Tutorial 6 - Asymptotic Stability

May 20, 2019

Introduction

In this tutorial we will prove the asymptotic stability theorem. This theorem shows the relation between the dynamics of the linear approximation of a system around an attracting fixed point to the dynamics of the full system: if the linearized system is asymptotically stable, this implies that the full, non-linear system is also asymptotically stable.

Recap of relevant information

Lyapunov stability An equilibrium x^* is Lyapunov stable is for every neighborhood N of x^* there exists a smaller neighborhood $M \subset N$ such that $x \in M$ implies $x(t) \in N$ for all $t \geq 0$.

Asymptotic stabillity An equilibrium x^* is asymptotically stable if it is Lyapunov stable and there exists a neighborhood N of x^* such that every point $x \in N$ approaches x^* as $t \to \infty$.

*** Examples for the difference between these definitions:** if time permits, at the end.

A simple criterion for linear systems (Theorem 2.10 (M) - Asymptotic linear stability): In a linear system described by $\dot{x} = Ax$, all the eigenvalues of A have a negative real part iff $x^* = 0$ is asymptotically stable.

The theorem and its proof - 4.6 in (M)

Consider a dynamical system $\dot{x} = f(x)$, with $x \in \mathbb{R}^n$. Then the following theorem holds:

Theorem: Asymptotic linear stability implies asymptotic stability. *Let $f : E \to \mathbb{R}^n$ be C^1 and have an equilibrium x^* such that all the eigenvalues of $Df(x^*)$ have real parts less than zero. Then x^* is asymptotically stable.*

Proof. Without loss of generality, assume $x^* = 0$. (If not, we can define $y = x - x^*$ and the same proof holds with the required modifications).

Define $A = Df(x^*)$, $g(x) = f(x) - Ax$, and rewrite the differential equation as

$$\dot{x} = Ax + g(x). \tag{1}$$

Define $\eta(t) = e^{-tA}x(t)$, then the differential equation obtains the form $\dot{\eta} = e^{-tA}g(x(t))$. This equation can be integrated by time from 0 to t to obtain $\eta(t) - \eta(0) = e^{-tA}x(t) - x(0) = \int_0^t ds\, e^{-sA}g(s(t))$, and returning to x we obtain

$$x(t) = e^{tA}x(0) + \int_0^t dse^{(t-s)A}g(s(t)) \tag{2}$$

Now we shall estimate the two terms on the right:

By our assumption, there is an $\alpha > 0$ such that every eigenvalue of A, denoted λ, satisfies $Re(\lambda) < -\alpha < 0$. Therefore, we can use the following estimate (2.): for any vector v there is a $K \geq 1$ such that

$$|e^{tA}v| \leq Ke^{-\alpha t}|v|. \tag{3}$$

(and in fact this is true for $K = 1$.)

Since f is C^1, then close enough to its fixed point, the linear approximation is good and the remainder is small, i.e. for any $\varepsilon > 0$ there is a $\delta > 0$ such that, if $x \leq \delta$, then $|g(x)| \leq \varepsilon|x|$.

Together, these provide an estimate for (2):

$$|x(t)| \leq Ke^{-\alpha t}|x(0)| + \varepsilon \int_0^t ds\, |e^{(t-s)A}x| \leq Ke^{-\alpha t}\delta + K\varepsilon \int_0^t ds\, e^{(t-s)\alpha}|x| \tag{4}$$

Let $\xi(t) = e^{\alpha t}|x(t)|$:

$$\xi(t) \leq K\delta + K\varepsilon \int_0^t ds\, \xi(s). \tag{5}$$

Use Gronwall's lemma (2.) to obtain:

$$\xi(t) \leq K\delta e^{K\varepsilon t} \Rightarrow |y(t)| \leq K\delta e^{(K\varepsilon - \alpha)t}. \tag{6}$$

Then, provided ε is chosen small enough so that $K\varepsilon < \alpha$, then $|y(t)| \to 0$ and is bounded by $K\delta$ for all $t \geq 0$.

