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When complexity leads to simplicity: Ocean surface mixing simplified by vertical convection

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The effect of weak vertical motion on the dynamics of materials that are limited to move on the ocean surface is an unresolved problem with important environmental and ecological implications (e.g., oil spills and larvae dispersion). We investigate this effect by introducing into the classical horizontal time-periodic double-gyre model vertical motion associated with diurnal convection. The classical model produces chaotic advection on the surface. In contrast, the weak vertical motion simplifies this chaotic surface mixing pattern for a wide range of parameters. Melnikov analysis is employed to demonstrate that these conclusions are general and may be applicable to realistic cases. This counter-intuitive result that the very weak nocturnal convection simplifies ocean surface mixing has significant outcomes. © 2012 American Institute of Physics. [http://dx.doi.org/10.1063/1.4719147]

I. INTRODUCTION

Ocean circulation and mixing are driven by the wind, tides, thermohaline fluxes, and biological activity. Thus, mixing in the ocean occurs over multiple spatial and temporal scales, from millimeters to basin wide and from microseconds to decades and beyond. Numerous studies applied dynamical system methods (e.g., Refs. 1–13 and references therein) to investigate mixing in the ocean. Motivated by the shallow water approximation, these studies often employed two-dimensional (2D) incompressible flow models. However, in many applications, for example, when studying submesoscale (100 m–20 km) dynamics, the flow field is three dimensional (3D): vertical velocities, such as those found near fronts or induced by night convection, become more significant.14, 15 Three-dimensional effects change the character of surface mixing.16, 17 Concretely, it was demonstrated that 3D random motion or turbulent motion exhibit complex aggregates of particles.17–21 Here we study surface mixing when the underlying 3D flow is quasi-periodic in time, namely, deterministic and nonturbulent. Such models are appropriate for studying the main Lagrangian mixing mechanisms.6, 22, 23 Other effects, such as turbulent eddy mixing induced by winds and thermohaline fluxes, may be viewed as a stochastic component that is superimposed on the deterministic underlying flow, see Refs. 24–26 and references therein.

In 2D steady flows we can distinguish between two typical types of stagnation points: elliptic points (centers) and hyperbolic points (saddles). Intuitively, the particles encircle elliptic points (e.g., the centers of the gyres in our model) whereas saddles separate the flowing particles to different regions. More precisely, associated with the saddles are two special material lines that cross at the saddle point and are called the stable and unstable manifolds. The stable manifolds consist of the particles that move toward the hyperbolic points whereas the unstable manifolds consist of particles that emanate from the saddle (move toward the saddle when integrated backward in time). The global structure of these material lines defines the different regions in the flow. For example, in the

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steady double-gyre model, the demarcation line between the two gyres corresponds to the unstable manifold of the right hyperbolic point and to the stable manifold of the left hyperbolic point, thus dividing the domain to an upper and a lower gyre.

The stable and unstable manifolds of hyperbolic trajectories (for example, of hyperbolic periodic trajectories) of time-dependent flows are similarly defined. Chaotic Lagrangian 2D mixing is commonly associated with tangles of stable and unstable manifolds of hyperbolic trajectories through the mechanism of lobe dynamics. Notably, in the realm of geophysical flows, structures such as gyres normally exist for a limited time (a couple of days for submesoscale flows, weeks for mesoscale). Thus, concepts associated with infinite time intervals such as chaos, stable and unstable manifolds, and strange attractors are ill defined. The theory of Lagrangian coherent structures (LCS) provides the proper generalization of the stable and unstable manifolds concept to finite time noisy data sets. It shows that the LCS role in controlling particles transport is very similar to that played by the stable and unstable manifolds in the traditional theory. Here, we argue that by characterizing the geometrical structure of the stable and unstable manifolds of an idealized model we may extract information regarding mixing on the relevant finite time period of two-three days.

II. MODEL DESCRIPTION AND ITS GEOPHYSICAL SCALES

Simple double gyres and other 2D models are often employed to study chaotic mixing in the ocean. Although these models were mainly used to study the basin-scale, wind driven gyres, similar structures have been observed and investigated on smaller scales as well (e.g., Refs. 2, 36, and 38–41). Such models mimic important characteristics of ocean dynamics and thus provide insights into mixing. Following this approach we introduce a modified double-gyre model to study the effect of vertical convection on chaotic mixing and then argue that the principles we reveal apply to more realistic settings. The underlying 2D incompressible flow is a non-symmetric time-periodic double gyre. The time periodicity represents here the tidal effects and results in chaotic mixing. We introduce into this classical model a second perturbation which adds a vertical component to the flow with a non-trivial time average (mimicking, for example, night convection). The structure of the flow field is shown schematically in Fig. 1 (see Appendix A for a detailed derivation of the equations). Here, the 3D flow is incompressible and the resulting surface flow becomes non-area preserving. Naively, one would suspect that the small vertical component

![Figure 1](http://example.com/figure1.png)

**FIG. 1.** Schematic description of the flow field. The strength of the horizontal velocities (represented by double arrows) is much larger than the velocities associated with the vertical gyres. The horizontal demarcation line oscillates in time along the y axis as shown. The amplitude of the vertical motion is maximal at night and is negligible at daytime.
that appears in geophysical flows will not play a significant role in finite time surface mixing. Below, we show by dimensional analysis that when \( \frac{U_{\text{vert}}}{U_{\text{hor}}} \approx \mathcal{O}(1) \), where \( U_{\text{hor}}, U_{\text{vert}} \) denote typical horizontal and vertical velocities, respectively, and \( L_{\text{hor}}, L_{\text{vert}} \) denote typical horizontal and vertical (mixed layer or thermocline) lengthscales, respectively, the surface mixing character is strongly influenced by the vertical motion.

