Final Homework

Submit in writing, till Feb 10, 2021

1. For the below list of models, using **only elementary tools** (dimensionality consideration and simple geometric consideration - no need to compute fixed points and their stability) classify the systems as type **(O)**,**(C)** or **(PC)** and explain your choice shortly;

   (O) Ordered: All solutions can have only simple asymptotic behavior - explain what kinds of behaviors are possible for the corresponding class of systems.

   (C) Chaotic: the system has an invariant set on which the motion is chaotic.

   (PC) Possibly chaotic: the system may have complicated and possibly chaotic dynamics or it may be ordered. Further analysis is needed to find out which of the two cases occurs; suggest methods (analytic and numerical) for checking whether the system is chaotic or ordered.

   (a) $x_{n+1} = 5x_n - 0.2\sin\pi x_n, \qquad x_n \in \mathbb{R}^1$

   (b) $x_{n+1} = 5x_n - 0.2\sin\pi x_n$, mod 1 $\qquad x_n \in [0,1]$

   (c) $\frac{dx}{dt} = y^2$, $\frac{dy}{dt} = \sin(x+y)$, $\frac{dz}{dt} = z^2 - xy, \qquad (x,y,z) \in \mathbb{R}^3$

   (d) $\frac{dx}{dt} = y^2$, $\frac{dy}{dt} = \sin(x+y+z)$, $\frac{dz}{dt} = z^2 - xy, \qquad (x,y,z) \in \mathbb{R}^3$

   (e) $z_{n+1} = z_n^3 \;, \qquad z_n \in \mathbb{C}.$

   (f) $\frac{dx}{dt} = y, \frac{dy}{dt} = x^3 - x + \sin(x), \qquad (x,y) \in \mathbb{R}^2$

   (g) $\frac{dx}{dt} = y, \frac{dy}{dt} = x^3 - x + \sin(x) + 0.1\sin(t), \qquad (x,y) \in \mathbb{R}^2 x_n \in [0,1]$

   (h) $\frac{dx}{dt} = x + 0.1y$, $\frac{dy}{dt} = x - 11y$, $\frac{dz}{dt} = 10z - x - y, \qquad (x,y,z) \in \mathbb{R}^3$

2. Consider the Lorenz system, with positive parameters ($r, b > 0,\ \sigma > b+1$):

$$\begin{cases} \frac{dx}{dt} = \sigma(y-x) \\ \frac{dy}{dt} = rx - y - xz \\ \frac{dz}{dt} = xy - bz \end{cases} \qquad (x,y,z) \in \mathbb{R}^3 \tag{1}$$

(a) Find the fixed points, their stability, and the corresponding eigenvector/eigen spaces and draw the local phase portraits near them.

(b) Which bifurcation occurs at $r = 1$? Suggest how to analyze this bifurcation (it is sufficient to list the main steps).

(c) What are the symmetries of (1)? Explains your results in (a,b) in view of these.

(d) Show that for all positive parameter values the Lorenz system has a trapping region E_C:

$$E_C = \{rx^2 + \sigma y^2 + \sigma(z - 2r)^2 \leqslant C, \tag{2}$$

namely, that for sufficiently large C, every trajectory eventually enters E_C.

(e) **Bonus:** Study numerically the Lorenz system at $\sigma = 10, b = 8/3$, close to the bifurcation point $r = 1$ and close to the Lorenz attractor at $r = 28$. Find the trapping regions for these parameters.

3. Forced and damped asymmetric Duffing equation:

   Consider the forced and damped motion of a particle in a slightly asymmetric double well potential (so $0 < a \ll 1, 0 \leqslant \varepsilon \ll 1$):

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = x - ax^2 - x^3 + \varepsilon\gamma \sin(\omega t) - \varepsilon\delta y \tag{3}$$

(a) For $\varepsilon = 0$, hereafter, called the unperturbed system, show that this system is Hamiltonian of mechanical form. Find the Hamiltonian function $H(x,y)$, draw schematically the potential $V(x)$, draw schematically the level curves of H in the (x,y) plane (the phase space) for $a = 0$ and for small a and explain what are the possible types of motion the particle may exhibit.

(b) For small a, $\gamma = 0, \delta \geq 0$, find the fixed points and their stability (find them to first order in a).

(c) Find the symmetries of (1) for:

   i. $a = \gamma = \delta = 0$.
   (e.g. :$(x,y,t) \to (x, -y, -t)$ is a time-reversal symmetry)

   ii. $a = \delta = 0$.

iii. $a > 0, \quad \gamma = \delta = 0.$

(d) **Bonus:** Find the form of the Melinikov functions $M_{L,R}(t_0; a, \omega, \delta)$; Denote by $q_{L,R}(t;a) = (x_{L,R}(t;a), y_{L,R}(t;a))$ the left/right homoclinic orbits of the unperturbed Hamiltonian system satisfying $y_{L,R}(0;a) = 0$.

i. What are the symmetries of $y_{L,R}(t;a) = 0$ as a function of t?

ii. "Compute" the Melnikov function, up to quadratures, by using the solutions symmetries. In particular, show that

$$M_{L,R}(t_0; a, \omega, \delta) = \gamma C_{L,R}(\omega)\cos(\omega t_0) - \delta K_{L,R} \tag{4}$$

and provide the integral expressions for $C_{L,R}(\omega), K_{L,R}$ in terms of $q_{L,R}(t;a)$ (no need to compute the integrals).
[hint: use the fact that the perturbation dependence on t is trigonometric so $\sin(\omega(t+t_0)) = \sin(\omega t)\ \cos(\omega t_0) + \cos(\omega t)\ \sin(\omega t_0)$, together with the symmetries of the homoclinic solution]

iii. For a fixed ω and a values, find the curves in the (γ, δ) parameter plane beyond which the Left/Right Melnikov function has simple zeroes.

iv. Draw schematically the stable and unstable manifolds of the time $T = \frac{2\pi}{\omega}$ Poincaré map for the three different regimes in the (γ, δ) parameter plane.

(e) **Bonus:** For $\gamma = 0, \delta > 0$ and small ε, show that for $\nu < \varepsilon\delta$ and C sufficiently large, $E_C = \{H(x,y) + \nu xy \leq C\}$ is a trapping region.

(f) **Bonus:** What are the dynamical consequences of your findings in d+e?

(g) **Bonus:** Identify a region in the schematic drawing of the phase space which is mapped onto itself as a horseshoe. Explain the consequences of your findings.

(h) **Bonus:** verify your predictions by numerical integrations. Try to find numerically the stable and unstable manifolds. Examine numerically the behavior at fixed, interesting values of (γ, δ), as you increase ε.

4. Reflections:

(a) Summarize, in your own words, 3 of the most important results you learned in class.

(b) Which subjects would you skip/add?

(c) Which subjects you thought needed more in-depth study?