1 Symmetries

A flow is said to have a symmetry if there is a diffeomorphism, $S : M \to M$, that conjugates the flow to itself:

$$\varphi_t(S(z)) = S(\varphi_t(z)), \quad t \in \mathbb{R}.$$  \hfill (6.24)

Since we assume that $S$ is smooth, we can take the time derivative of this relation to obtain an equivalent requirement on the vector field associated with $\varphi$:

$$f(S(z)) = DS(z)f(z).$$  \hfill (6.25)
1.1 Continuous and discrete symmetries

\[ \dot{r} = rh(r), \dot{\theta} = 1 \quad (6.20) \]

Some symmetries, like a rotation symmetry, depend continuously upon a parameter and are thus called continuous symmetries. For example, the system (6.20) is obviously symmetric under the rotation

\[ S_\psi(r, \theta) = (r, \theta + \psi) \quad (6.26) \]

for any angle $\psi$. For this case $DS$ is the identity matrix, so (6.25) becomes $f(r, \theta + \psi) = f(r, \theta)$, which is satisfied for all $\psi$ when $f$ is a function of $r$ only.

The collection of symmetries of a flow forms a group. This follows because the identity map is always a symmetry, and if $S_1$ and $S_2$ are symmetries of $\varphi$, then so is their composition $S_3 = S_1 \circ S_2$. Similarly, the inverse of a symmetry also satisfies (6.24) and therefore is also a symmetry. For example, the rotation symmetry (6.26) is a representation of the abstract rotation group, $O(2)$.

Discrete symmetries can also occur. For example, the system (6.11) is symmetric under the transformation $S(x, y) = (-x, -y)$, a rotation by $\pi$. To see this, note that for this case $DS = -I$, so (6.25) becomes $f(-x, -y) = -f(x, y)$, which is obviously satisfied by (6.11). The symmetry group in this case has two elements, the identity and $S$, and is called $\mathbb{Z}^2$. Much more about the implications of the existence of a nontrivial symmetry group can be found in (Field and Golubitsky 1995; Golubitsky and Stewart 2002).

\[ \dot{x} = y^2x - x^2y, \dot{y} = x^3 + y^3 \quad (6.11) \]
1.2 Reversors

Another type of symmetry that commonly occurs is a time reversal or reversing symmetry—when the motion backward in time is equivalent to that forward in time. Thus, a system is said to have reversing symmetry if there is a diffeomorphism, $S$ (the reversor), that conjugates the flow to its inverse so that $\varphi_-(S(z)) = S(\varphi_+(z))$. Again, this is equivalent to a requirement on the vector field

$$-f(S(z)) = DS(z)f(z). \quad (6.27)$$

This implies that in the new coordinate system, $\zeta = S(z)$, the differential equation $\dot{\zeta} = f(z)$ becomes

$$\dot{\zeta} = DS(z)\dot{z} = DS(z)f(z) = -f(S(z)) = -f(\zeta),$$

which is the same differential equation going backward in time.

In many cases the reversor $S$ is an involution, i.e., $S^2 = S \circ S = \text{id}$. For example, for mechanical Hamiltonian systems (recall §1.4) of the form

$$H(x, y) = \frac{1}{2}y^2 + V(x),$$

the involution $S(x, y) = (x, -y)$ reverses the momentum, $y$, and is equivalent to reversing time. Note also that in this case $S$ is orientation reversing, $\det(DS) = -1 < 0$.

The fixed set of a reversor $S$ is

$$\text{Fix}(S) = \{z : z = S(z)\}.$$ 

An orbit that intersects $\text{Fix}(S)$ is a symmetric orbit. In particular, a symmetric equilibrium is a point $z^* \in \text{Fix}(S) \cap \{f(z) = 0\}$. Not every orbit is symmetric; however, every orbit has a symmetric partner (see Exercise 5).

It can be shown that the fixed set of any orientation-reversing involution in $\mathbb{R}^2$ is a curve, $C = \text{Fix}(S)$ (MacKay). If this is the case, then whenever $z^*$ is a symmetric, linear center, it must be a true center of the nonlinear system.

**Lemma 1.** Suppose $\dot{z} = f(z)$ is reversible with reversor $S$ and $\text{Fix}(S)$ is a curve that contains an equilibrium $z^*$ that is a linear center. Then $z^*$ is a topological center.

Recall, for a linear center at the origin:

$\triangleright$ **Topological center:** there is a $\delta > 0$ such that every trajectory in $B_\delta(0) \setminus \{0\}$ is a closed loop enclosing the origin.
**Proof idea:** Close to the linear center, in polar coordinates, the angle \( \theta \) must increase monotonically (see Meiss). Hence, for a point \( z(0) \in \text{Fix}(S) \) in this neighborhood the orbit must return to \( \text{Fix}(S) \) (roughly after an increase by \( \pi \)). Denote the time at which this first return happens \( \tau \). Then the reflection \( \zeta(t) = S(z(t)) \) of this orbit segment touches \( \text{Fix}(S) \) at \( z(0) \) and \( z(\tau) \). However \( \zeta(t) \) is a solution beginning at \( z(0) \) and going backwards in time, and so the curve \( \gamma = \{ \phi_r(z(0)) : \theta \leq t \leq \tau \} \) is a closed loop and by uniqueness must be a periodic orbit with period \( 2\tau \).