More precisely, consider an incompressible 3D flow in the non-dimensional box \((L_x, L_y, L_z)\) (all of which are of order one) which induces a 2D surface flow of the form (see Appendix A for model description)

\[
\mathbf{u}(x, y, t) = u_{\text{double-gyre}} \begin{pmatrix} u_{0x}(x, y) \\ u_{0y}(x, y) \end{pmatrix}
+ \epsilon u_{\text{tide}} \begin{pmatrix} u_{1x}(x, y, \omega_{\text{tide}}t + \theta_{\text{tide}}; \epsilon) \\ u_{1y}(x, y, \omega_{\text{tide}}t + \theta_{\text{tide}}; \epsilon) \end{pmatrix}
+ \epsilon u_{\text{vert}} \begin{pmatrix} u_{2x}(x, \omega_{\text{day}}t; \epsilon) \\ u_{2y}(y, \omega_{\text{day}}t; \epsilon) \end{pmatrix},
\]

where \( u_i = \begin{pmatrix} u_{i,x} \\ u_{i,y} \end{pmatrix}, \) \( i = 0, 1, 2, \) are the normalized velocity fields with maximal amplitude 1. The steady double-gyre flow component is \( u_{\text{double-gyre}}(0) \), the tidal component is \( \epsilon u_{\text{tide}} u_1 \), and the vertical convection induces the \( \epsilon u_{\text{vert}} u_2 \) component. Both \( u_1 \) and \( u_2 \) are periodic functions in \( t \) (with periods \( 2\pi/\omega_{\text{tide}} \) and \( 2\pi/\omega_{\text{day}} \), respectively) and \( \theta_{\text{tide}} \) represents the phase of the tide relative to the vertical motion. The first two terms \( u_0, u_1 \) are area preserving (i.e., divergence free) whereas \( u_2 \) breaks the area preservation character of the surface flow while keeping the incompressibility of the underlying 3D flow. Most importantly, in the rescaled non-dimensional formulation, the magnitude of the second term is \( \epsilon u_{\text{tide}} = \mathcal{O}(\frac{U_{\text{vert}}}{U_{\text{hor}}}) \) whereas the last term’s magnitude is \( \epsilon u_{\text{vert}} = \mathcal{O}(\frac{U_{\text{vert}}}{U_{\text{hor}}}) \). Thus, even though geophysical flows have \( U_{\text{vert}} \approx U_{\text{tide}} \), because \( \frac{U_{\text{vert}}}{U_{\text{hor}}} \approx 1 \), the two time-dependent terms \( \epsilon u_{\text{tide}}, \epsilon u_{\text{vert}} \) may be of the same order. Finally, the time-dependent terms have significant influence on particle mixing in the considered geophysical time scale only when their period is of the same order as the typical gyre time scale, namely, of \( \mathcal{O}(\frac{L_{\text{vert}}}{\omega_{\text{day}}}) \).

In many submesoscale flows (including bays, coastal regions, and more, \( \frac{U_{\text{vert}}}{U_{\text{hor}}} \approx \mathcal{O}(10^{-3} \sim 10^{-1}) \) at \( 1000 \text{~m} \sim \mathcal{O}(1) \text{~m} \) and \( 50~\text{~m} \leq 200~\text{~m} \approx \mathcal{O}(10) \). Moreover, for these flows the diurnal and tidal frequencies are both of the same order as the typical time scale associated with the steady gyres \( \frac{L_{\text{vert}}}{U_{\text{hor}}} \approx 4 \times 10^3 \text{~s} \approx 11 \text{~h} \). Thus, the current analysis is relevant to geophysical submesoscale flows.

Figure 2 shows the dramatic influence of the vertical component on surface particles mixing by comparing the mixing after two days when \( \epsilon u_{\text{vert}} \) is zero or small (Figs. 2(a) and 2(b)) and when \( \epsilon u_{\text{vert}} \) is large (Fig. 2(c)). Particles that started in the lower gyre are colored in blue and those that were initially in the upper gyre are colored in red. The clear distinction between the two mixing patterns is evident. In the purely 2D case (Fig. 2(a)) or for tiny vertical convection (Fig. 2(b)) bidirectional flux emerges—blue particles penetrate the upper gyre and red particles invade the lower one. On the other hand, when significant convection is introduced (Fig. 2(c)) unidirectional flux appears—some blue particles go up but no red particles are found in the lower gyre at this time point. These plots demonstrate the relevance of the qualitative results to geophysical scales: the integration is performed over the geophysical time scale of two days and with a relative large \( \epsilon \), in the geophysical range (in which the tidal induced velocity can be similar to the ambient velocity).

