Homework #6

Due: November 24

1. Show that the map:

$$F_\mu : x_{n+1} = x_n + \mu - x_n^2 \qquad x_n \in R \tag{1}$$

   undergoes a saddle-node bifurcation at $(x,\mu) = (0,0)$; show that for $\mu < 0$ it has no fixed points whereas for $\mu > 0$ it has two fixed points, one stable and one unstable. Draw the bifurcation diagram for F_μ.

2. Consider a general one dimensional family of maps $F_\mu(x) : R \to R$. Find conditions under which $F_\mu(x)$ undergoes a saddle-node bifurcation at some value $(x,\mu) = (x^*,\mu^*)$: define $G(x,\mu) = F_\mu(x) - x$, and using the implicit function theorem find conditions under which $G(x,\mu) = 0$ has a unique parabola like solution $G(x,\mu(x))$ for $\mu > \mu^*$ (and $|\mu - \mu^*|$ and $|x - x^*|$ small). Verify that (1) indeed satisfies these conditions.

3. Consider the logistic map:

$$x_{n+1} = r x_n (1 - x_n) = F(x; r) \tag{2}$$

   In class, we have shown that the map has two fixed points, $x^* = 0$ and $x^* = 1 - \frac{1}{r}$, both of which unstable for $r > 3$.

   (a) Show that the map undergoes a period doubling bifurcation at $r = 3$, by finding the fixed points of the second-iterate map F^2.

   (b) For which values of r is the 2-cycle stable?

   (c) Bonus: what happens when the 2-cycle becomes unstable?

4. Bonus: find a paper in your field of interest in which local bifurcations play a role. Write the equations of the bifurcation and show the bifurcation diagram.