Secondary homoclinic bifurcation theorems

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We develop criteria for detecting secondary intersections and tangencies of the stable and unstable manifolds of hyperbolic periodic orbits appearing in time-periodically perturbed one degree of freedom Hamiltonian systems. A function, called the "Secondary Melnikov Function" (SMF) is constructed, and it is proved that simple (resp. degenerate) zeros of this function correspond to transverse (resp. tangent) intersections of the manifolds. The theory identifies and predicts the *rotary number* of the intersection (the number of "humps" of the homoclinic orbit), the *transition number* of the homoclinic points (the number of periods between humps), the existence of tangencies, and the scaling of the intersection angles near tangent bifurcations *perturbationally*. The theory predicts the minimal transition number of the homoclinic points of a homoclinic tangle. This number determines the relevant time scale, the minimal stretching rate (which is related to the topological entropy) and the transport mechanism as described by the TAM, a transport theory for two-dimensional area-preserving chaotic maps. The implications of this theory on the study of dissipative systems have yet to be explored. © *1995 American Institute of Physics.*

I. INTRODUCTION

We consider two-dimensional autonomous Hamiltonian system possessing a hyperbolic fixed point, a homoclinic orbit connecting it to itself, and a family of periodic orbits nested in the homoclinic orbit. This situation is generic for one degree of freedom nonlinear Hamiltonian systems. When such a system is perturbed by a time-periodic perturbation (which *may be* or *may not be* Hamiltonian) chaotic behavior may appear. The study of such systems, called one and a half degrees of freedom systems, began more than 100 years ago with Poincaré, and has been the subject of numerous papers ever since because of its relevance to a wide variety of physical problems and because of its fundamental nature; These systems model the simplest physical settings which give rise to a Smale horseshoe. Hence, the study of these simple systems serves as a building block for the analysis of chaotic systems in large.

To study the dynamics of the perturbed system, one considers the Poincaré map F which is constructed from the flow by sampling the solutions every period of the perturbation,[1,2] T. For small perturbation the hyperbolic fixed point and its stable and unstable manifolds persist, however, generically they do not coincide as in the autonomous case. In fact, they may intersect each other transversely, creating a homoclinic tangle. Since both manifolds are invariant, once they intersect at a point they must intersect at all its forward and backward iterates. As line elements are also stretched in the vicinity of the hyperbolic point, the intersection of the manifolds gives rise to a complicated structure, the homoclinic tangle[3–5] (see Fig. 1). The *primary* distance function between the stable and unstable manifolds is given, to first order in the perturbation expansion parameter, ϵ, by the Melnikov function, multiplied by a scaling factor.[2,6] The *primary* distance function is the distance between the manifolds taken along the direction perpendicular to the unperturbed homoclinic orbit, taking the *first intersection in elapsed time* of the manifolds with this cross section, see Fig. 1 or Wiggins[7] for discussion.

Theorem 4.5.3 of Guckenheimer and Holmes[2] states that if the Melnikov function is independent of ϵ and has simple zeros in t then, for ϵ sufficiently small, the manifolds intersect transversely. Hence, by calculating the Melnikov function one can prove that transverse PIPs (primary intersection points) exist. This implies by the Smale–Birkhoff homoclinic theorem that the dynamical system is chaotic. Theorem 4.5.4 in Guckenheimer and Holmes states that if, additionally, the Melnikov function depends on a parameter μ and it has a quadratic zero in t at μ_0, then, provided some generic conditions are satisfied, there exists a bifurcation value $\mu=\mu_0+O(\epsilon)$ for which quadratic homoclinic tangencies occur. For dissipative systems, it is implied by Newhouse results,[8] that tangencies of the manifolds occur in a neighborhood of μ, and associated with these tangencies are complicated behaviors such as wild hyperbolic sets and strange attractors.[8,9]

In this paper we prove similar results regarding the "secondary" distance function. Observe that a secondary homoclinic point, like r_0 in Fig. 1, belongs to $F^jK_0\cap J_0$ for some j (K_0 and J_0 are defined as in the figure, by segments of the stable and unstable manifolds with end points which are PIPs). Easton called j the *transition number* of the orbit r_i, and proved it is a well-defined quantity (independent of the choice of r_0 in the figure).[10] Hockett and Holmes[11] defined the *rotary number* of a homoclinic orbit as, roughly speaking, the number of times the orbit encircles the perturbed elliptic orbit (counting the infinite time encircling as well). For example, the orbit r_i is called a *secondary (2-rotary) intersection point (SIP) of index l* if r_0 has a transition number l, and the orbit r_i encircles around the interior once as i increases from $-l-1$ to 0. Let $r(t)=(x(t),y(t))$ denote the homoclinic solution of the perturbed flow corresponding to Fig. 1 so that $r(iT)=r_i$. Then the rotary number counts the number of humps in the graph of $|x(t)|$ vs t and

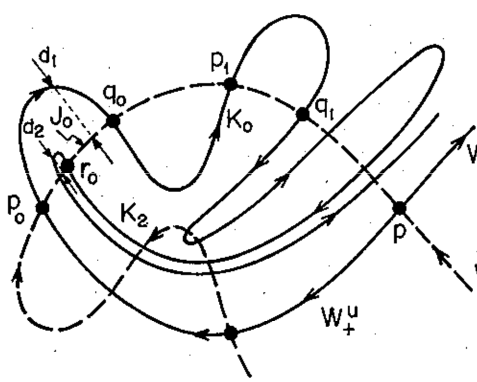

FIG. 1. The homoclinic tangle. d_1 is a primary distance, d_2 is a secondary distance.

the gap between the two humps is given approximately by the transition number times the Poincaré map period, T.

The existence of an n-rotary intersection point for some $n>1$ may be easily established using geometrical arguments in the area-preserving maps case (such as the Poincaré map of the Hamiltonian flow perturbed by Hamiltonian perturbation) in which the homoclinic loop "encloses" a bounded domain, or, in general by using the structure near the primary homoclinic point. Moreover, using linearization about the perturbed hyperbolic periodic orbit and using the Melnikov function to extract information regarding the splitting distance between the manifolds, it is possible to estimate the asymptotic dependence of the transition number on ϵ as $\epsilon \to 0$. Holmes and Marsden[12] and Hockett and Holmes[11] have used these ideas to estimate the number of iterates needed to guarantee hyperbolicity on the invariant set near primary homoclinic points. Judd[13] has used this method to estimate the transition number of secondary intersection points for the Duffing equation.