### III. CHARACTERIZING THE MIXING PATTERNS BY MELNIKOV ANALYSIS

To explain these results we calculate the Melnikov function which provides the leading order approximation for the distance between the stable and unstable manifolds in time-dependent flows (see Refs. 31 and 47–49 for its formulation in the quasi-periodic case and Refs. 2, 22, and 50 for applications of these methods in the fluids and geophysical contexts when the underlying flow is two dimensional and incompressible). Denote the demarcation line between the gyres by
FIG. 2. Bidirectional and unidirectional cross-gyre mixing after two days. Initially, particles (e.g., larvae) are evenly distributed and are colored blue in the lower gyre ($y < y_h$) and red in the upper gyre ($y > y_h$). When there is no vertical convection (a) or when it is very weak (b), bidirectional mixing by tangles is observed: red particles go down and blue particles go up. In (a) $u_{vert} = 0$ and in (b) $u_{vert} = 0.1$, and similar behavior appears up to $u_{vert} = 0.15$ (not shown). When the vertical convection is increased to a geophysical relevant range ((c), where $u_{vert} = 1$) the stable and unstable manifolds do not cross, and unidirectional mixing occurs: blue particles cross to the upper gyre yet no red particles go down during this simulation. In all simulations we take $\epsilon = 0.4$, $u_{tide} = 1$, $y_{hc} = \frac{2}{3}$ whereas $u_{vert}$ and $\theta_{tide}$ vary as indicated. Panels (d) and (e) demonstrate that at some predicted parameter regime (see below and Fig. 4), the bidirectional cross-gyre mixing may be observable for some tidal phases (d) and not observable in two days time for others (e). Finally, panel (f) shows that when the gyres are very energetic, bidirectional flux reigns even when the vertical convection vigor is increased.

\[ (x_h(s), y_h = \frac{2}{3} L_y), \quad \frac{dx_h}{ds} = u_{double-gyre} u_0(x_h(s), y_h) \text{ and } s \text{ parameterizes the line } (x_h(s), y_h). \]

At a given tidal phase $\theta_{tide}$, the distance between the manifolds near the point $(x_h(-s), y_h)$ (see Ref. 50, equation 2.7) is

\[ d(s, \theta_{tide}; \epsilon) = \frac{\epsilon M(s, \theta_{tide})}{\|u_0(x_h(-s), y_h)\|} + O(\epsilon^2), \]  

(2)

where the Melnikov function $M(s, \theta_{tide})$ is explicitly computable (see below). This calculation provides a simple criterion for detecting the existence of tangles that produce chaotic mixing in the perturbative regime: tangles exist in the parameter regimes at which this function has simple zeroes.\(^{27,50}\) Many studies demonstrated that incompressible time-periodic (and quasi-periodic) 2D flows usually produce tangles.\(^{2,6,22,23,31,34,35,50}\) The tangles emerge in these studies since the forced
system is Hamiltonian, the fluid domain is bounded, and there is a single turnstile between the gyres (see Refs. 3 for a flow in an unbounded domain where tangles do not appear). Here we show that three dimensionality may destroy the tangles.

Note that the geometrical characterization of the manifolds explains the nature of finite time mixing of surface particles. When the Melnikov function has simple zeroes it implies that there is a bidirectional flux between the two gyres (Fig. 2(a)), whereas when the Melnikov function is always positive it means that particles move in a unidirectional manner, only from the lower to the upper gyre (Fig. 2(b)) (see also Refs. 3, 51, and 52). This distinction between unidirectional flux and bidirectional flux mechanisms may be applied also to the corresponding LCS and is thus relevant and important even for the geophysical finite time intervals regime as is demonstrated by Figs. 2 and 3.

The quasi-periodic Melnikov function for the specific flow (Eq. (2)) is of the form (Appendix B)

\[
M(s, \theta_{\text{tide}}) = u_{\text{tide}} f_1(\Omega_{\text{tide}}) \cos(\omega_{\text{tide}} s + \theta_{\text{tide}})
+ u_{\text{vert}} u_{\text{av}} (\Delta y) \left( f_2(\Omega_{\text{day}}) \sin(\omega_{\text{day}} s) + u_{\text{av}} \right)
= u_{\text{tide}} f_1(\Omega_{\text{tide}}) \left( \cos(\omega_{\text{tide}} s + \theta_{\text{tide}}) + A(\sin(\omega_{\text{day}} s) + B) \right).
\] (3)
Here $\Delta y_h = \frac{\Delta y - y_h}{L}$ denotes the scaled difference between the location of the demarcation lines of the vertical (y_h) and horizontal (y_h) gyres so $u_{y_h}(\Delta y_h)|_{\Delta y_h = 0} = 0$. The term $u_{2av}$ corresponds to the non-trivial time average of the vertical component $u_{2av}(\cdot)$ at $y_h$. The effective frequencies $\Omega_{\text{tide, day}} / \Omega_{\text{double-gyre}}$ compare the characteristic time scale associated with the steady gyres $(\cdot)_{\text{double-gyre}}$ to the tide and convection characteristic periods (see Ref. 53 for a general discussion regarding such scales), and the functions $f_1(\Omega_{\text{tide}}), f_2(\Omega_{\text{day}})$ are explicitly computed, see Appendix B for details. As the Melnikov function here is a sum of two oscillatory terms and a constant, three different parameter regimes emerge:

