Tutorial 4 - Contraction Principle

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Introduction
In today’s tutorial we’ll prove the contraction map theorem, that gives sufficient conditions for existence of a unique fixed point in a mapping of a metric space to itself. For more details, see (M) chapter 3, up to 3.3.

Definitions

Metric space
A metric set is a set $M$ equipped with a metric $d$ on $M$, i.e. a function

$$d : M \times M \to \mathbb{R}$$

such that for every $x, y, z \in M$ the following holds:

1. $d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality)

Examples:
- $M = \mathbb{R}^n$, $d(x, y) = \sqrt{\sum (x_i - y_i)^2}$
- $M = \mathbb{R}^+$, $d(x, y) = |\log(y/x)|$
- Any vector space $V$ with a norm $||·||$, with the metric defined as $d(x, y) = ||x - y||$.

For example, $M = \{f : [a, b] \to \mathbb{R} : \int_a^b |f(x)|^2 dx < \infty\}$, $d(f, g) = \int_a^b |f(x) - g(x)|^2 dx$

Cauchy sequence (in a metric space)  Given a metric space $(M, d)$, a sequence $\{x_n\}_{n=1}^\infty \subset M$ is Cauchy if, for any $\varepsilon > 0$ there exists an integer $N \in \mathbb{N}$ such that for all positive integers $n, m > N$,

$$d(x_n, x_m) < \varepsilon.$$  

Property: Every convergent sequence is a Cauchy sequence.

Complete metric space  A complete metric space is a metric space $(M, d)$ in which all Cauchy sequences converge to limits in $M$.

Examples: $\mathbb{R}$, $[0, 1]$ with $d(x, y) = |x - y|$.  

Counter-examples: $\mathbb{Q}$, $(0, 1]$ with $d(x, y) = |x - y|$.  

**Contraction mapping**  Let \((X, d)\) be a metric space. Then a map \(T : X \to X\) is called a contraction map on \(X\) if there exists \(c \in [0, 1)\) such that
\[
d(T(x), T(y)) \leq cd(x, y)
\]
for all \(x, y \in X\).

**Examples:**
- \(X = \mathbb{R}, d(x, y) = |x - y|, T(x) = x/2\). Who is \(c\)?
- The logistic map for \(r < 1\):
  \[
x_{n+1} = rx_n(1 - x_n).
  \]

**Obvious property:** A contraction map is Lipschitz continuous, i.e. \(f : X \to X\) with
\[
d(f(x), f(y)) \leq Kd(x, y)
\]
for \(K \geq 0\).

**Contraction Map Theorem**

AKA Banach-Caccioppoli fixed-point theorem.

**Contraction Map Theorem**  Let \((X, d)\) be a non-empty complete metric space with a contraction mapping \(T : X \to X\). Then \(T\) admits a unique fixed point \(x^* = T(x^*) \in X\).

**Proof:**  Choose some \(x_0 \in X\) and define the following sequence:
\[
\{x_n\}_{n=1}^{\infty} : x_n = T(x_{n-1}).
\]
Then, for all \(n \in \mathbb{N}\), from the contraction mapping
\[
d(x_n, x_{n-1}) \leq qd(x_{n-1}, x_{n-2}) \leq q^2d(x_{n-2}, x_{n-3}) \leq \ldots \leq q^n d(x_1, x_0).
\]
with \(0 \leq q < 1\).

**This sequence is a Cauchy sequence:**  For \(m, n \in \mathbb{N}\) with \(m > n\),
\[
d(x_m, x_n) \leq d(x_m, x_{m-1}) + \ldots + d(x_{n+1}, x_n)
\]
\[
\leq (q^{m-1} + q^{m-2} + \ldots + q^n)d(x_1, x_0)
\]
\[
\leq q^n d(x_1, x_0) \sum_{k=0}^{m-n-1} q^k
\]
\[
\leq q^n d(x_1, x_0) \sum_{k=0}^{\infty} q^k = q^n d(x_1, x_0) \frac{1}{1-q}
\]

Given \(\varepsilon > 0\), since \(q \in [0, 1)\) there exists \(N \in \mathbb{N}\) large enough such that
\[
q^N < \frac{\varepsilon (1-q)}{d(x_1, x_0)}.
\]
Thus, for \(n, m > N\),
\[
d(x_m, x_n) \leq q^n d(x_1, x_0) \frac{1}{1-q} < \frac{\varepsilon (1-q)}{d(x_1, x_0)} d(x_1, x_0) \frac{1}{1-q} = \varepsilon.
\]
Thus this sequence is a Cauchy sequence, and by completeness of \((X, d)\) the sequence has a limit \(x^* \in X\).
This limit is a fixed point of $T$:

\[ x^* = \lim_{n \to \infty} x_n = \lim_{n \to \infty} T(x_{n-1}) = T(\lim_{n \to \infty} x_{n-1}) = T(x^*), \]

where the third equality results from the continuity of $T$.

Another way to prove this: Suppose $N$ is large enough such that $d(x_n, x^*) < \varepsilon$ for all $n > N$, then:

\[ d(T(x^*), x^*) \leq d(T(x^*), x_{n+1}) + d(x_{n+1}, x^*) \leq d(T(x^*), T(x_n)) + d(x_{n+1}, x^*) \leq (q + 1)\varepsilon. \]

Since this is true for any $\varepsilon > 0$, the distance is 0 and $T(x^*) = x^*$.