The main result of this paper is Theorem 1 of Sec. II, stating, roughly, that for sufficiently small ϵ, simple zeros in t_0 of the SMF $\bar{h}_2(t_0,\epsilon)$ correspond to transverse secondary homoclinic intersections with transition number $j(t_0,\epsilon)$, where $\bar{h}_2(t_0,\epsilon)$ is defined by

$$\bar{h}_2(t_0,\epsilon)=M(t_0)+M(t_0+P(\epsilon M(t_0))), \quad 0<t_0<T, \tag{1.1}$$

$M(t)$ denotes the Melnikov function, $P(H)$ the period of the unperturbed periodic orbits with energy H, t_0 parametrization of the unstable manifold as in the Melnikov theory, and WNLG (with no loss of generality) we assume $H=0$ on the separatrix and $M(0)=0$. The transition number is defined by

$$j(t_0,\epsilon)=\left[\frac{t_0+P(\epsilon M(t_0))}{T}\right], \quad 0<t_0<T, \tag{1.2}$$

where $[x]$ indicates the integer part of x and T is the period of the time-dependent perturbation.

The importance of this theorem is not in the existence result, nor in the general asymptotic of the transition number as these may be found using other tools. Its importance lies in the explicit, rigorous relation between the number and nature of the zeros of the SMF and the homoclinic points and the explicit formula for the transition number. This formula has no unknown constants, and in practice it supplies accurate estimates for transition numbers as low as two.[14] Furthermore, this theorem leads naturally to

—construction of the set of ϵ intervals for which transverse secondary homoclinic solutions of transition number l exist (Theorem 2);

—construction of the set of ϵ values close to which secondary homoclinic tangencies of transition number l occur, and prediction of the structure of the manifolds near the bifurcation values (Theorem 3).

—in the presence of additional parameters, construction of surfaces close to which homoclinic tangencies occur (Theorem 4);

—generalization of the criterion $\bar{h}_2(t,\epsilon)=0$ and the proof of its validity to criterion for the existence of an n-rotary homoclinic point with transition vector $(j_1,\dots,j_{n-1})$, as is outlined in Sec. III and the proof of Theorem 1.

Since $P(H)\to\infty$ as $H\to 0$, it follows from (1.1) that $\bar{h}_2$ depends singularly on ϵ. This singular dependence on ϵ makes the proof of the bifurcation theorems nontrivial, as the implicit function theorem may not be used near $\epsilon=0$. A trivial (yet important) result of this dependence and the periodicity of $M(t)$ is that there are families of countable infinity sets of ϵ values for which secondary homoclinic tangencies of the same nature occur. In contrast, varying one parameter, generically, will cause at most finite number of *primary* (1-rotary) homoclinic tangencies. Hence, generically, the following scenario occurs: for given parameter values, let l be the smallest integer for which $F^n K_0 \cap J_0 \neq 0$ (l is called the type number of the homoclinic tangle.[10,14] As ϵ is decreased, the "tip" of $F^n K_0$ gets shorter, until, at the bifurcation point we have a secondary tangency of transition number l. As ϵ is further decreased, the scenario repeats itself with a larger l value. Incorporating the influence of additional parameters results in a topological bifurcation diagram[15] dividing the parameter space to regions indexed by l, the tangle type number, and to the various kinds of bifurcations occurring for each l. Moreover, there is evidence that the perturbed flow exhibits *self-similar behavior* near these bifurcation values.[14]

What is the significance of proving the existence of SIP of transition number l? First, we propose that the prediction of the rotary number and the transition numbers are important in applications for predicting stretching and transport rates. In particular, we propose that the type number of the homoclinic tangle determines a lower bound on the stretching rate[13,15] and determines the oscillation period of the escape rates and the elongation rates.[14,15] Hence, the described theoretical results regarding global bifurcations of certain homoclinic points have direct bearings on transport and mixing properties of the flow. Moreover, using the SMF, we can prove that tangencies occur at SIPs on a countable infinity set of parameter values. In the dissipative case, near each of these parameter values Newhouse results apply, hence this theory gives an initial construction of the set of parameter values for which we expect to see wild hyperbolic sets as described by Newhouse.[8] For these parameter values, we also expect a change[16] in the "Pruning fronts" which describe the growth rate of the number of periodic orbits in the system (see Cvitanović *et al.*[17]). Indeed, Hockett and Holmes[11] showed that the rotary number of a transverse homoclinic point determines the properties of the rotation set of the hyperbolic nonwandering invariant set associated with the homoclinic point.

Finally, we draw two speculative analogies to the theory for the primary intersection points. The first is concerned with the limit of periodic orbits to the homoclinic orbits and the second concerns the angle of intersection. Theorem 4.6.4 of Guckenheimer and Holmes implies that the primary homoclinic bifurcations are the limit as $m\rightarrow\infty$ of sequences of *subharmonic saddle–node bifurcations with rotation number* $m/1$. Using the typical structure near a homoclinic bifurcation one concludes that there is a sequence of saddle–node bifurcations of periodic orbits with rotation number m/n converging onto the n-rotary homoclinic bifurcation as $m\rightarrow\infty$. These sequences, however, seem to be undetectable by the subharmonic Melnikov function! A subharmonic SMF may be needed for their detection. The angle of intersection between the stable and unstable manifolds at a primary intersection point is proportional to $\epsilon M'(t_0)$, where t_0 satisfies $M(t_0)=0$. In analogy we assume (without proving) that the angle between the manifolds at the intersection point near (t_0,ϵ_0) is proportional to $\epsilon_0[\partial\bar{h}_2(t_0,\epsilon_0)/\partial t_0]$. The scaling of this angle with ϵ near the bifurcation points is found in Theorems 3 and 4.

This paper is organized as follows: in Sec. II we list the assumptions on the form of the Hamiltonian system suitable for our theory and formulate the four theorems we prove regarding secondary intersections of the manifolds in these systems. In Sec. III we motivate the choice of the "Secondary Melnikov Function" (SMF) and the stated results by introducing the Whisker map[18,19] and by interpreting its variable geometrically following the lines of Escande's work.[20] In Sec. IV we summarize our results in a nonformal way and discuss further developments. The appendices consist of the proofs of Theorems 1–4. In Appendices A and B we prove Theorems 1 and 2 dealing with conditions for transverse intersection of the manifolds. In Appendix C we prove Theorems 3 and 4, dealing with conditions for tangencies and other degenerate behavior of the manifolds.

