Tutorial 4 - Contraction Principle

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Introduction

In today's tutorial we'll prove the contraction map theorem, that gives sufficient conditions for existence of a unique fixed point in a mapping of a metric space to itself. For more details, see (M) chapter 3, up to 3.3.

Definitions

Metric space A metric set is a set M equipped with a metric d on M, i.e. a function

$$d : M \times M \to \mathbb{R}$$

such that for every $x, y, z \in M$ the following holds:

1. $d(x,y) = 0 \iff x = y$
2. $d(x,y) = d(y,x)$
3. $d(x,y) \leq d(x,z) + d(z,y)$ (triangle inequality)

Examples:

- $M = \mathbb{R}^n$, $d(x,y) = \sqrt{\sum(x_i - y_i)^2}$
- $M = \mathbb{R}^+$, $d(x,y) = |\log(y/x)|$
- Any vector space V with a norm $||\cdot||$, with the metric defined as $d(x,y) = ||x-y||$. For example, $M = \{f : [a,b] \to \mathbb{R} : \int_a^b |f(x)|^2 dx < \infty\}$, $d(f,g) = \int_a^b |f(x) - g(x)|^2 dx$

Cauchy sequence (in a metric space) Given a metric space (M,d), a sequence $\{x_n\}_{n=1}^{\infty} \subset M$ is Cauchy if, for any $\varepsilon > 0$ there exists an integer $N \in \mathbb{N}$ such that for all positive integers $n, m > N$,

$$d(x_n, x_m) < \varepsilon.$$

Property: Every convergent sequence is a Cauchy sequence.

Complete metric space A complete metric space is a metric space (M,d) in which all Cauchy sequences converge to limits in M.

Examples: $\mathbb{R}$, $[0,1]$ with $d(x,y) = |x-y|$.

Counter-examples: $\mathbb{Q}$, $(0,1]$ with $d(x,y) = |x-y|$.

Contraction mapping Let (X,d) be a metric space. Then a map $T : X \to X$ is called a contraction map on X if there exists $c \in [0,1)$ such that

$$d(T(x),T(y)) \leq cd(x,y)$$

for all $x,y \in X$.

Examples:

- $X = \mathbb{R}$, $d(x,y) = |x-y|$, $T(x) = x/2$. Who is c?
- The logistic map for $r < 1$: $x_{n+1} = rx_n(1-x_n)$.

Obvious property: A contraction map is Lipschitz continuous, i.e. $f : X \to X$ with $d(f(x), f(y)) \leq Kd(x,y)$ for $K \geq 0$.

Contraction Map Theorem

AKA Banach-Caccioppoli fixed-point theorem.

Contraction Map Theorem Let (X,d) be a non-empty complete metric space with a contraction mapping $T : X \to X$. Then T admits a unique fixed point $x^* = T(x^*) \in X$.

Proof: Choose some $x_0 \in X$ and define the following sequence:

$$\{x_n\}_{n=1}^{\infty} : x_n = T(x_{n-1}).$$

Then, for all $n \in \mathbb{N}$, from the contraction mapping

$$d(x_n, x_{n-1}) \leq qd(x_{n-1}, x_{n-2}) \leq q^2 d(x_{n-2}, x_{n-3}) \leq \ldots \leq q^n d(x_1, x_0).$$

with $0 \leq q < 1$.

This sequence is a Cauchy sequence: For $m,n \in \mathbb{N}$ with $m > n$,

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + \ldots + d(x_{n+1}, x_n) \\ &\leq (q^{m-1} + q^{m-2} + \ldots + q^n) d(x_1, x_0) \\ &\leq q^n d(x_1, x_0) \sum_{k=0}^{m-n-1} q^k \\ &\leq q^n d(x_1, x_0) \sum_{k=0}^{\infty} q^k = q^n d(x_1, x_0) \frac{1}{1-q} \end{aligned}$$

Given $\varepsilon > 0$, since $q \in [0,1)$ there exists $N \in \mathbb{N}$ large enough such that

$$q^N < \frac{\varepsilon(1-q)}{d(x_1, x_0)}.$$

Thus, for $n, m > N$,

$$d(x_m, x_n) \leq q^n d(x_1, x_0) \frac{1}{1-q} < \frac{\varepsilon(1-q)}{d(x_1, x_0)} d(x_1, x_0) \frac{1}{1-q} = \varepsilon.$$

Thus this sequence is a Cauchy sequence, and by completeness of (X,d) the sequence has a limit $x^* \in X$.

This limit is a fixed point of T:

$$x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T(x_{n-1}) = T(\lim_{n\to\infty} x_{n-1}) = T(x^*),$$

where the third equality results from the continuity of T.

Another way to prove this: Suppose N is large enough such that $d(x_n, x^*) < \varepsilon$ for all $n > N$, then:

$$d(T(x^*),x^*) \le d(T(x^*),x_{n+1}) + d(x_{n+1},x^*) \le d(T(x^*),T(x_n)) + d(x_{n+1},x^*) \le (q+1)\varepsilon.$$

Since this is true for any $\varepsilon > 0$, the distance is 0 and $T(x^*) = x^*$.

