Tutorial 2 - Perturbation Theory

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Introduction

Perturbation theory is a large collection of iterative methods for obtaining approximate solutions to problems involving a small parameter ε.

When will we use perturbation theory? When we can formulate a problem that we don't know how to solve as a problem we do know how to solve plus a small perturbation. The canonical physical example is the three-body gravitational problem, which is also the canonical example of where perturbation theory can fail.

Persistent properties

A central theme in perturbation theory is to continue equilibrium and periodic solutions to the perturbed system, applying the Implicit Function Theorem. For example, consider a system of differential equations

$$\dot{x} = f(x, \varepsilon),\ x \in \mathbb{R}^n,\ \varepsilon \in \mathbb{R},$$

$f : \mathbb{R}^{n+1} \to \mathbb{R}$. Equilibria are given by the equation $f(x, \varepsilon) = 0$. Assume $x_0 \in \mathbb{R}^n$ such that $f(x_0, 0) = 0$, and that $D_x f(x_0, 0)$ has maximal rank. Then the Implicit Function Theorem guarantees the existence of a mapping $\varepsilon \mapsto x(\varepsilon)$ in the neighborhood of x_0 with $x(0) = x_0$ such that

$$f(x(\varepsilon), \varepsilon) = 0.$$

This expresses persistence of equilibria given some conditions on the dynamical system.

A similar argument can be given for periodic orbits: Let the system with $\varepsilon = 0$ have a periodic orbit γ_0. Let Σ be a local transversal section of γ_0 and $P_0 : \Sigma \to \Sigma$ the corresponding Poincare map. Then P_0 has a fixed point $x_0 \in \gamma_0 \cap \Sigma$. For small ε, a local Poincare map $P_\varepsilon : \Sigma \to \Sigma$ is well defined. Its fixed points x_ε correspond to periodic orbits γ_ε. The equation $P_\varepsilon(x(\varepsilon)) = x(\varepsilon)$ with $x(0) = x_0$ can be also solved via the Implicit Function Theorem.

So the implicit function theorem guarantees persistence of equilibrium and periodic solutions under some conditions; these can generally be approximated using series expansions in the small parameter ε. Note that when the critical elements are not persistent, bifurcations occur.

Useful definition: Asymptotic order relation

Define $f(x) = \mathscr{O}(g(x))$ for $x \to x_0$ if

$$|f(x)/g(x)| < M$$

for $x \sim x_0$. For example, $x\sin(x) = \mathcal{O}(x)$ because $x\sin(x)/x < 1$.

Regular perturbation theory

Let's start with an example:

Example 1 ((BO) 7.1, example 1): Find the approximate roots of

$$x^3 - 4.001x + 0.002 = 0.$$

To solve perturbatively, introduce a small parameter ε and consider the 1-parameter family of polynomial equations

$$x^3 - (4+\varepsilon)x + 2\varepsilon = 0.$$

It turns out that it is easier to compute approximate roots because by considering roots as functions of ε, we may assume

$$x(\varepsilon) = \sum_{n=0}^{\infty} a_n \varepsilon^n.$$

Note that here we make an implicit assumption that x is analytic in ε - this may not always be satisfied by the actual solution.

Insert $x(\varepsilon)$ into the equation and collect orders of ε:

$$(\sum_{n=0}^{\infty} a_n \varepsilon^n)^3 - (4+\varepsilon)\sum_{n=0}^{\infty} a_n \varepsilon^n + 2\varepsilon = 0 = \sum_{n=0}^{\infty} b_n \varepsilon^n :$$

It is because ε is a variable that we can conclude $b_n = 0$ for all n. Thus, the equation can be solved iteratively:

$$b_0 = a_0^3 - 4a_0 = 0 \Rightarrow a_0 = -2, 0, 2.$$

$$\rightarrow b_1 = 3a_0^2 a_1 - 4a_1 - a_0 + 2 = 0 \rightarrow a_1 = \frac{a_0 - 2}{3a_0^2 - 4}$$

and so on... Thus, for $a_0 = -2$, we obtain

$$x = -2 - \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 + \mathcal{O}(\varepsilon^3).$$

Setting $\varepsilon = 0.001$ we can obtain the solution of the original polynomial up to 10^{-9} accuracy.

