Tutorial 8 - Lyapunov functions

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Introduction

In this tutorial we will discuss Lyapunov functions: Lyapunov devised another technique that can potentially show that an equilibrium is stable - the construction of what is now called a Lyapunov function. An advantage of this method is that it can sometimes prove stability of a nonhyperbolic equilibrium; a disadvantage is that there is no straightforward construction of Lyapunov functions.

Lyapunov functions

Lyapunov functions are nonnegative functions that decrease in time along the orbits of a dynamical system.

Definition: A continuous function $L : \mathbb{R}^n \to \mathbb{R}$ is a *strong* Lyapunov function for an equilibrium x^* of a flow φ_t on $\mathbb{R}^n$ if there is an open neighborhood U of x^* such that $L(x^*) = 0$, $L(x) > 0$ for $x^* \neq x \in U$, and

$$L(\varphi_t(x)) < L(x) \quad for\ all\ x \in U/\{x^*\}\ and\ t > 0. \tag{1}$$

The function L is a *weak* Lyapunov function if the strong inequality is replaced by a weak inequality, $L(\varphi_t(x)) \leq L(x)$.

If $L \in C^1$, the strong condition is equivalent to the condition $\frac{dL}{dt} < 0$: If (1) is satisfied, then for any $x \in U$, $\frac{dL}{dt} \equiv \lim_{t\to 0} \frac{L(\varphi_t(x))-L(x)}{t} < 0$, and conversely, if $\frac{dL}{dt} < 0$ then at every point $\lim_{t\to 0} \frac{L(\varphi_t(x))-L(x)}{t} < 0$ so necessarily, since L is continuous, then the first condition follows.

Using the chain rule:

$$0 > \frac{dL}{dt} = \nabla L \cdot \dot{x} = \nabla L \cdot f(x)$$

so in the smooth case, the condition that L is a Lyapunov function is that its gradient vector points in a direction opposed to that of the vector field f.

Some intuition: Assume an unstable fixed point $x^* = 0$ has a trajectory leading out from it in the infinitesimal region around 0 - a trajectory with x_0 infinitesimally close to 0, that satisfies in some region $||\varphi_t(x_0)|| > ||x_0||$. Therefore, a function L that satisfies $L(0) = 0$ and $L(x \neq 0) > 0$ cannot satisfy $L(\varphi_t(x)) < L(x)$ on this trajectory - this can be visualized in $\mathbb{R}^2$.

Assume L is a strong Lyapunov function for an $\mathbb{R}^2$ flow. Recall the gradient of L at x is always directed in the direction of the greatest increase in L. In $\mathbb{R}^2$, this allows for

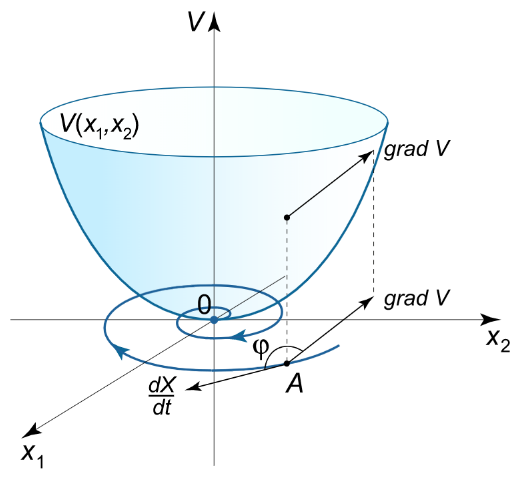

Figure 1.