Left to prove:

1. *Estimate 2.44 in chapter 2.7 (M): for any vector v there is a* $K \geq 1$ *such that* $|e^{tA}v| \leq Ke^{-\alpha t}|v|$.

   Proof for the case A is diagonal: $A_ij = \lambda_i\delta_{ij}$. Then

   $$|e^{tA}v| = \sqrt{\sum_{j=1}^{n}(e^{t\lambda_j}v_j)^2} \leq \sum_{j=1}^{n}|e^{t\lambda_j}v_j| \leq \sum_{j=1}^{n}|e^{t(a_j+ib_j)}v_j| \leq e^{-t\alpha}\sum_{j=1}^{n}|v_j| \leq Ke^{-t\alpha}|v|,$$

   where in the last inequality we use $(a+b)^2 \leq 2a^2 + 2b^2$. For the general case in which A is not diagonalizable, see chapter 2.7, estimate 2.44.

   In fact, it can be shown that $K = 1$.

2. *Gronwall's lemma:* Suppose $g, k : [0, a \to \mathbb{R}$ are continuous, $k(t) \geq 0$, and $g(t)$ satisfies

   $$g(t) \leq C + \int_0^t k(s)g(s)ds \equiv G(t) \tag{7}$$

   for all $t \in [0, a]$. Then

   $$\dot{G}(t) = k(t)g(t) \leq k(t)G(t) \Rightarrow \dot{G} - kG \leq 0. \tag{8}$$

Multiply by $e^{-\int_0^t k(s)ds}$:

$$e^{-\int_0^t k(s)ds}\dot{G} - e^{-\int_0^t k(s)ds}kG = \frac{d}{dt}(G(t)e^{-\int_0^t k(s)ds}) \leq 0. \tag{9}$$

Integrate by time from 0 to t to obtain $G(t)e^{-\int_0^t k(s)ds} \leq G(0) = C$, therefore

$$G(t) \leq Ce^{-\int_0^t k(s)ds}. \tag{10}$$

□

Example (M): Consider the system:

$$\dot{x} = -x - y - r^2$$

$$\dot{y} = x - y + r^2$$

where r is the polar radius, $r = \sqrt{x^2 + y^2}$. The origin is obviously a fixed point, and linear stability analysis shows that it is a stable focus:

$$Df(0,0) = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$$

has eigenvalues $\lambda_{\pm} = -1 \pm i$. We want to show that indeed at a neighborhood of the origin, the full system is bounded and asymptotically reaches the origin. It is easier to study the differential equation for r, using the fact that $2r\dot{r} = 2x\dot{x} + 2y\dot{y}$:

$$\dot{r} = r(-1 + y + x).$$

Since by definition of r, $-r \leq x, y \leq r$, if $r < 1/2$ then $-1 + y + x < 0$ and $\dot{r} < 0$ for $r < 1/2$. Thus, r is monotonically decreasing and this implies that the origin is asymptotically stable.

Example 4.16 (M): All IC end up at the stable fp, but the system is not Lyapunov stable: Consider the system

$$\dot{r} = r(1 - r)$$

$$\dot{\theta} = \sin^2(\theta/2)$$

where r, θ are polar coordinates in the plane. The system has two fixed points: $(0,0)$ (unstable) and $(1,0)$ (stable). The two equations are uncoupled and easy to analyze as separate 1D systems. For r, easy to see that every $r > 0$ is asymptotic to $r = 1$. For θ, $\sin^2(\theta/2) \geq 0$ so θ is semi-stable. However since θ is a periodic angle coordinate, even in the unstable direction $\theta = \delta > 0$ it reaches the same point. Therefore, every IC is attracted to $(1,0)$. However, this point is not Lyapunov stable - for any $\varepsilon < 2$ there are nearby points, f.e. $(1, \delta)$, that leave the ball of radius ε about the equilibrium.