The remaining question is, does the obtained series series converge to the solution of the original equation? For which values of ε?

Leaving for now the convergence question on the side, this example illustrates the 3 steps of perturbative analysis:

1. Convert the original problem into a perturbation problem by introducing a small parameter ε.

   * Introduce ε such that the 0th-order solution is obtainable as a closed-form analytic expression (unless ε is already given...)

2. Assume an expression for the answer in the form of a perturbation series and compute coefficients of that series.

   * Generally, the existence of a closed-form 0th-order solution ensures that higher-order terms may be determined in closed-form analytical expressions.

3. Recover answer to the original problem by summing the perturbation series for the appropriate value of ε.

   * If the perturbation series converges, its sum is the desired solution. If it converges at a fast rate, first terms are enough. However, the series may diverge:

Example 2 (Tabor, 3.1(c)): Consider the equation

$$\dot{x} = x + \varepsilon x^2.$$

Use the expansion

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + ...,$$

plug into the equation and collect orders:

$$\mathcal{O}(\varepsilon^0): \ \dot{x}_0 = x_0 \Rightarrow x_0 = Ae^t.$$

$$\mathcal{O}(\varepsilon^1): \ \dot{x}_1 = x_1 + x_0^2 \Rightarrow x_1 = A^2 e^t (e^t - 1)$$

$$\mathcal{O}(\varepsilon^2): \ \dot{x}_2 = x_2 + 2x_1 x_0 \Rightarrow x_2 = A^3 e^t (e^t - 1)^2$$

and in fact this procedure can be continued to obtain the series:

$$x(t) = Ae^t \sum_{n=0}^{\infty} (\varepsilon A(e^t - 1))^n,$$

as you will check in your homework. The radius of convergence of this series is $\varepsilon A(e^t - 1) < 1$, and this produces a critical time after which the perturbed solution is no longer valid,

$$t_c = \log\left(\frac{1+\varepsilon A}{\varepsilon A}\right).$$

Note: this equation is a particular example of Bernoulli equations $\dot{x} = x + \varepsilon x^\alpha$, a family of non-linear differential equations that can be solved exactly in analytical form.

Comments:

* Regular perturbation theory is a perturbation problem whose perturbation series is a power series in ε with a non-vanishing radius of convergence.

* A basic feature of regular perturbation theory is that the exact solution for a small but non-zero ε smoothly approaches the unperturbed/zero-order solution as $\varepsilon \to 0$.

* It is most useful when the first few steps reveal the important features of the solution, and the remaining steps give small corrections - related to rate of convergence.

* Sundman's theorem - A solution to the 3-body problem as a convergent series in $t^{1/3}$. However, this series converges incredible slowly - for astronomical observations, $10^{8000000}$ terms of the series are required - and fail to improve our understanding of the problem. Check out https://sites.math.washington.edu/ $\sim$ morrow/336_13/papers/peter.pdf for a summary of this solution.

Singular perturbation theory

Comments:

* Singular perturbation theory is a perturbation problem whose perturbation series either is not analytic in ε, or, if it does have a power series in ε then the power series has a vanishing radius of convergence.

* A basic feature of singular perturbation theory is that the exact solution for $\varepsilon = 0$ is fundamentally different in character from the neighboring solutions obtained in the limit $\varepsilon \to 0$.

Let's start with an example:

Example 3 (Tabor): Consider the polynomial:

$$\varepsilon x^2 + x - 1 = 0.$$

In the limit $\varepsilon \to 0$ the zero-order system has only one root, whereas the perturbed problem has 2. Therefore, this is singular perturbation theory. The solution to the apparent "paradox" is that the extra root goes to ∞ as $\varepsilon \to 0$.

Solving the unperturbed equation, we obtain $x = 1$. This root seems to behave regularly as $\varepsilon \to 0$, so we can expand around it:

$$x_1 = 1 + \sum_{n=1}^{\infty} a_n \varepsilon^n.$$

Plugging this into the polynomial, obtain:

$$x_1 = 1 - \varepsilon + 2\varepsilon^2 + \mathcal{O}(\varepsilon^3).$$

Indeed, around 1, the term εx^2 is indeed small - as small as ε, generally.