II. ASSUMPTIONS AND RESULTS

We consider two-dimensional time-periodic near-integrable equations:

$$\frac{dx}{dt}=f_1(x,y)+\epsilon g_1(x,y,t;\epsilon),$$
$$\frac{dy}{dt}=f_2(x,y)+\epsilon g_2(x,y,t;\epsilon), \tag{2.1a}$$

where

$$f=(f_1,f_2)=\left(\frac{\partial H(x,y)}{\partial y},-\frac{\partial H(x,y)}{\partial x}\right),$$
$$g=(g_1,g_2),\quad q=(x,y)\in M, \tag{2.1b}$$

and M is a two-dimensional C^r manifold $r\geqslant 4$. Assume the following on (2.1):

A0 f,g are C^r functions of their variables (x,y,t), $r\geqslant 4$, and g is periodic in t with period T (independent of ϵ):

$$g(x,y,t+T;\epsilon)=g(x,y,t;\epsilon),\quad T=\frac{2\pi}{\omega},$$
$$\text{for all } (x,y)\in M, t\in R. \tag{2.2}$$

g is analytic in ϵ and possibly in other parameters μ. We drop its explicit dependence on ϵ and μ when appropriate.
A1 The origin is a hyperbolic fixed point of (2.1), hence, WNLG we assume

$$H(x,y)=\frac{y^2}{2}-\frac{x^2}{2}+O(3),\quad \text{for } |x|,|y|\ll 1, \tag{2.3}$$

where $O(3)$ stands for terms of order 3 in x and y.

A2 For $\epsilon=0$, the left branches of the stable and unstable manifolds of the origin coincide, forming a homoclinic orbit, Γ, parametrized by the solution $q^0(t)$. The right branches may either extend to infinity (*open flow*), or stay bounded (*closed flow*), forming a homoclinic connection or heteroclinic connections, but in any case they do not extend to the left half-plane. WNLG we assume that $q^0(0)=(x^0(0),0)$, and that $H(q^0(t))=0$.

A3 For $\epsilon=0$ the interior of Γ is foliated by periodic orbits $q^H(t)$, with period $P(H)$, labeled by their energies $H=H(q^H(t))<0$. Assume that the nth derivative of $P(H)$ satisfies $P^{(n)}(H)=(1/H^n)(C(n)+o(H))$ as $H\rightarrow 0^-$ for $n=1,2$, where $C(n)$ is an order one constant and $C(1)<0$. If the system is closed, $P(H)$ satisfies similar conditions for $H>0$.

A4 Near the origin g is of the form: $g(x,y,t,\epsilon;\mu)=xp(t,\epsilon;\mu)+yq(t,\epsilon;\mu)+\tilde{g}(x,y,t,\epsilon;\mu)$, where p,q are periodic in t and $\tilde{g}$ is $O(2)$ in x,y (uniformly in ϵ).

A5 The Melnikov function, a periodic function in t of period T, defined by

$$M(t_0)=M(t_0;\mu)=\int_{-\infty}^{\infty} f\wedge g\Big|_{(q^0(t-t_0),t)}dt$$
$$=\int_{-\infty}^{\infty}(f_1g_2-f_2g_1)\Big|_{(q^0(t-t_0),t)}dt, \tag{2.4}$$

satisfies the following assumptions:

(i) $M(t)$ has at least two simple zeros in $[0,T)$. WNLG assume $M(0)=0$, $M'(0)<0$.
(ii) For all real c in $[\min M(t),\max M(t)]$, $M(t)=c$ has a finite number of solutions $t\in[0,T)$, denoted by $M^{-1,i}(c)$.
(iii) $M'(t)$ has a finite number of zeros, all of which are of finite order.

Remarks

(1) For ϵ sufficiently small conditions **A0** and **A1** guarantee that the Poincaré map has a hyperbolic fixed point near the origin. Since g is analytic in ϵ by the implicit function theorem there exists a change of coordinates such that **A4** is satisfied in the new coordinate systems. However, in practice **A4** is the most restrictive assumption; if it is not satisfied, one needs to solve for the perturbed periodic orbit (to all orders in ϵ) and make the appropriate, time-dependent shift of coordinates to transform the system to this form. To avoid this transformation, one may derive similar theory, valid on shorter time scales.

(2) For clarity of presentation, we will concentrate on the behavior of the left branches of the stable and unstable manifolds, which create the homoclinic loop at $\epsilon=0$. Nonetheless, the theory applies to other cases, e.g., closed flows or homoclinic loops which are composed of several heteroclinic connections. There, additional secondary intersections between various branches of the manifolds may appear, and one needs to keep track of the different cross sections and the different Melnikov integrals corresponding to each broken heteroclinic or homoclinic connection.

(3) Assumptions **A5**(ii) and **A5**(iii) essentially state that the Melnikov function is not pathological, and specifically that it does not have plateaus in any subsegments of $[0,T)$.

(4) For simplicity of notation, we assume in all the figures (but not in the analysis) that $M(t)$ has exactly two simple zeros every period of the perturbation, hence, for small ϵ the Poincaré map F has exactly two primary intersection orbits denoted by q_i and p_i in Fig. 1.

(5) For example, a system satisfying **A0–A5** is the system describing the motion of a particle in a forced cubic potential[14] with the Hamiltonian

$$H^{\epsilon}(x,y,t)=\tfrac{1}{2}y^2+(-\tfrac{1}{2}x^2-\tfrac{1}{3}x^3)[1+\varepsilon\cos(\omega t)]. \tag{2.5}$$

In Appendices A–C the following results are proved:

Theorem 1: *Let (2.1) satisfy* ***A0–A5****. Let* $\eta,\eta'>0$ *and let* $0<\beta<\alpha<1/2$ *be given constants. Then there exists an* $\hat{\epsilon}=\hat{\epsilon}(\alpha,\beta,\eta,\eta')$ *such that if* (t_0,ϵ_0) *satisfy conditions C1 and C2 listed below, and* $\epsilon_0<\hat{\epsilon}$ *then the stable and unstable manifolds intersect transversely in a secondary homoclinic point of transition number* $l(t_0,\epsilon_0)$ *defined by (1.2) for all* $\tilde{\epsilon}\in(\epsilon_0-\Delta\epsilon,\epsilon_0+\Delta\epsilon)$*, where* $\Delta\epsilon=O(\epsilon_0^{1+\alpha})$ *and* $\tilde{t}=t_0+O(\epsilon_0^{\beta})$*. The conditions C1 and C2 are*

$$\text{C1:}\quad \bar{h}_2(t_0,\epsilon)=0 \quad\text{and } M(t_0)\leqslant-\eta,$$

$$\text{C2:}\quad \left|\frac{\partial\bar{h}_2}{\partial t_0}(t_0,\epsilon_0)\right|>\eta'.$$

(see Fig. 2.)