The limit is unique: Assume two fixed points $p_1 = T(p_1)$ and $p_2 = T(p_2)$. From the contraction property of $T$,

\[ d(T(p_1), T(p_2)) \leq q \cdot d(p_1, p_2) = q \cdot d(T(p_1), T(p_2)). \]

Since $q < 1$, this means that $d(p_1, p_2) = 0$ and from the definition of a metric space this means $p_1 = p_2$.

Examples:

- $M = [a, b], d(x, y) = |x - y|$. This example can be analyzed graphically:

  Consider a mapping $T : [0, 1] \to [0, 1]$ satisfying the contraction condition. This means that its slope is never above 1, and therefore it must cross the $y = x$ line once and only once:

![Figure 1: A general 1D contraction map.](image)

- Importance of completeness of the metric space: Consider $M = (0, 1], d(x, y) = |x - y|, T(x) = x/2$. Then all of the conditions for the contraction map theorem hold except for completeness of the metric space, and indeed the map does not have a fixed point in $M$.

- $M = C^0(S^1)$ is the space of continuous functions on the unit circle that are periodic with period 1: $f(x) = f(x + 1)$. The distance between two functions is defined by the sup norm: $d(f, g) = \sup_x |f(x) - g(x)|$. 


Consider the mapping operator

\[ T(f)(x) = \frac{1}{2} f(2x). \]

Then \( T(f) \in C^0(S^1) \), and \( d(T(f), T(g)) = \frac{1}{2} d(f, g) \). Therefore \( T \) is a contraction map and hence it has a fixed point, which in this case is a function \( f^*(x) : [0, 1] \to \mathbb{R} \) that satisfies \( T(f^*) = \frac{1}{2} f^*(2x) \) for all values of \( x \).

What is \( f^* \)? According to the theorem, any initial function will converge to \( f^* \), so it suffices to check one. For example, take \( f_0(x) = \sin 2\pi x \). Then \( f_1(x) = T(f_0)(x) = \frac{1}{2} \sin 4\pi x \), and \( f_n(x) = \frac{1}{2^n} \sin 2^{n+1} \pi x \). In the sup norm, this sequence converges to 0:

\[
\sup_x \{ f_n(x) - 0 \} = \sup_x \{ f_n(x) \} = \frac{1}{2^n} \sup_x \{ \sin 2^{n+1} \pi x \} = \frac{1}{2^n} \to 0.
\]

- A more interesting example: Consider the same \( M \) and \( d \) as the previous example, with the following mapping:

\[ T(f)(x) = \cos 2\pi x + \frac{1}{2} f(2x). \]

This is still a contraction mapping (check!), so the theorem holds and the map has a single fixed point \( f^*(x) \). Again, to find the fixed point let’s consider the initial function \( f_0 = \sin(2\pi x) \). Then

\[
\begin{align*}
f_1(x) &= \cos 2\pi x + \frac{1}{2} \sin(4\pi x), \\
f_2(x) &= \cos 2\pi x + \frac{1}{2} \cos(4\pi x) + \frac{1}{4} \sin(8\pi x), \\
f_j(x) &= \sum_{n=1}^{j-1} \frac{\cos(2^{n+1} \pi x)}{2^n} + \frac{1}{2^j} \sin(2^{j+1} \pi x)
\end{align*}
\]

The last term goes to zero in the sup-norm, and by the contraction-mapping theorem, the result is guaranteed to be unique and continuous. The fixed point is not an elementary function; it is an example of a Weierstrass function (continuous everywhere, differentiable nowhere), see Fig. 2.

Figure 2: The Weierstrass function (from (M) chapter 3).
Uses:

- Proof of the theorem about persistence of period-m maps given small perturbations.
- Proof of existence and uniqueness of solutions to ODEs:

**Theorem - Picard-Lindelof Existence and Uniqueness**

Consider the initial value problem

\[ \dot{x} = f(x), x(t_0) = x_0 \]

for \( x : \mathbb{R} \to \mathbb{R}^n, f : \mathbb{R}^n \to \mathbb{R}^n. \)

Suppose that for \( x_0 \in \mathbb{R}^n, \) there is a \( b \) such that \( f : B_b(x_0) \to \mathbb{R}^n \) is Lipschitz with constant \( K. \) Then the initial value problem has a unique solution \( x(t) \) for \( \theta \in J = [t_0 - a, t_0 + a], \) provided that \( a = b/M \) where \( M = \max_{x \in B_b(x_0)} |f(x)|. \)

**Proof idea:**

Look at the integrated problem \( x(t) = x_0 + \int_{t_0}^{t} f(x(\tau))d\tau, \) and define the map:

\[ T(u)(t) = x_0 + \int_{t_0}^{t} f(u(\tau))d\tau. \]

Then \( x^*(t) \) is a solution to the initial value problem iff it is a fixed point of \( T. \) One can show that \( T \) is a contraction for short enough times and therefore a solution \( x^*(t) \) exists and is unique. The full proof appears in (M), chapter 3.