**Bidirectional flux.** where $M(s; \theta_{\text{tide}})$ has simple zeroes in $s$ for all $\theta_{\text{tide}}$ values (the red region in Fig. 4). For all possible tidal phases, bidirectional flux, of order $\epsilon |u_{\text{tide}, f_1}|$, appears (as shown in Figs. 2(a) and 2(b)). Asymptotically, the motion is chaotic and there is a mixing layer at which sensitive dependence on initial conditions emerges. **Unidirectional flux regime.** where $M(s; \theta_{\text{tide}})$ has no simple zeroes in $s$ for all $\theta_{\text{tide}}$ values (the blue region in Fig. 4). Then, for all possible tidal phases, unidirectional flux of order $\epsilon |u_{\text{vert}} u_{2av}(\Delta y_h) u_{2av}|$, appears (as shown in Fig. 2(c)). The asymptotic behavior here includes attractors and repellers (possibly chaotic). **Tidal dependent regime.** where $M(s; \theta_{\text{tide}})$ has simple zeroes in $s \in [0, 2\pi/\omega_{\text{day}}]$ only for some specific ranges of $\theta_{\text{tide}}$ values whereas for other values it may have zeros only for larger $s$ values (the white region in Fig. 4). Then, for some tidal phases bidirectional flux appears whereas for others, at least for the two day period, all particles flow unidirectionally from one gyre to the other.

The division into these three regions in terms of some of the physical parameters (lumped into A, B in Eq. (3)) is shown in Fig. 4. These bifurcation diagrams that rely on the perturbative Melnikov
analysis have predictive power for sufficiently small $\epsilon$ values. Indeed, numerical integrations of the stable and unstable manifolds show that for sufficiently small $\epsilon$ (e.g., $\epsilon = 0.001$) the manifolds indeed have these three distinct regimes as predicted by the Melnikov analysis (see Ref. 56, notice some notational differences). Here, we show that even when we take a much larger $\epsilon$ value of 0.4, the principle division to the three regimes remains (though clearly the bifurcation lines shift). Three important principles emerge from this analysis:

**Realistic vertical motion usually makes the surface mixing unidirectional:** The Melnikov function has no zeroes when

$$\frac{|u_{vert}|}{|u_{tide}|} > |u_{2y}(\Delta y_h)|^{-1} \left| \frac{f_1(\Omega_{tide})}{f_2(\Omega_{day})} \right| \left( \frac{|u_{2v}|}{|f_2(\Omega_{day})|} - 1 \right)^{-1}$$

(equivalently $|B| > 1 + 1/|A|$). Thus, unidirectionality appears when $\Delta y_h \neq 0$ (the horizontal and vertical gyres are not aligned) provided $|u_{2v}| > |f_2(\Omega_{day})|$. We argue below that this latter inequality is satisfied when there is no day-time vertical convection. Hence, typical submesoscale vertical velocities eliminate the tangles and make the flux unidirectional. The blue regions in Fig. 4 show the dependence of this bound on $u_{\text{double-gyre}}, u_{\text{vert}}/u_{\text{tide}}$ and on $\Delta y_h$. We see that in the geophysical regime of $u_{\text{double-gyre}} \approx 1$, unidirectional motion reigns: only very strong alignment between the horizontal and vertical gyres (Fig. 4(a)) or very small vertical component (Fig. 4(b)) lead to bidirectional/chaotic motion.

**The asymmetric nature of the night convection leads to unidirectional flux:** The above-mentioned underlying assumption, that $|u_{2v}| > |f_2(\Omega_{day})|$, is natural in the geophysical context. The convection occurs during the night at certain sites, so that at a fixed surface location $u_2(\cdot, t)$ does not change sign with time and is essentially zero during the day. Hence, the integrated effect of $u_2(\cdot, t)$ on the steady flow (i.e., when we set $u_{\text{tide}} = 0$) will be to split the manifolds with no tangles. On the other hand, in a non-geophysical case where the convection pattern may reverse periodically during one day cycle so that $u_{2v} \approx 0$ (as in many lab experiments, see Ref. 54), $M(s, \theta_{\text{tide}})$ has a zero average in $s$ and therefore must have simple zeroes. Here we demonstrate that a small amount of day convection does not alter the results: the integrated effect here is still unidirectional as long as the steady gyres are not too energetic as explained next.

**The mixing zone is strongly influenced by the vigor of the base flow:** The integrals $f_1$ and $f_2$ in Eq. (3) have oscillatory integrands depending on the effective frequencies $\Omega_{\text{tide}}$ and $\Omega_{\text{day}}$ which are inversely proportional to $u_{\text{double-gyre}}$. Hence, when $u_{\text{double-gyre}}$ is small (so the base flow is weak), the first oscillatory term is exponentially smaller than the second one, and both are much smaller than the constant term (as long as $u_{\text{vert}} u_{2y}(\Delta y_h)$ is bounded away from zero). Thus, as shown in Fig. 4, at small $u_{\text{double-gyre}}$ the unidirectional flux dominates (blue region)—the upwelling leads to motion from the lower to the upper gyre. The narrow exceptional region of bidirectional flux at these $u_{\text{double-gyre}}$ values (red region) appears only if the upwelling position aligns with the separatrix (as in Ref. 54) or if the vertical component amplitude is very small (i.e., when $u_{\text{vert}} u_{2y}(\Delta y_h) \approx 0$). As $u_{\text{double-gyre}}$ increases so the base flow becomes vigorous, the dominance between these terms changes. When $\Omega_{\text{day}} = O(1)$, bidirectional flux may emerge (here when $u_{\text{double-gyre}} \gtrsim 3$, corresponding, in the submesoscale regime, to $U_{\text{double-gyre}} = u_{\text{double-gyre}} \frac{L_{\text{Lobe}}}{12 \text{hours}} \approx 0.54 \text{ m/s}$).