Generically, we expect the angle between the stable and unstable manifolds to be proportional to leading order in ϵ to $\epsilon_0[\partial\bar{h}_2(t_0,\epsilon_0)/\partial t_0]$.

Theorem 2: *Let (2.1) satisfy* ***A0–A5*** *and let* $l>0$ *and* $\eta\in(0,\bar{M}]$ *be given, where*

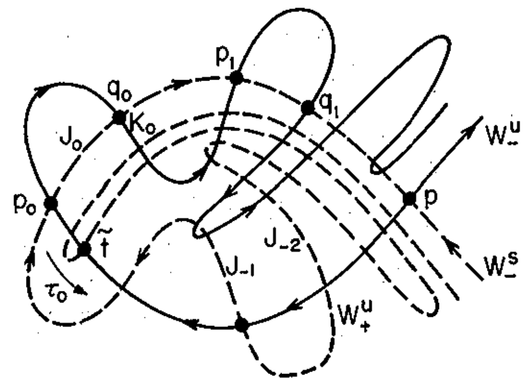

FIG. 2. Transverse secondary intersection. SIP with transition number $l=2$ at $\tau_0=\tilde{t}$.

$$\bar{M}=\min\{\max_t M(t),-\min_t M(t)\}. \tag{2.6}$$

Then, there exists a set of nonempty intervals, $\bar{I}_\eta\subset(0,T)$*, and there exist k curves [the number of curves is determined by the multiplicity of M(t) and some of them may be defined only on subsegments of* $\bar{I}_\eta$*]* $(\epsilon^i(t_0;l),t_0)$*,* $i=1,\dots,k$*, satisfying condition C1 of theorem 1 for all* $t_0\in\bar{I}_\eta$ *with* $j(t_0,\epsilon^i(t_0;l))=l$*. Moreover, given* $\eta'>0$ *sufficiently small and* $\hat{\epsilon}$*, there exists an* $l_0(\hat{\epsilon})$ *such that for* $l\geqslant l_0$ *there exist open intervals of positive lengths of* t_0 *and curves* $\epsilon^i(t_0,l)<\hat{\epsilon}$ *satisfying conditions C1 and C2. Finally, if* $\max M(t)=-\min M(t)$ *there exist countable infinity values of* $(t_0,\epsilon(t_0,l))$ *satisfying C1 but not C2:* $[\partial\bar{h}_2(t_0,\epsilon(t_0,l))/\partial t_0]=0$.

Remarks:

(1) $l_0(\hat{\epsilon})$ *is the type number of the homoclinic tangle*[10] at $\hat{\epsilon}$, and plays an important role in determining the nature of stretching and transport in such flows.[14]

(2) If $\max M(t)\neq-\min M(t)$ there may still exist countable infinity values of $(t_0(l),\epsilon_0(t_0(l),l))$ satisfying C1 but not C2. (See Fig. 3.)

Corollary: *Let (2.1) satisfy* ***A0–A5****. Then, for* $\epsilon>0$ *sufficiently small there exist secondary homoclinic intersection points with finite transition numbers.*

Theorem 3: *Define* $t_1(t_0,\epsilon_0)=t_0+P(\epsilon_0M(t_0))$*. Let (2.1) satisfy* ***A0–A5****, and let* l*,* $t_0\in(0,T)$ *and* $\epsilon_0=\epsilon^l(t_0,l)$ *be given so that* (t_0,ϵ_0) *satisfy C1 and* $(\partial\bar{h}_2/\partial t_0)(t_0,\epsilon_0)=0$*. Then, for sufficiently small* ϵ_0*:*

S1 *Generically (the generic conditions to be satisfied in each of the cases listed below are given in Appendix C, with the proof of the theorem) the manifolds undergo a "secondary homoclinic saddle–node bifurcation" for* $\tilde{\epsilon}=\epsilon_0+O(\epsilon_0^{1+\alpha})$*,* $0<\alpha<1/2$*; Namely, at* $\epsilon=\tilde{\epsilon}$ *there exists a secondary homoclinic tangency of transition number l, with generically, quadratic tangency (but possibly higher-order even-order tangency). For* $\epsilon<\tilde{\epsilon}$ *(or* $\epsilon>\tilde{\epsilon}$*) the manifolds do not have secondary intersection with transition number l for t near* t_0 *whereas for* $\epsilon>\tilde{\epsilon}$ *(correspondingly* $\epsilon<\tilde{\epsilon}$*) two secondary homoclinic orbits with transition number l are created with* $\tilde{t}=t_0+O(\epsilon_0^{\alpha/2})$ *and with angle of intersection* $\epsilon[\partial\bar{h}_2(\tilde{t},\tilde{\epsilon})/\partial\tilde{t}]=O(\epsilon_0^{1+\alpha/2})$*.*

S2 *If additionally* $M'(t_0)=M'(t_1(t_0,\epsilon_0))=0$*, then, generically, the manifolds undergo a "perturbed transcritical bifur-*