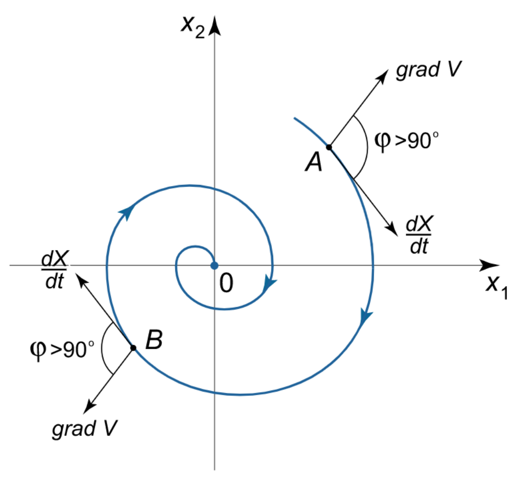

Figure 2.

some graphical intuition as to what is the Lyapunov function: $\dot{L} < 0$ implies that the angle between the gradient vector and the velocity vector is larger than 90°. It is fairly intuitive, at least in $\mathbb{R}^2$, that if this is the case everywhere along a phase trajectory then since L itself is increasing away from the origin (which is the fixed point), then the trajectory will tend towards the origin, i.e. the origin is a stable fixed point, see Fig. 1, Fig. 2.

Indeed, this can be proved for the general case $\mathbb{R}^n$:

Theorem 4.7 - Lyapunov Functions (M): Let x^* be an equilibrium point of a flow $\varphi_t(x)$. If L is a weak Lyapunov function in some neighborhood U of x^*, then x^* is stable. If L is a strong Lyapunov function, then x^* is asymptotically stable.

Example: Consider the system

$$\begin{aligned} f(y,z) &= \begin{pmatrix} z \\ -y-2z \end{pmatrix}: \\ y' &= z \\ z' &= -y-2z \end{aligned} \tag{2}$$

and the function $L(y,z) = (y^2+z^2)/2$. Then $x^* = (0,0)$ is a fixed point, $f(0,0) = 0$, and indeed $L(x^*) = 0$. Also, $L > 0$ for $x \neq (0,0)$. Finally, the negative condition is satisfied: $\nabla L \cdot f = yz + z(-y-2z) = -2z^2 \leq 0$. Therefore $x^* = (0,0)$ is a stable fixed point.

Proof of stability: Consider a flow $\varphi_t(x)$ defined by the function $f(x) : \mathbb{R}^n \to \mathbb{R}^n$, and a weak Lyapunov function L. Assume without loss of generality $x^* = 0$.

We want to prove that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $x_0 < \delta$ then $\varphi_t(x_0) < \varepsilon$ for all times $t > 0$.

Choose $\varepsilon > 0$ such that the ball of radius ε around the origin is in U. Then, let's define $m = \min_{||x||=\varepsilon} L(x)$. The constant m exists since $||x|| = \varepsilon$ is compact. Since L is positive definite, $m > 0$.

From continuity of L and $L(0) = 0$, there exists some $0 < \delta < \varepsilon$ such that for any $||x|| < \delta$, $L(x) < m$. Choose an initial condition $||x_0|| < \delta$. Since $L(\varphi_t(x))$ is decreasing, $L(\varphi_t(x_0)) < m$. We claim this implies $\varphi_t(x_0) < \varepsilon$:

Assume by contradiction that there exists a time t_1 such that $\varphi_{t_1}(x_0) > \varepsilon$. Then there is a time t_2 such that $\varphi_{t_2} = \varepsilon$, but $L(\varphi_{t_2}) < m = \min_{||x||=\varepsilon} L$, which is a contradiction.

Proof of asymptotic stability: Consider the case that L is a strong Lyapunov function. Then it is also a weak Lyapunov function and for $||x_0|| < \delta$, $\varphi_t(x_0) < \varepsilon$. We need to prove $\varphi_t(x_0) \to 0$ for $t \to \infty$.

Since L is strictly decreasing and non-negative, a sequence $\{\varphi_{t_n}(x_0)\}_{n=1}^{\infty}$ has a limit in the region, $L(\varphi_{t_n}(x_0)) \to c \geq 0$ when $n \to \infty$. Assume by contradiction $c > 0$, and have $z \in B_\varepsilon(0)$ the point at which this value is acquired. Then $L(\varphi_t(z)) < c$, and $\varphi_{t_n+s}(x_0) \to \varphi_t(z)$. Therefore, for a large enough n, $L(\varphi_{t_n+s}(x_0)) < c$. Finally, find $m > n$ such that $t_m > t_n + s$ to obtain the contradiction. □