IV. CONCLUSIONS

Summarizing, we showed that typical night convection motion simplifies surface ocean mixing in the double-gyre model, making it unidirectional. Moreover, we showed that despite the fact that the vertical motion in submesoscale geophysical flows is much smaller than the horizontal velocities, its influence on the surface mixing is dramatic. Our counter intuitive result is that usually this 3D effect makes the surface mixing simpler and not more complicated as one would naively expect. Finally, we note that it can be shown that the main three principles that emerged here are rather general and do not depend on the specific geometry and symmetry properties of the double-gyre
model nor on the particular form of the vertical night convection. These principles distinguish typical geophysical surface mixing from mixing by typical time-dependent two-dimensional flows and from other three-dimensional flows.

This result that very weak nocturnal convection typically simplifies ocean surface mixing has wide-range implications. In the modified double-gyre model, for typical gyres strength, we showed that a unidirectional flux that leads to engulfment of the upper gyre particles by the lower particles be of order one, whereas another three-dimensional flows.

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APPENDIX A: MODEL DESCRIPTION

While we employ a specific model, we stress that much of our results are quite general and mainly depend on the breaking of the separatrix by the averaged flow and on the scaling of the effective frequencies.

Consider passive surface particles that move within a 3D flow in the box \([0, L_x] \times [0, L_y] \times [0, 1]\) yet are restricted to the plane \(z = 0\) (all variables are nondimensional and are rescaled to be of order one)

\[
\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v, \quad \frac{dz}{dt} = 0. \tag{A1}
\]

Here \(\vec{r} = (x, y, z = 0)\) is the particle’s surface position, and \(\vec{V} = (u(x, y, z = 0), v(x, y, z = 0), w(x, y, z = 0) = 0)\) is the velocity field experienced by the particles at the surface. Notice that generally \(w \neq 0\) for “regular” particles that are not constrained to stay at the surface. The 3D kinematic volume preserving velocity field \((u, v, w)\) is derived from three planar stream functions \(\Psi_{xy}(x, y, t), \Psi_{yz}(y, z, t),\) and \(\Psi_{xz}(x, z, t)\) as, e.g., in Ref. 55,

\[
\begin{align*}
    u &= \frac{\partial \Psi_{xy}}{\partial y} - \frac{\partial \Psi_{xz}}{\partial z}, \\
    v &= -\frac{\partial \Psi_{xy}}{\partial x} + \frac{\partial \Psi_{yz}}{\partial z}, \\
    w &= \frac{\partial \Psi_{xz}}{\partial x} - \frac{\partial \Psi_{yz}}{\partial y}. \tag{A2}
\end{align*}
\]

The model consists of a surface double gyre \(\psi_{xy}\), perturbed by weak periodic surface currents representing the tides and by weak vertical motion representing convection. The stream functions are of the form

\[
\begin{align*}
    \Psi_{xy}(x, y, t) &= [u_{\text{double-gyre}} \psi_{xy}(x, y) + \epsilon \cdot u_{\text{tide}} \cdot c_1 (t; \theta_{\text{tide}}) \tilde{\psi}_{xy}(x, y)], \\
    \Psi_{yz}(y, z, t) &= \epsilon \cdot u_{\text{vert}} \cdot c_2 (t) \tilde{\psi}_{yz}(y, z; y_{\text{h1}}), \\
    \Psi_{xz}(x, z, t) &= \epsilon \cdot u_{\text{vert}} \cdot \tilde{\psi}_{xz}(x, z, t), \tag{A3}
\end{align*}
\]

where \(\epsilon\) is a small parameter, \(u_{\text{double-gyre}}, u_{\text{tide}},\) and \(u_{\text{vert}}\) control the relative sizes of the averaged horizontal, tidal, and vertical velocities, respectively, \(c_1 (t; \theta_{\text{tide}}), c_2 (t)\) are periodic functions with distinct periods and \(c_1\) may be shifted by the phase \(\theta_{\text{tide}}\). Typically, \(u_{\text{double-gyre}}\) and \(u_{\text{tide}}\) are taken to be of order one, whereas \(u_{\text{vert}}\) serves to control the effect of the vertical velocity.
The steady double gyre, that represents the large scale geophysical vortical structures in a rectangular domain, is taken to be symmetric in $x$ and asymmetric in $y$ (the longer leg)

$$\psi_{xy} = \frac{1}{2\pi} \left( \sin \left( 2\pi \frac{y}{L_y} \right) + \sin \left( \pi \frac{y}{L_y} \right) \right) \sin \left( \pi \frac{x}{L_x} \right). \quad (A4)$$

Here, the numbers $L_x < L_y$ define the rectangular domain and are assumed to be both of order one. The two gyres are separated by a separatrix line in the $x$ direction located at $y_h = 2L_y/3$, so one gyre extends from $y = 0$ to $y_h$ and the other gyre extends between $y_h$ and $L_y$. In all the computations we take $L_x = 3/4$ and $L_y = 5/4$ (see Ref. 56 for motivation and details).