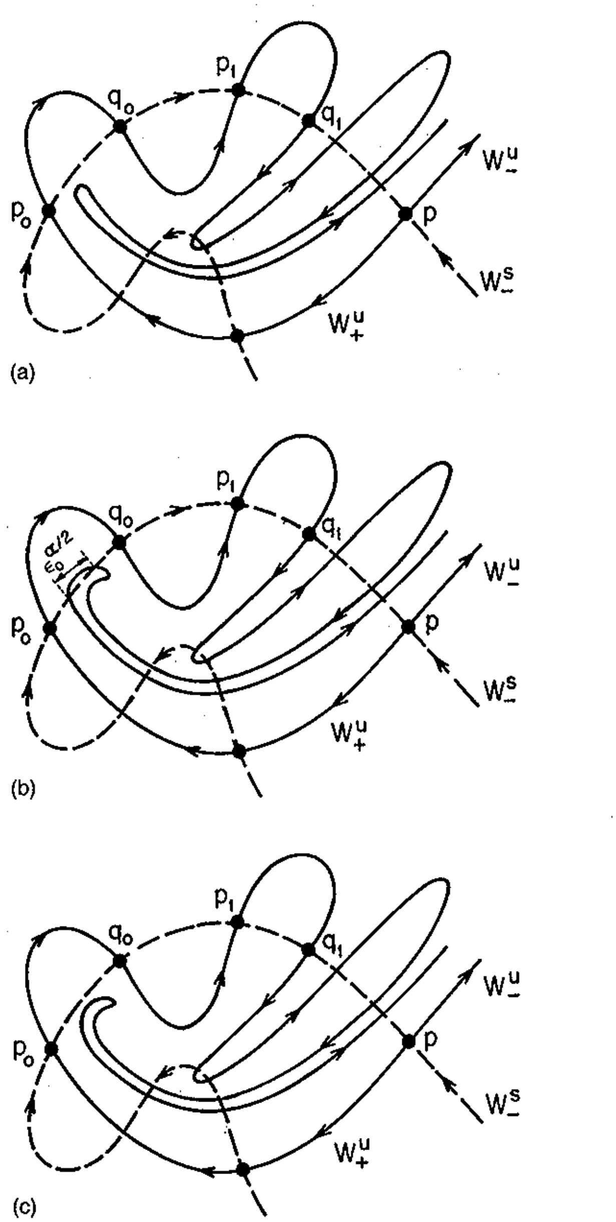

FIG. 3. Secondary homoclinic bifurcations. (a) $\epsilon<\tilde{\epsilon}$, $l=2$. (b) $\epsilon>\tilde{\epsilon}$, case **S1** of Theorem 3. (c) $\epsilon>\tilde{\epsilon}$ case **S2**(ii) of Theorem 3.

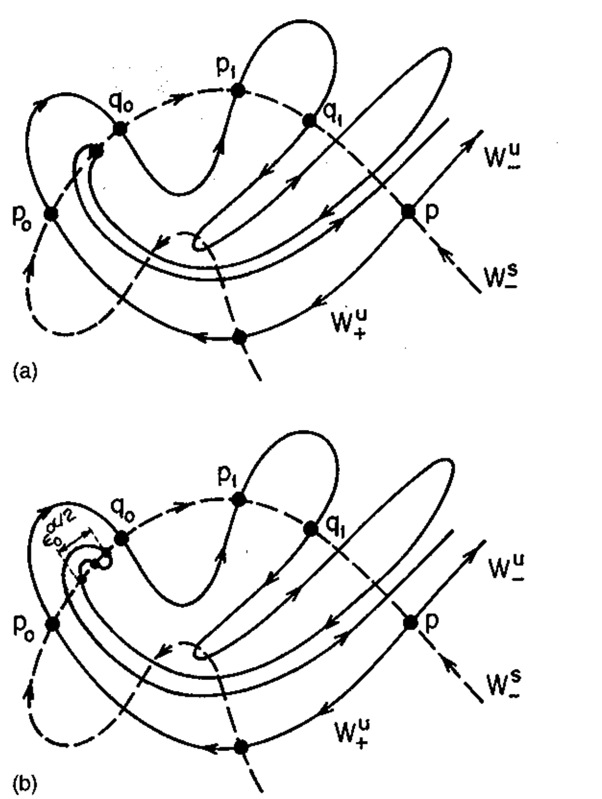

FIG. 4. Secondary homoclinic bifurcation with symmetry. (a) $\epsilon<\tilde{\epsilon}$, $l=2$. (b) $\epsilon>\tilde{\epsilon}$, case **S3** of Theorem 3.

cation"; depending on the relations between the second-order derivatives of $M(t)$ at t_0 and t_1, there are two possibilities:

(i) *The manifolds intersect topologically transversely at two homoclinic points for both $\epsilon<\epsilon_0$ and $\epsilon>\epsilon_0$, scaling near the bifurcation, at $\tilde{\epsilon}=\epsilon_0+O(\epsilon_0^{1+\alpha})$, $0<\alpha<1/4$ like $\tilde{t}=t+O(\epsilon_0^{\alpha})$ with the angle $\tilde{\epsilon}\,[\partial\bar{h}_2(\tilde{t},\tilde{\epsilon})/\partial t]=O(\epsilon_0^{1+\alpha})$. Generically, there may be none, one or two values of $\tilde{\epsilon}$ in the interval $(\epsilon_0-\Delta\epsilon,\epsilon_0+\Delta\epsilon)$ at which the manifolds are tangent.*

(ii) *The manifolds do not intersect for $(\tilde{t},\tilde{\epsilon})=[t_0+O(\epsilon_0^{\alpha}),\epsilon_0+O(\epsilon_0^{1+\alpha})]$, $0<\alpha<1/4$, but may intersect on smaller scales, with $\tilde{\epsilon}[\partial\bar{h}_2(\tilde{t},\tilde{\epsilon})/\partial\tilde{t}]=o(\epsilon_0^{5/4})$.*

S3 *If $M'(t_0)=M'(t_1(t_0,\epsilon_0))=0$ and additionally $M''(t_0)=-M''(t_1)$ then generically, the manifolds undergo topologically a "pitchfork bifurcation" (the description of which may be deduced similarly to previous cases) for $\epsilon=\epsilon_0+O(\epsilon_0^{1+\alpha})$, $0<\alpha<1/3$, and $\tilde{t}=t_0+O(\epsilon_0^{\alpha/2})$, and $\tilde{\epsilon}\,[\partial\bar{h}_2(\tilde{t},\tilde{\epsilon})/\partial\tilde{t}]=O(\epsilon_0^{\alpha})$. If $M(t)$ is odd in t $(M(t)=-M(-t))$ then only conditions **S1** and **S3** are possible.* (See Fig. 4.)

Theorem 4: *Let g of (2.1) depend on the parameters $\mu\in R^p$ and let (2.1) satisfy* **A0–A5** *for an open ball of μ values centered at μ_0 (of size independent of ϵ). Let l, $t_0\in\bar{I}_\eta(\mu_0)$ and $\epsilon_0=\epsilon^l(t_0,l;\mu_0)$ be given so that C1 is satisfied and $(\partial\bar{h}_2/\partial t_0)(t_0,\epsilon_0;\mu_0)=0$. Then for ϵ_0 sufficiently small, generically, the manifolds undergo a topological saddle–node bifurcation along a p-dimensional surface, defined for $\epsilon\in(\epsilon_0-\Delta\epsilon,\epsilon_0+\Delta\epsilon)$, $\mu\in(\mu_0-\Delta\mu,\mu_0+\Delta\mu)$, $\Delta\epsilon=O(\epsilon_0^{1+\alpha})$, $\alpha<1/2$, and $\Delta\mu$ sufficiently small. The bifurcation surface of the manifolds intersections is $O(\epsilon_0^{1+\alpha},\epsilon_0\Delta\mu^2)$ close to the bifurcation surface of $\bar{h}_2(t_0,\epsilon_0)$.*

Remark: Similarly, one can consider the parameter dependence of cases **S2** and **S3** of Theorem 3.