Example: Any linear system $\dot{x} = Ax$ that is asymptotically stable has a strong Lyapunov function, of the form $L = x^T Sx$, where S is a symmetric matrix: Note that $\dot{L}$ is negative if

$$\dot{x}^T Sx + x^T S\dot{x} = x^T(A^T S + SA)x < 0$$

for all $x \neq 0$. To solve this, we can require $A^T S + SA = -I$ (called the Lyapunov equation), and then $\dot{L} = -|x|^2 < 0$ for $x \neq 0$. Then this equation always has a solution when A's eigenvalues have negative real parts, by $S = \int_0^\infty e^{\tau A^T} e^{\tau A} d\tau$. This can be checked by plugging this S into the Lyapunov equation, multiplying the equation by e^{tA^T} from the left and by e^{tA} from the right, and noticing that the left hand side becomes a full derivative.

Finding a Lyapunov function

In general, finding a Lyapunov function for a nonlinear system is a matter of guessing. However, when the equilibrium is asymptotically stable, a Lyapunov function is guaranteed to exist, and therefore the two conditions, asymptotic stability and existence of a strong Lyapunov function, are equivalent:

Theorem 4.23 (M): If x^* is an asymptotically stable equilibrium that attracts a neighborhood U, then the function

$$L(x) = \int_0^\infty e^{-s} \sup_{t \geq s} |\varphi_t(x) - x^*| ds$$

is a strong Lyapunov function on U.

We shall not show the proof.

Although this theorem guarantees that a strong Lyapunov function exists for an asymptotically stable equilibrium, it is not possible to construct it using this method unless the flow can be obtained analytically - in which case there is no reason to find L! However, there are cases in which it is not hard to find a Lyapunov function and for which stability is not obvious. You will see some examples in the HW.

Lorenz system - an example for a non-hyperbolic fixed point

The Lorenz system is

$$\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz, \end{aligned} \tag{3}$$

where we assume the parameters r, σ and b are positive. The equilibrium at the origin a Jacobian with an eigenvalue $\lambda = -b$ for the eigenvector in the z direction, and two other eigenvalues determined by

$$\lambda^2 + (\sigma+1)\lambda + \sigma(1-r) = 0.$$

Thus, the origin is attracting when $r < 1$ but is a saddle when $r > 1$. We know all this from linear stability analysis.

What happens if $r = 1$? Linear analysis cannot tell us. We construct the following Lyapunov function:

$$L = \frac{1}{2}\left(\frac{x^2}{\sigma} + y^2 + z^2\right),$$

and we can check that $\frac{dL}{dt} = -(x-y)^2 - bz^2$. Therefore, $\dot{L} = 0$ on the line $\{x = y, z = 0\}$ and is not a strong Lyapunov function. So what can we do?

In this case, the LaSalle invariance principle saves us:

Theorem 4.25 (LaSalle's Invariance Principle): Suppose x^* is an equilibrium of a flow $\varphi_t(x)$, and L a weak Lyapunov function for a neighborhood U of x^*. Let $Z = \{x \in U : \dot{L} = 0\}$ be the set where L is not decreasing. Then, if x^* is the only forward-invariant subset of Z (i.e. there are no other fixed points, periodic orbits, etc.), then it is asymptotically stable and attracts every point in U.

An example

The equations for the damped pendulum, in dimensionless variables, can be written in the following form:

$$\dot{x} = y, \quad \dot{y} = -y - \sin x.$$

(a) Show that the origin is a stable fixed point using the energy function $V(x,y) = \frac{1}{2}y^2 + (1-\cos x)$.

Solution:

$$\nabla V \cdot f = \sin xy + y(-y - \sin x) = -y^2 \leq 0.$$

(b) Show that the origin is an asymptotically stable fixed point using a "better" Lyapunov function $V(x,y) = \frac{1}{2}(x+y)^2 + x^2 + \frac{1}{2}y^2$ than the energy function.

Solution:

$$\nabla V \cdot f = 2xy - y^2 - (x+2y)\sin x \approx 2xy - y^2 - x^2 - 2xy = -(x^2+y^2) < 0.$$

except if $x = y = 0$.