The steady flow is perturbed by two periodic processes: First, a transient horizontal current, which represents, for example, the tides$^{1,3}$ Second, a transient vertical motion, which represents, for example, the nocturnal convection. Hence, the two periodic processes usually have a different period where often the dominant tidal period is the $M_2$ (approximately 12 h 40 min) and the convection has a periodicity of one day. The transient perturbation in the surface current is time periodic with frequency $\omega_{tide}$. In all the computations $\omega_{tide} = 36\pi/19$, so one day corresponds to $t = 2$ (similar results appear when we repeat the calculations for $\omega_{tide} = \sqrt{3}\pi$, namely, the irrationality effects are not significant here, see the end of Appendix B for more details).

$$\tilde{\psi}_{xy} = \frac{1}{2\pi} \sin \left( 2\pi \frac{y}{L_y} \right) \sin \left( \pi \frac{x}{L_x} \right), \quad (A5)$$

$$c_1 (t; \theta_{tide}) = \sin (\omega_{tide} t + \theta_{tide}). \quad (A6)$$

The time periodic two-dimensional flow ($u_{tide} \neq 0$ and $u_{vert} = 0$) exhibits the typical chaotic mixing patterns that are controlled by the relative amplitude of the oscillating component $\epsilon u_{tide}/u_{double-gyre}$ and by $\Omega_{tide}$, the effective frequency.

The vertical motion is known to occur along some localized chimneys at which strong downwelling occurs, and the upwelling is distributed over wider regions. Therefore, we model the vertical convection by a steep double gyre with an adjustable parameter $y_{hc}$ that controls the location of the demarcation line between the two vertical gyres

$$\tilde{\psi}_{xz} = \frac{1}{2\pi} \sin (\pi z) \cdot \tanh \left( \frac{y \cdot \alpha}{L_y} \right) \cdot \tanh \left( \left( y_{hc} - y \right) \cdot \frac{\alpha}{L_y} \right)$$

$$\cdot \tanh \left( \left( y - L_y \right) \cdot \frac{\alpha}{L_y} \right),$$

$$c_2 (t) = \frac{1}{2} \left( \tilde{c}_2 + \frac{4}{\pi} \sin \left( \omega_{day} \cdot t \right) \right). \quad (A7)$$

Here, for simplicity, $\alpha$ dictates the width of both the upwelling and downwelling regions. Additionally, the convection pattern is known to be sustained throughout the night and to be essentially quiescent during the day. Hence, the time dependency corresponds to the first two Fourier components of a square wave with a bottom at ($\tilde{c}_2 - 1$). Geophysically, one should set $\tilde{c}_2 = 1$, to reflect the threshold characteristic of the diurnal cycle of convection: the convection occurs only at nights and ceases to occur at day time. Nonetheless, we keep the parameter $\tilde{c}_2$ free to demonstrate the important effect of this special process (the principle that the asymmetric nature of the night convection leads to unidirectional flux). Finally, we show below that the main conclusions of the paper are independent of the form of $\tilde{\psi}_{xz}$ (in the simulations we take it to be zero). In all the calculations here we take $\omega_{day} = \pi$, $\tilde{c}_2 = 1$, $\alpha = 10$.

Summarizing, the 3D flow (A3) induces the following equations of motion for the surface particles:

$$\dot{x} (t) = u_{double-gyre} \frac{\partial \psi_{xy}}{\partial y} + \epsilon u_{tide} \cdot c_1 (t; \theta_{tide}) \frac{\partial \tilde{\psi}_{xy}}{\partial y} - \epsilon u_{vert} \frac{\partial \tilde{\psi}_{xz}}{\partial z},$$

$$= u_{double-gyre} u_{0x} (x, y) + \epsilon (u_{tide} \cdot c_1 (t; \theta_{tide}) \cdot \hat{u}_{1x}(x, y) + u_{vert} \cdot \hat{u}_{2z}(x, z = 0, t)), \quad (A8)$$
\[ \hat{y}(t) = -u_{\text{double-gyre}} \frac{\partial \psi_{xy}}{\partial x} - \epsilon u_{\text{tide}} c_1 \left( \epsilon \theta_{\text{tide}} \right) \frac{\partial \tilde{\psi}_{xy}}{\partial x} + \epsilon u_{\text{vert}} \cdot c_2(t) \frac{\partial \tilde{\psi}_{xz}}{\partial z} \]

\[ = u_{\text{double-gyre}} M_{0y}(x, y) + \epsilon \left( u_{\text{tide}} \cdot c_1(\epsilon \theta_{\text{tide}}) \cdot \tilde{u}_{1y}(x, y) + u_{\text{vert}} \cdot c_2(t) \cdot \tilde{u}_{2y}(y, z = 0) \right). \]

We assume that the velocity of the surface particles is identical to that of the water, i.e., they have no inertia (see, e.g., Ref. 57, for the Lagrangian coherent structure approach to the modified equations of motion for inertial particles). In addition, we assume that the particles do not interact with each other, and they are not affected by surface tension.