III. THE WHISKER MAP

The motivation for choosing the function $\bar{h}_2(t_0,\epsilon_0)$ comes from combining the geometrical viewpoint of the structure of the manifolds introduced above and the analytical, perturbative approach, in which one looks at the change in energy along paths of the perturbed system.[18,19] Following Escande[20] geometrical interpretation of the Whisker map, we define the separatrix map, W, as the return map of the energy and time variables (H_n,τ_n) to the cross sections Σ_h and Σ_τ respectively, where Σ_h is composed of a segment of the x

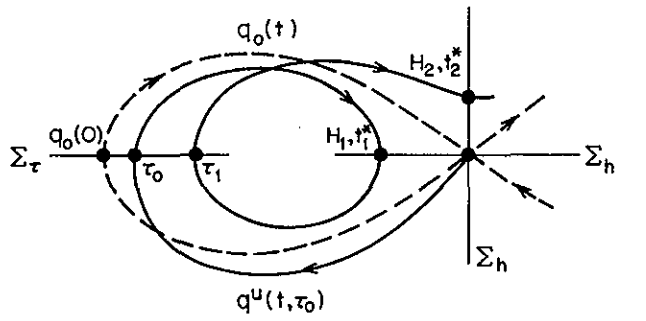

FIG. 5. The separatrix map. --- Unperturbed separatrices; —— an orbit belonging to the unstable manifold.

axis and a segment of the y axis, centered at the origin, and Σ_t consists of a segment of the x axis centered at $q^0(0)$ (see Fig. 5):

$$\begin{gathered} W:(H_n,\tau_n)\to(H_{n+1},\tau_{n+1}), \\ q(\tau_n,\tau_0)\in\Sigma_t, \\ H_n=H(q(t_n^*,\tau_0),t_n^*), \quad q(t_n^*,\tau_0)\in\Sigma_h, \\ \tau_{n-1}<t_n^*<\tau_n, \end{gathered} \tag{3.1}$$

where $q(t,\tau_0)$ is a solution to Eq. (2.1), $q(\tau_0,\tau_0)\in\Sigma_t$. In Fig. 5 we draw a solution $q^u(t,\tau_0)$ belonging to the unstable manifold so $\lim_{t\to-\infty}q^u(t,\tau_0)=(0,0)$.

In the neighborhood of the separatrix the cross sections Σ_h and Σ_t are transverse to the unperturbed trajectories. Therefore, for ϵ sufficiently small, the separatrix map is well defined there and so is the parametrization of the unstable manifold by τ_0. It follows from Eqs. (2.1) and (2.3) that $H_n \ll 1$ near the separatrix. The Whisker map is defined to be the leading-order approximation in ϵ and H to the separatrix map and can be calculated explicitly[19] as

$$h_{n+1}=h_n+\epsilon M(t_n), \quad t_{n+1}=t_n+P(h_{n+1}), \tag{3.2}$$

where $M(t)$ is the Melnikov function and $P(h)$ is the period of the unperturbed orbit with energy h at Σ_h. Assume for now that (3.1) is given identically by (3.2) and that as in the case of the unperturbed problem negative energies correspond to orbit passing to the left of the origin (intersect Σ_h with $y=0$), whereas orbits with positive energy pass above or below the origin (intersect Σ_h with $x=0$). Then it is natural to associate with the orbits which never reach Σ_h, namely the homoclinic orbits, zero energy. In Theorem 1 it is proven that this choice makes H_n a continuous function of t_0.

In terms of this formulation, a primary intersection point exists if there exist initial conditions (H_0,t_0) such that $H_0=H_1(t_0)=0$. Replacing (3.1) with (3.2) we obtain $h_1=\epsilon M(t_0)=0$. Hence, once the above assumptions are made rigorous, one may construct an alternative proof of Theorems 4.5.3 and 4.5.4 of Guckenheimer and Holmes.

Similarly, a secondary intersection point of transition number l exists if there are initial conditions (H_0,τ_0) such that $H_0=H_2=0$, $H_1<0$, $0<\tau_0<T$, and $lT<\tau_1<(l+1)T$. Replacing H_i with h_i and τ_i with t_i, setting $h_0=0$, and using (3.2), we obtain

$$\begin{gathered} h_1(t_0)=\epsilon M(t_0), \quad 0<t_0<T, \\ t_1(t_0,\epsilon)=t_0+P(\epsilon M(t_0)), \\ h_2(t_0,\epsilon)=\epsilon M(t_0)+\epsilon M(t_1(t_0,\epsilon))=\epsilon\bar{h}_2(t_0,\epsilon)=0. \end{gathered} \tag{3.3}$$

Hence, to prove Theorem 1, we need to prove that H_2 is continuous and differentiable, that simple zeros of H_2 imply transverse secondary homoclinic intersections, that $\bar{h}_2$ and t_1 are good approximations to H_2 and τ_1 and that $[t_1/T]=[\tau_1/T]$ for open intervals in ϵ. These assertions are proved in Appendices A and B. Moreover, since $M(t)$ is periodic, the solutions to (3.3) can be easily constructed and analyzed as is demonstrated in the proof of Theorem 2 in Appendix A.

