Homework #3 - Perturbation Theory

Due: November 10

1. (BO) Solve the following problem only up to its leading order ($\mathcal{O}(\varepsilon^0)$, i.e. for $\varepsilon = 0$) using perturbation theory:

$$y''(x) + y(x) = \varepsilon \frac{\cos(x)}{30 + y^2},\ y(0) = y(\pi/2) = 2.$$

Considering $\varepsilon = 1$, here are a few solutions obtained for the full equation by a high-order numerical solver:

$y(\pi/100) = 2.06117$, $y(\pi/10) = 2.51489$, $y(\pi/4) = 2.82097$.

Compare to your very approximate solution at these x values to the full equation's solutions - how good is the perturbation approximation? Can you give a reason why?

2. Solve the differential equation

$$\dot{y} = y + \varepsilon$$

in two ways:

a) solve using perturbation theory, assuming y is given by a power series in the perturbation parameter ε,

b) solve by a topologically conjugate flow: find a transformation $x = h(y;\varepsilon)$ such that the resulting equation for x is immediately soluble, and conclude the solution for y by using h^{-1}. Did you obtain the same solution in both cases?

3. (BO 7.2) Obtain a perturbative series for the roots of

$$x^2 - 2.0004x + 0.9998 = 0$$

as we did in class, by introducing a small parameter ε and assuming $x = \sum a_n \varepsilon^n$. Solve to first order $\mathcal{O}(\varepsilon)$, and compare with the exact roots (found by solving the quadratic equation). Why does the most straightforward application of perturbation fail?

Clue: Study the exact solutions.

4. a) (Tabor) Find the exact solution of the equation

$$\dot{x} = x + \varepsilon x^2,\ x(0) = A,$$

by the change of parameters $x(t) = 1/y(t)$.

b) In class, we showed that the corresponding perturbation series does not converge beyond $t_c = \log\left(\frac{1+\varepsilon A}{\varepsilon A}\right)$ if $A > 0$. Considering the exact solution, what happens at this value?

c) Describe the type of solution $x(t)$ for different initial values $x(0) = A$ by a graphic illustration of the phase space.