**APPENDIX B: MELNIKOV FUNCTION CALCULATION**

The heteroclinic orbit (here, the demarcation line between the two gyres, see Ref. 27 for the more general setting) of the 2D steady flow may be explicitly found as follows:

\[ x_h(t) = \frac{L_x}{2} \left( 1 + \frac{4}{\pi} \tan^{-1} \left( \tanh \left( \frac{k \tau t}{2L_x} \right) \right) \right), \]

\[ y_h(t) = \frac{2}{3} L_y := y_h, \] (B1)

where

\[ k = -\frac{3}{4} \frac{u_{\text{double-gyre}}}{L_y}. \] (B2)

In particular, notice that only the \( x \) component depends on time and that its dependence on time is symmetric with respect to reflection about the midpoint and time reversal: \((x_h(t) - \frac{L_x}{2}) = -(x_h(-t) - \frac{L_x}{2}).\)

Denoting each of the additive velocity perturbation term in (A8) by \((\tilde{u}_i, \tilde{v}_i)\), by

\[ \hat{x}_h(\tau) = \frac{\pi}{2} \left( 1 + \frac{4}{\pi} \tan^{-1} \left( \tanh \left( \tau \right) \right) \right). \] (B3)

and by \( \theta \) any phase-dependent terms, the Melnikov integral is composed of a sum of terms of the form

\[ M_i(s; \theta) = u_{\text{double-gyre}} \int_{-\infty}^{\infty} \left( u_{0i} \tilde{u}_{iy} - u_{0y} \tilde{u}_{ix} \right) \left| \left. x_h(t), y_h(t) + s; \theta \right| \right. dt \]

\[ = k \int_{-\infty}^{\infty} \sin \left( \frac{\pi x_h(t)}{L_x} \right) \tilde{u}_{iy}(x_h(t), y_h(t), t + s; \theta) dt \]

\[ = -\frac{2L_x}{\pi} \int_{-\infty}^{\infty} \sin \left( \hat{x}_h(\tau) \right) \tilde{u}_{iy} \left( \frac{L_x}{k\pi} \hat{x}_h(\tau), y_h(t), \frac{2L_x}{k\pi} \tau + s; \theta \right) d\tau, \] (B4)

in which \( s \) is length along the unperturbed orbit and where the second equality follows from the form of \( u_0 \) and the observation that the heteroclinic orbit is along the \( x \) axis, namely, \( u_{0x}(x_h(t), y_h) \equiv 0 \).

We thus immediately conclude that the form of the perturbed velocity field in the \( x \)-direction (any of the \( \tilde{u}_{ix}, i > 0 \)) does not influence the Melnikov integral at all. In particular, convection motion in the \( xz \) plane does not influence the separatrix splitting to leading order in \( \epsilon \). This result should be intuitively clear—the velocity component in the \( x \)-direction does not break the symmetry of the heteroclinic orbit which is aligned with the \( x \)-direction.

More specifically, for perturbations of the form (A3), the Melnikov function is composed of the two terms

\[ M_1(s, \theta_{\text{tide}}) = -\frac{2u_{\text{tide}} L_x}{\pi} \int_{-\infty}^{\infty} \sin \left( \hat{x}_h(\tau) \right) \cdot c_1 \left( \frac{2L_x}{k\pi} \tau + s + \theta_{\text{tide}} \right) \cdot \tilde{u}_{1y} \left( \frac{L_x}{k\pi} \hat{x}_h(\tau), y_h(t) \right) d\tau \]

(B6)
\[ M_2(s) = -\frac{2 u_{\text{vert}} L_x}{\pi} \hat{u}_{2y}(y_h, z = 0) \int_{-\infty}^{\infty} \sin(\hat{s}(\tau)) \cdot \cosh \left(\frac{2 L_x}{k_{\pi}} \tau + s\right) d\tau, \] (B7)

where \( \theta_{\text{tide}} \) is the tidal phase—the shift of the tide cycle with respect to the diurnal cycle. The \( \hat{u}_{2y} \)'s correspond to the spatial dependence of the velocity perturbation terms of (A8). Plugging in the form of the \( c_i \)'s and of \( \hat{u}_{1y} \), notice that the only property of \( \hat{u}_{2y} \) that matters for the Melnikov function calculation is its value on the surface at the separatrix) we obtain

\[
M(s, \theta_{\text{tide}}) = u_{\text{tide}} f_1(\Omega_{\text{tide}}) \cos(\omega_{\text{tide}} s + \theta_{\text{tide}}) + u_{\text{vert}} \hat{u}_{2y}(y_h, z = 0) \left( f_2(\Omega_{\text{day}}) \sin(\omega_{\text{day}} s) + \tilde{c}_2 IC \right) 
\]

\[
= u_{\text{tide}} f_1(\Omega_{\text{tide}}) \cos(\omega_{\text{tide}} s + \theta_{\text{tide}}) + u_{\text{vert}} u_{2y}(\Delta y_h) \left( f_2(\Omega_{\text{day}}) \sin(\omega_{\text{day}} s) + u_{2av} \right).
\] (B8)

Setting \( \Delta y_h = \frac{\bar{y} - y_h}{L_y} \), \( L_x = 3/4 \), \( L_y = 5/4 \), \( \alpha = 10 \) we obtain

\[
f_1(\Omega_{\text{tide}}) = -\frac{1}{\pi} \sin \left( \frac{4\pi}{3} \right) \int_{-\infty}^{\infty} \sin(\hat{s}(\tau)) \cdot \cos(\hat{s}(\tau)) \cdot \sin \frac{8L_x L_y}{3\pi u_{\text{double-gyre}}} \omega_{\text{tide}} \tau \, d\tau,
\]

\[
f_2(\Omega_{\text{day}}) = -\frac{4}{\pi^2} \int_{-\infty}^{\infty} \sin(\hat{s}(\tau)) \cdot \cos \frac{8L_x L_y}{3\pi u_{\text{double-gyre}}} \omega_{\text{day}} \tau \, d\tau,
\]

\[
u_{2av} = \tilde{c}_2 IC = -\tilde{c}_2 \frac{1}{\pi} \int_{-\infty}^{\infty} \sin(\hat{s}(\tau)) \, d\tau = -\frac{\tilde{c}_2}{4},
\]

\[
u_{2y}(\Delta y_h) = L_x \hat{u}_{2y}(y_h, z = 0) = -\frac{L_x}{2} \cdot \tanh \left( \frac{2\alpha}{3} \right) \cdot \tanh \left( \frac{\alpha}{3} \right) \cdot \tanh(\alpha \Delta y_h)
\]

\[= -0.374 \tanh(10\Delta y_h).\]