The limit is unique: Assume two fixed points $p_1 = T(p_1)$ and $p_2 = T(p_2)$. From the contraction property of T,

$$d(T(p_1),T(p_2)) \le q \cdot d(p_1,p_2) = q \cdot d(T(p_1),T(p_2)).$$

Since $q < 1$, this means that $d(p_1, p_2) = 0$ and from the definition of a metric space this means $p_1 = p_2$. $\square$

Examples:

- $M = [a,b]$, $d(x,y) = |x-y|$. This example can be analyzed graphically:

  Consider a mapping $T : [0,1] \to [0,1]$ satisfying the contraction condition. This means that its slope is never above 1, and therefore it must cross the $y = x$ line once and only once:

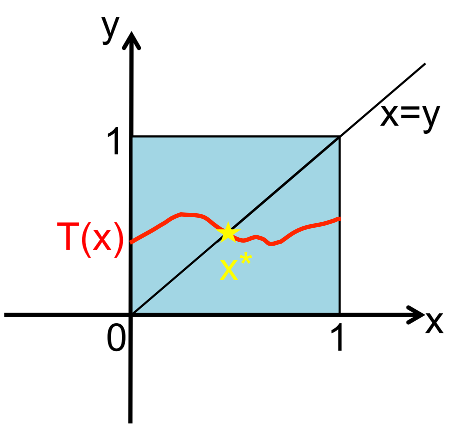

Figure 1: A general 1D contraction map.

- *Importance of completeness of the metric space:* Consider $M = (0,1]$, $d(x,y) = |x-y|$, $T(x) = x/2$. Then all of the conditions for the contraction map theorem hold except for completeness of the metric space, and indeed the map does not have a fixed point in M.

- $M = C^0(\mathbb{S}^1)$ is the space of continuous functions on the unit circle that are periodic with period 1: $f(x) = f(x+1)$. The distance between two functions is defined by the sup norm: $d(f,g) = \sup_x |f(x) - g(x)|$.

Consider the mapping operator

$$T(f)(x) = \frac{1}{2} f(2x).$$

Then $T(f) \in C^0(\mathbb{S}^1)$, and $d(T(f), T(g)) = \frac{1}{2} d(f, g)$. Therefore T is a contraction map and hence it has a fixed point, which in this case is a function $f^*(x) : [0,1] \to \mathbb{R}$ that satisfies $T(f^*)(x) = \frac{1}{2} f^*(2x)$ for all values of x.

What is f^*? According to the theorem, any initial function will converge to f^*, so it suffices to check one. For example, take $f_0(x) = \sin 2\pi x$. Then $f_1(x) = T(f_0)(x) = \frac{1}{2} \sin 4\pi x$, and $f_n(x) = \frac{1}{2^n} \sin 2^{n+1} \pi x$. In the sup norm, this sequence converges to 0:

$$\sup_x \{f_n(x) - 0\} = \sup_x \{f_n(x)\} = \frac{1}{2^n} \sup_x \{\sin 2^{n+1} \pi x\} = \frac{1}{2^n} \underset{n \to \infty}{\longrightarrow} 0.$$

- A more interesting example: Consider the same M and d as the previous example, with the following mapping:

$$T(f)(x) = \cos 2\pi x + \frac{1}{2} f(2x).$$

This is still a contraction mapping (check!), so the theorem holds and the map has a single fixed point $f^*(x)$. Again, to find the fixed point let's consider the initial function $f_0 = \sin(2\pi x)$. Then

$$\begin{aligned} f_1(x) &= \cos 2\pi x + \frac{1}{2} \sin(4\pi x), \\ f_2(x) &= \cos 2\pi x + \frac{1}{2} \cos 4\pi x + \frac{1}{4} \sin(8\pi x)), \\ f_j(x) &= \sum_{n=1}^{j-1} \frac{\cos(2^{n+1} \pi x)}{2^n} + \frac{1}{2^j} \sin\left(2^{j+1} \pi x\right) \end{aligned}$$

The last term goes to zero in the sup-norm, and by the contraction-mapping theorem, the result is guaranteed to be unique and continuous. The fixed point is not an elementary function; it is an example of a Weierstrass function (continuous everywhere, differentiable nowhere), see Fig. 2.

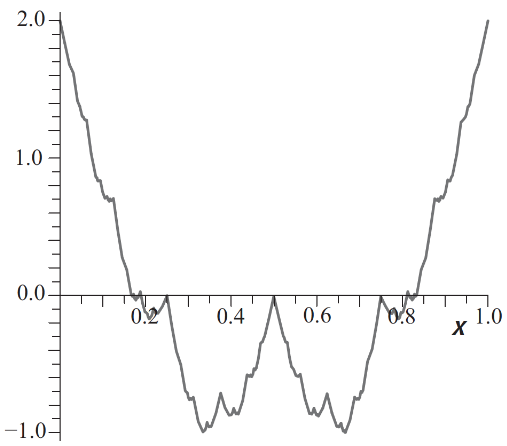

Figure 2: The Weierstrass function (from (M) chapter 3).

Uses:

- Proof of the theorem about persistence of period-m maps given small perturbations.
- Proof of existence and uniqueness of solutions to ODEs:

Theorem - Picard-Lindelof Existence and Uniqueness Consider the initial value problem

$$\dot{x} = f(x), x(t_0) = x_0$$

for $x : \mathbb{R} \to \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$.

Suppose that for $x_0 \in \mathbb{R}^n$, there is a b such that $f : B_b(x_0) \to \mathbb{R}^n$ is Lipschitz with constant K. Then the initial value problem has a unique solution $x(t)$ for $\theta \in J = [t_0 - a, t_0 + a]$, provided that $a = b/M$ where $M = \max_{x \in B_b(x_0)} |f(x)|$.

Proof idea:

Look at the integrated problem $x(t) = x_0 + \int_{t_0}^{t} f(x(\tau))d\tau$, and define the map:

$$T(u)(t) = x_0 + \int_{t_0}^{t} f(u(\tau))d\tau.$$

Then $x^*(t)$ is a solution to the initial value problem iff it is a fixed point of T. One can show that T is a contraction for short enough times and therefore a solution $x^*(t)$ exists and is unique. The full proof appears in (M), chapter 3.