In particular, the family of solutions described in Theorem 2 is simply given by

$$\begin{gathered} \epsilon^i(t_0,l)=\frac{1}{M(t_0)}P^{-1}(-t_0+M^{-1,i}(-M(t_0))+lT), \\ t_0\in\bar{I}_\eta, \end{gathered} \tag{3.4}$$

where $M(x)=-M(t_0)$ has $k=k(t_0)\geq 1$ solutions for $t_0\in\bar{I}_\eta=\{t_0|\bar{M}\geq -M(t_0)\geq\eta\}$, given by $x=M^{-1,i}(-M(t_0))\in(0,T)$. Moreover, by (1.2) $j(t_0,\epsilon^i(t_0,l))=l$, proving the first claim in the theorem. As for the primary intersection points, one can use the linearized structure near the origin to establish that the angle between the manifolds near the SIP at (t_0,ϵ_0) is proportional to $\epsilon_0[\partial\bar{h}_2(t_0,\epsilon_0)/\partial t_0]$:

$$\begin{aligned} \frac{\partial\bar{h}_2(t_0,\epsilon_0)}{\partial t_0}&=M'(t_0)+M'(t_1) \\ &\times[1+\epsilon M'(t_0)P'(\epsilon M(t_0))]. \end{aligned} \tag{3.5}$$

The type number of the homoclinic tangle for a given ϵ value is given by the minimal value of l for which there exists $t_0\in(0,T)$ such that $\epsilon^i(t_0,l)\leq\epsilon$. This number is fixed between the bifurcation values of (3.4), which can be found easily from (3.4) and (3.3) and the periodicity of $M(t)$. Hence the additional significance of Theorems 3 and 4 in supplying the topological bifurcation curves across which the type numbers may change.

Finally, using the separatrix map (3.2), we construct an nth-order Melnikov function and the associated $n-1$ dimensional transition vector $l=(l_0,\ldots,l_{n-2})$:

$$\begin{gathered} \bar{h}_n(t_0,\epsilon)=\sum_{i=0}^{n-1}M(t_i(t_0,\epsilon)), \\ t_0(t_0,\epsilon)=t_0, \quad 0<t_0<T, \\ t_i(t_0,\epsilon)=t_{i-1}(t_0,\epsilon)+P\left(\epsilon\sum_{j=0}^{i-1}M(t_j(t_0,\epsilon))\right), \\ i=1,\ldots,n-1, \\ l_i(t_0,\epsilon)=[t_{i+1}(t_0,\epsilon)/T], \quad i=0,\ldots,n-2. \end{gathered} \tag{3.6}$$

Similar theorems to Theorems 1–4, stating that simple zeros of $\bar{h}_n$ correspond to homoclinic intersections with rotary number n and with a vector type number l may be formulated (though the values of α and β change, see Sec. IV). For any finite n a similar proof to the $n=2$ case applies. Moreover, in Appendix A we present the outline of the proof that the corresponding function H_n is continuous for $n=o(1/\epsilon)$.

IV. SUMMARY AND DISCUSSION

Under assumptions **A0–A5** on the form of f and g, we define a function, $\bar{h}_2(t_0,\epsilon_0)$ by (1.1) and an integer number $j(t_0,\epsilon_0)$ by (1.2), and we prove the following results (the mathematical formulation of the assumptions and theorems is given in Sec. II):

(1) For sufficiently small ϵ, simple zeros in t of $\bar{h}_2(t,\epsilon)$ correspond to transverse secondary intersections of the manifolds with transition number $j(t,\epsilon)$.

(2) We construct families of solutions $\epsilon^i(t,l)$ $l=1,\ldots,\infty$, $i=1,\ldots,k(t)$ satisfying:
(i) $\bar{h}_2(t,\epsilon^i(t,l))=0$ and $j(t,\epsilon^i(t,l))=l$.
(ii) The solutions supply simple zeros in t of $\bar{h}_2$ on some nonempty open subintervals of $(0,T)$.
(iii) $\epsilon^i(t,l)\to 0$ as $l\to\infty$.

(3) For sufficiently large l, the existence of these solutions implies the existence of a secondary transverse intersection of transition number l. Moreover, given an $\hat{\epsilon}$ there exists a minimal value of l, denoted by l_0 such that there exists a t_0 solving $\bar{h}_2(t_0,\epsilon^i(t_0;l_0))=0$ and $\epsilon^i(t_0;l_0)<\hat{\epsilon}$. l_0 is the type number of the homoclinic tangle.

(4) We prove that under some generic conditions, degenerate zeros of $\bar{h}_2(t_0,\epsilon_0)$ correspond to tangencies or other homoclinic bifurcations of the manifolds. We find the scaling of the bifurcating solutions with ϵ from which we find the scaling of the angle of intersection there.

(5) We establish that these bifurcations occur on countable infinity sets of parameters, indexed by l.

(6) We prove the persistence of the secondary tangent homoclinic bifurcations when other parameters are introduced.

We foresee two interesting directions for continuation. The first is extending these ideas to higher dimensional systems. This study may reveal new phenomena as the mechanism for the creation of secondary tangencies is not well understood in more than two dimensions. Since the Melnikov technique has been already derived for higher dimensional systems,[7] at least for some cases the extension of the theory should be trivial, yet its consequences intriguing. Developing the theory for more general structures is related to the nontrivial extension of the transport theory to higher dimensions and is challenging. Recently, Haller and Wiggins[21] and Kaper and Kovacic[22] have developed criteria for proving the existence of multihump homoclinic orbits near hyperbolic-resonant two degrees of freedom Hamiltonian systems, spending time of, respectively, order $\log(\epsilon)$ and order $1/\sqrt{\epsilon}$, near the slow manifold on which the resonance occurs. The orbits detected in this paper spend $O(\log\epsilon)$ near the fixed point, and their existence in this two degrees of freedom setting, and in particular the analogous transition number is yet to be revealed.

The second direction is investigating the implications and relations of the SMF applied to slightly dissipative systems with other phenomena:

(a) *Formation of strange attracting sets and density variation on them:* In two dimensions the mechanism for the collapse of phase space into the strange attracting set must occur through the motion of the lobes.[15] Detailed study of the lobe properties, such as boundary length and changes in the lobe width as the lobe is iterated corresponds to a systematic study of the transient behavior. The attractor is obtained by considering the lobe images after an infinite number of iterations. Such a study involves the delicate issue of asymptotic behavior of the manifolds. Generalizing the corresponding transport theory, the TAM,[14] to the non-area-preserving case will enable us to predict the rate at which these lobes collapse. It will be interesting to relate this study to the recent theory regarding existence and "robustness" of strange attractors near homoclinic tangencies[9] and the construction of SRB measures on the attractors on one hand, and to the physicist characterization of density distributions on attractors on the other hand.[23] For example, it seems that the density variations on the attractor are related to the accumulation rate of initial conditions along the unstable manifolds. Incorporating the finite time results, the magnitude of the derivative of the nth crossing time and its scaling near bifurcations with the TAM may give some information regarding this accumulation rate. If such a relation can be established, it may give a method for estimating the invariant measure and $f(\alpha)$ on the attractors and may be valuable in improving numerical methods for computing the manifolds.