Noticing that \( \nu_{2y}(\Delta y_h) \nu_{2av} = \left[ \nu_{2y}(y_h, \omega_{\text{day}}t; \epsilon) \right]_t \) and recalling that \( \omega_{\text{tide}} = 36\pi/19 \), \( \omega_{\text{day}} = \pi \) we obtain all the coefficients that appear in Eq. (3) of the paper

\[
\Omega_{\text{tide}, \text{day}} = \omega_{\text{tide}, \text{day}} \approx \frac{4.74}{u_{\text{double-gyre}}} \cdot \frac{2.5}{u_{\text{double-gyre}}},
\]

\[
u_{2y}(\Delta y_h) = -0.374 \tanh(10\Delta y_h),
\]

\[
u_{2av} = \left[ \nu_{2y}(y_h, \omega_{\text{day}}t; \epsilon) \right]_t \text{ for } \Delta y_h \neq 0,
\] (B9)

\[
A = \frac{u_{2y}(\Delta y_h) u_{\text{vert}} f_2(\Omega_{\text{day}})}{u_{\text{tide}} f_1(\Omega_{\text{tide}})},
\]

\[
B = \frac{u_{2av}}{f_2(\Omega_{\text{day}})},
\]

where the functions \( f_{1,2} \) are calculated numerically and are shown in Fig. 5.

Now, we need to find when the Melnikov function has zeroes, namely, when does the equation

\[
\cos(\omega_{\text{tide}} s + \theta_{\text{tide}}) = -A(\sin(\omega_{\text{day}} s) + B)
\] (B10)

has zeroes in \( s \) for a fixed \( \theta_{\text{tide}} \). Here, we should distinguish between several cases:

- For all \( s \), \( |A(\sin(\omega_{\text{day}} s) + B)| > 1 \), namely, \( |B| > \frac{1}{|A|} + 1 \). Then, the Melnikov function has no zeroes for any \( \theta_{\text{tide}} \).
- There exists a non-empty open interval \( I_{A,B} = [0, 2\pi/\omega_{\text{day}}] \) such that for \( \omega_{\text{day}} s \in I_{A,B} \), the function \( |A(\sin(\omega_{\text{day}} s) + B)| < 1 \). Such an interval exists when \( |B| < \frac{1}{|A|} + 1 \). Then, for any \( s \in I_{A,B} \), there exists a \( \theta_{\text{tide}} \) for which \( s \) corresponds to a zero of the Melnikov function, and, generically, this zero is simple. The more interesting question is whether for any \( \theta_{\text{tide}} \) there are simple zeros in \( s \). If \( \omega_{\text{tide}}/\omega_{\text{day}} \) is irrational, then, such zeroes indeed always exist. On the other hand, if \( \omega_{\text{tide}}/\omega_{\text{day}} \) is rational and the open interval \( I_{A,B} = [0, 2\pi/\omega_{\text{day}}] \) at which \( |A(\sin(\omega_{\text{day}} s) + B)| > 1 \) is non-empty, we may have intervals of \( \theta_{\text{tide}} \) values for which there
FIG. 5. The Melnikov function coefficients $f_1(\Omega_{\text{tide}}(u_{\text{double}} - \text{gyre}))$ (blue) and $f_2(\Omega_{\text{day}}(u_{\text{double}} - \text{gyre}))$ (red).

are no simple zeroes. The interval $I_{A,B}$ is non-empty if $|B| > \frac{1}{|A|} - 1$. Thus, in the rational case we have three regimes: $|B| > \frac{1}{|A|} + 1$ where there are no zeros (blue region in Fig. 4), $\frac{1}{|A|} - 1 < |B| < \frac{1}{|A|} + 1$ where some values of $\theta_{\text{tide}}$ may not have zeros (white region in Fig. 4), and $|B| < \frac{1}{|A|} - 1$ (red region in Fig. 4) where for all $\theta_{\text{tide}}$ we have zeros. When $\omega_{\text{tide}}/\omega_{\text{day}}$ is an irrational, which is well approximated by a rational, for $\theta_{\text{tide}}$ values at which the rational case has no zeroes, the zeros occur only at large $s$ values. Here, since $\omega_{\text{tide}}/\omega_{\text{day}} \approx 2$, we indeed see that such a behavior is observed (Figs. 2, 3(e), and 3(f)).

27 J. Guckenheimer and P. Holmes, Non-Linear Oscillations, Dynamical Systems and Bifurcations of Vector Fields (Springer-Verlag, New York, 1983).
42 The specific dependence of $\alpha(x)$ on $(x,y)$ is convenient for defining a 3D incompressible flow (see Appendix A) but it is inessential for the applicability of our results.