Homework #6
Due: November 24

1. Show that the map:

\[ F_\mu : x_{n+1} = x_n + \mu - x_n^2 \quad x_n \in \mathbb{R} \tag{1} \]

undergoes a saddle-node bifurcation at \((x, \mu) = (0, 0)\); show that for \(\mu < 0\) it has no fixed points whereas for \(\mu > 0\) it has two fixed points, one stable and one unstable. Draw the bifurcation diagram for \(F_\mu\).

2. Consider a general one dimensional family of maps \(F_\mu(x) : \mathbb{R} \to \mathbb{R}\). Find conditions under which \(F_\mu(x)\) undergoes a saddle-node bifurcation at some value \((x, \mu) = (x^*, \mu^*)\): define \(G(x, \mu) = F_\mu(x) - x\), and using the implicit function theorem find conditions under which \(G(x, \mu) = 0\) has a unique parabola like solution \(G(x, \mu(x))\) for \(\mu > \mu^*\) (and \(|\mu - \mu^*|\) and \(|x - x^*|\) small). Verify that (1) indeed satisfies these conditions.

3. Consider the logistic map:

\[ x_{n+1} = r x_n (1 - x_n) = F(x; r) \tag{2} \]

In class, we have shown that the map has two fixed points, \(x^* = 0\) and \(x^* = 1 - \frac{1}{r}\), both of which unstable for \(r > 3\).

(a) Show that the map undergoes a period doubling bifurcation at \(r = 3\), by finding the fixed points of the second-iterate map \(F^2\).

(b) For which values of \(r\) is the 2-cycle stable?

(c) Bonus: what happens when the 2-cycle becomes unstable?

4. Bonus: find a paper in your field of interest in which local bifurcations play a role. Write the equations of the bifurcation and show the bifurcation diagram.