**Example:** The system

\[
\begin{align*}
\dot{x} &= -y + \alpha x^2 y, \\
\dot{y} &= x + \beta y^2 x^2
\end{align*}
\]

has the reversor \( S(x, y) = (x, -y) \) since

\[
DSf(x, y) = (-y + \alpha x^2 y, -x - \beta y^2 x^2) = -(\alpha x^2 y, x - \beta y^2 x^2) = -f(S(x, y)).
\]

Note that the fixed curve for \( S \) is the \( x \)-axis, and since the origin is a symmetric fixed point, Lemma 6.4 implies it is a center. A phase portrait is shown in Figure 6.11. When \( \alpha > 0 \), this system also has a pair of saddle equilibria.

### 1.3 Meiss Ex 6.5

A flow \( \phi \) has a reversor \( S \) and an orbit \( \Gamma = \{ \phi_r(x) : t \in \mathbb{R} \} \).

(a) Show that \( \tilde{\Gamma} = \{ S \circ \phi_{-r}(x) : t \in \mathbb{R} \} \) is also an orbit of \( \phi \).

Since \( S \) is a reversor we know that:

\[
\phi_{-r}(S(z)) = S(\phi_r(z))
\]

Hence:

\[
S \circ \phi_{-r}(x) = S(\phi_{-r}(x)) = \phi_{-(-r)}(S(x)) = \phi_r(S(x))
\]

Denote \( y = S(x) \), then \( \tilde{\Gamma} \) is of the form:

\[
\tilde{\Gamma} = \{ \phi_r(y) : t \in \mathbb{R} \}
\]

i.e. \( \tilde{\Gamma} \) is also an orbit of \( \phi \).

(c) Suppose \( \Gamma \cap \text{Fix}(S) \neq \emptyset \). Show that \( \Gamma \) and \( \tilde{\Gamma} \) coincide.
Fix(S) = \{z \mid S(z) = z\} \tag{4}

Consider $z^* \in \Gamma \cap \text{Fix}(S)$. As $z^* \in \Gamma$ we can write:

\[ \Gamma = \{\varphi_t(z^*) \mid t \in \mathbb{R}\} \tag{5} \]

We saw in (a) that $\bar{\Gamma} = \{\varphi_t(S(x)) \mid t \in \mathbb{R}\}$. Since $\varphi_{-t}(x) = \varphi_{-t}(z^*) \mid t \in \mathbb{R}$ we can reach in the same way the result that:

\[ \bar{\Gamma} = \{\varphi_t(S(z^*)) \mid t \in \mathbb{R}\} \tag{6} \]

Since $z^* \in \text{Fix}(S)$ we have:

\[ \bar{\Gamma} = \{\varphi_t(z^*) \mid t \in \mathbb{R}\} = \Gamma \tag{7} \]

i.e. the orbits coincide.

### 1.4 Meiss Ex 6.6

(a) Show that if $x^*$ is a symmetric equilibrium of a reversible system, then whenever $\lambda$ is an eigenvalue of the linearization at $x^*$, so is $-\lambda$.

Denote the system $\dot{x} = f(x)$ and the reversor of the system as $S$. Since the system is reversible, we know:

\[ -f(S(z)) = DS(z)f(z) \tag{8} \]

Differentiating:

\[ -Df(S(z)) \cdot DS(z) = D^2S(z)f(z) + DS(z) \cdot Df(z) \tag{9} \]

The linearization at $x^*$ is:

\[ \dot{x} = Df(x^*)x \tag{10} \]

Since $x^*$ is symmetric, we know $S(x^*) = x^*$. Substituting $z = x^* = S(z)$ in the previous equation, we have:

\[ -Df(x^*) \cdot DS(x^*) = D^2S(x^*)f(x^*) + DS(x^*) \cdot Df(x^*) \tag{11} \]
Since \( x^* \) is an equilibrium, \( f(x^*) = 0 \). Since \( S \), being a symmetry, is a diffeomorphism, we know that \( DS(z) \) is invertible and hence there exists \( (DS(x^*))^{-1} \). Multiplying the above equation from the left by \( (DS(x^*))^{-1} \), we see that:

\[
Df(x^*) = (DS(x^*))^{-1}(-Df(x^*))DS(x^*)
\]

Hence:

\[
(12)
\]

And so \( Df(x^*) \) is similar to \(-Df(x^*)\) and they have the same eigenvalues. However, the eigenvalues of \(-Df(x^*)\) are simply minus the eigenvalues of \( Df(x^*) \), and so if \( \lambda \) is an eigenvalue of \( Df(x^*) \) then \(-\lambda\) is an eigenvalue of \(-Df(x^*)\), and hence of \( Df(x^*) \) as well.

It is highly recommended that you solve the rest of exercises 6.5 and 6.6 at home for further practice on the subject of symmetries.

**Bibliography**