nu_g(\bar{h}) = h(g)$$

- (Π_{top}, ν) **is a boundary.**

  Let $\theta = \lim_{n\to\infty} w_1 \cdots w_n \nu$.

$$\begin{aligned}\int_{\Pi_{top}} \bar{h}(x)\, d\theta(x) &= \lim_{n\to\infty} \int \bar{h}(x)\, d(w_1 \cdots w_n \nu)(x) \\ &= \lim_{n\to\infty} \bar{L}_{w_1\cdots w_n}(\bar{h}) = \lim_{n\to\infty} L_{w_1\cdots w_n}(h) \\ &= \lim_{n\to\infty} h(w_1 \cdots w_n) = \tilde{h}(e, w)\end{aligned}$$

  Since $h \leftrightarrow \bar{h}$ is multiplicative, apply the last equations on $\tilde{h^2}$ to get the first equality:

$$\tilde{h}(e,w)^2 = \int_{\Pi_{top}} (\bar{h}(x))^2\, d\theta(x) \geq \left(\int_{\Pi_{top}} \bar{h}(x)\, d\theta(x)\right)^2 = \left(\tilde{h}(e,w)\right)^2$$

  So we have equality in Cauchy-Schwartz so $\bar{h}$ must be constant on $supp(\theta)$. But it holds for any $\bar{h} \in \mathcal{C}(\Pi_{top})$ so θ is Dirac.

- **Universality**

  Let (K, ν') be a compact boundary so the Furstenberg transform $F_K : \mathcal{C}(K) \to \mathcal{H}^{\infty}_{luc}$ is an isometry embedding. And we have $F_{\Pi_{top}} : \mathcal{C}(\Pi_{top}) \to \mathcal{H}^{\infty}_{luc}$ which is isometric isomorphism. Consider

  $$\mathcal{C}(K) \xrightarrow{F_K} \mathcal{H}^{\infty}_{luc} \xrightarrow{F^{-1}_{\Pi_{top}}} \mathcal{C}(\Pi_{top})$$

  It is a $*$-isomorphism embedding that maps $\mathbf{1}_K \mapsto \mathbf{1}_{\Pi_{top}}$so by Stone-Banach, it induces by a uniquely defined continuous map $\pi : \Pi_{top} \to K$. π define by: $\phi \in \mathcal{C}(K)$ maps to $F(\phi) \in \Pi_{top}$ so $F(\phi)(x) = \phi(\pi(x))$.

  This π is a G-factor map.