The second root goes to ∞ as $\varepsilon \to 0$ - this suggests a rescaling of the equation by substitution $x = y/\varepsilon^n$.

In order to determine the correct rescaling of the equation, we present the method of dominant balance:

Method of dominant balance:

We assume that x can be consistently rescaled by $1/\varepsilon^n$ around the roots of the equation. Thus, an order-of-magnitude calculation of the equation is:

$$\varepsilon \cdot \varepsilon^{-2n} + \varepsilon^{-n} - 1 = 0.$$

When $\varepsilon \to 0$, at least two of these terms must be of the same, largest, order of magnitude in order for the equation to have a solution. There are 3 options:

- Suppose $\varepsilon^{1-2n} \sim 1$. Then $n = 1/2$ and $x = \mathcal{O}(\varepsilon^{-1/2})$. If this is true, then the middle term scales like $\varepsilon^{-1/2}$, and tends to ∞ when $\varepsilon \to 0$. This is inconsistent with the assumption that the dominant balance is between the first and last terms.
- Suppose $\varepsilon^{-n} \sim 1$. Then $n = 0$ and $x = \mathcal{O}(1)$. If this is true, then the first term scales like ε^1 and is negligible with respect to the other terms. This is a consistent assumption, and recovers the solution that is close to the unperturbed solution.
- Suppose $\varepsilon^{1-2n} \sim \varepsilon^{-n}$. Then $n = 1$ and $x = \mathcal{O}(1/\varepsilon)$. This is consistent as it renders 1 negligible when $\varepsilon \to 0$.

Thus, the magnitude of the missing root is $\mathcal{O}(1/\varepsilon)$ when $\varepsilon \to 0$. This suggests a scale transformation: $x = y/\varepsilon$. Substituting this into the equation, we obtain:

$$y^2 + y - \varepsilon = 0.$$

This is now a regular perturbation problem for y, and no roots disappear in the limit $\varepsilon = 0$. One may find, solving perturbatively, that:

$$y_1 = \varepsilon - \varepsilon^2 + 2\varepsilon^3 + \mathcal{O}(\varepsilon^4),$$

$$y_2 = -1 - \varepsilon + \varepsilon^2 - 2\varepsilon^3 + \mathcal{O}(\varepsilon^4),$$

and transforming back to the original variable x, we find that the two roots are:

$$x_1 = 1 - \varepsilon + 2\varepsilon^2 + \mathcal{O}(\varepsilon^3),$$

$$x_2 = -\frac{1}{\varepsilon} - 1 + \varepsilon - 2\varepsilon^2 + \mathcal{O}(\varepsilon^3),$$

Note that the series for x_2 is a Laurent series in ε, and indeed tends to ∞ as $\varepsilon \to 0$.

Example 4 - Boundary layer:

$$\varepsilon y'' - y' = 0, y(0) = 0, y(1) = 1$$

Assume there is some region in which y'' is small, there we approximate $y' = 0$ so $y = C$. However, this cannot satisfy the boundary conditions, so it's not a perturbative solution. Therefore, there must be a region in which $\varepsilon y''$ is not negligible, therefore $y'' \to \infty$ when $\varepsilon \to 0$. Since y'' and y' have the same sign, there can only be one such region, and it must be close to one of the boundaries. We assume that the solution is approximately 0 for most of the regime $t \in [0,1]$ and begin growing towards $t \to 1$ in a sharp transition area in which y'' is large.

In order to track the behavior of the function in the transition area, we can rescale the time parameter $t = \xi/\varepsilon^n$. This is a type of ”zoom in” into the region of interest. Dominant balance shows that $n = 1$. Plugging this rescaling into the equation we obtain:

$$\frac{d^2y}{d\xi^2} - \frac{dy}{d\xi} = 0.$$

In fact, the perturbation seems to have disappeared.

Checking our guess: In this case, the equation can be solved. The exact solution to the equation is

$$y(t) = \frac{e^{t/\varepsilon} - 1}{e^{1/\varepsilon} - 1}.$$

Indeed, by plotting this function for small values of ε we obtain that y is almost constant except for a narrow interval around $t = 1$ of width of order ε, which is called the boundary layer.

Note: Boundary layers appear naturally in fluid flows, where a fluid can behave differently around a boundary than in the bulk.