(b) *Periodic orbits, pruning theory, and the SMF:* The pruning theory has been developed to describe the growth rate of the number of periodic orbits in dissipative systems using an approximate symbolic dynamics.[17] This symbolic dynamics is changing when secondary tangent bifurcations occur. Moreover, new invariant sets associated with the new transverse secondary intersections appear, and their rotation sets may be analyzed.[11] Therefore, the combination of the three theories may result in predictions for the changes in the pruning fronts for slightly dissipative systems.

(c) *Higher rotary number intersections:* Using the Whisker map as in (3.3), we can construct a third, fourth, and nth Melnikov function for detecting higher-order tangencies. Each of these will have infinitely many degenerate zeros. Proving that these degeneracies imply tangencies of the manifolds and measuring the set of all these bifurcation values is, in some sense, trying to prove the Newhouse result[8] (that tangencies persist) in a constructive fashion. This plan is very difficult to follow, and it might be impossible to complete because the error grows with the rotary number of the intersection (see conjectures A.1 and A.2). Nonetheless, such an investigation, and especially finding the rate at which the errors grow, may lead to better understanding of the development of the homoclinic tangles.

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APPENDIX A: TRANSVERSE SECONDARY HOMOCLINIC INTERSECTIONS

1. Proof of Theorem 1

The proof consists of three parts. In the first part we construct a function $H_2(t_0,\epsilon)$ and prove that it is continuous and differentiable and that its zeros correspond to secondary homoclinic points. In the second part we calculate the error in approximating $H_2(t_0,\epsilon)$ by $\epsilon\bar{h}_2(t_0,\epsilon_0)$. In the third part we establish that simple zeros of $\bar{h}_2(t_0,\epsilon_0)$ imply simple zeros of $H_2(\tilde{t},\tilde{\epsilon})$ for $\tilde{\epsilon}$ in the indicated interval. The condition C2 is used only in proving the third part of the theorem. Hence, we may use all the other results in proving Theorems 3 and 4.

A. Properties of $H_2(t_0,\epsilon)$

Strategy: *First, we define a three-dimensional region D in the extended phase space [(x,y,t) coordinate system] and establish that we can distinguish between trapped and untrapped orbits by using the energy function $H(q_\epsilon^u(t,t_0))$ measured on the cross section Σ_h. In Lemmas A.1 and A.2 we establish that this determines a function $H_i(x,y,t,\epsilon)$ for which positive/negative values imply untrapped/trapped orbits, revolving i times in the trapping region till time t. We identify $H_0=0$ with orbits which asymptote the origin as $t\to-\infty$, thus parametrizing the unstable manifold by $t_0=\tau_0$, the orbit's first crossing time of Σ_t. Then we define the functions $H_i(\tau_0,\epsilon)$ for these orbits. In Lemmas A.3 and A.4 we prove that for i=1,2 these become continuous functions of their arguments if we identify the zeros of the function H_i, $i>0$, with orbits which asymptote the origin as $t\to\infty$ and which cross Σ_t for the ith and last time at $t=\tau_i$ (i.e., these orbits do not reach Σ_h for the ith time in any finite time). We conclude that if we find orbits for which $H_i(\tau_0,\epsilon)$ changes sign as a function of τ_0 the unstable manifold crosses topologically transversely the stable manifold in an i-rotary intersection point [since we do not know whether $H_i(\tau_0)$ is C^1, we cannot verify actual transversality].*

We define the region D as shown in Fig. 6. Let Σ^θ denote the Poincaré section in t:

$$\Sigma^\theta=\{(x,y,t)|t=\theta \bmod T\}.$$

Let $u(\theta)$ denote the point of first intersection in elapsed time of W_+^u with Σ_t in Σ^θ. Similarly, let $s(\theta)$ denote the point of first intersection in elapsed time of W_+^s with Σ_t in Σ^θ [$s(0)=u(0)=p_0$ in Σ^0 of Fig. 6 since $M(0)=0$]. Let $D(\theta)$ denote the interior of the region enclosed by the segments of the stable and unstable manifolds extending from the fixed point to $s(\theta)$ and $u(\theta)$, respectively, and the line $I(u(\theta),s(\theta))$, the closed interval connecting $u(\theta)$ with $s(\theta)$ along Σ_t. Then we define

$$D=\{(x,y,t)|(x,y)\in D(\theta),\theta=t \bmod T\}. \tag{A1}$$

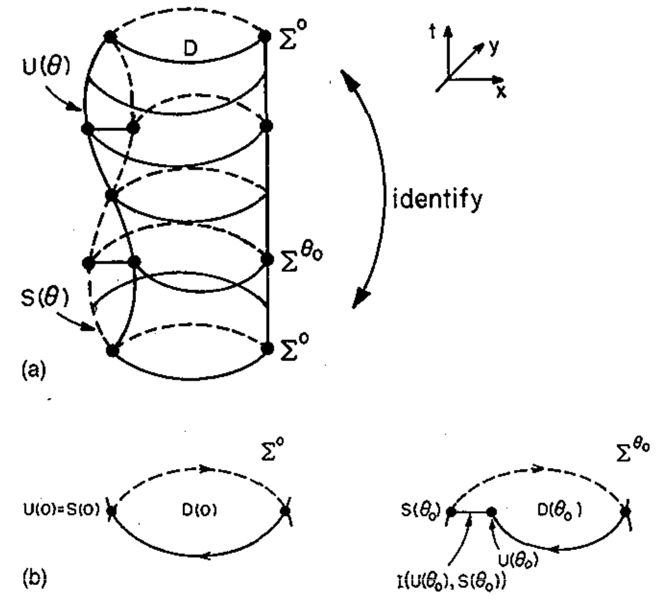

FIG. 6. Definition of the region D. The region D and two two-dimensional cross sections.

We say that an orbit is trapped at time t if $(q,t)\in D$ and that it is untrapped at time t if $(q,t)\notin\bar{D}$ [if (q,t) belongs to ∂D it is neither trapped nor untrapped].

Let us fix the parametrization of the unperturbed homoclinic orbit so that $q_0(0)\in\Sigma_t$. Let $q_\epsilon^u(t,\tau)$ denote a solution belonging to W_+^u. For $\hat{\epsilon}$ sufficiently small,[1,2]

$$q_\epsilon^u(t,\tau)=q_0(t-\tau)+\epsilon q_1^u(t,\tau)+O(\epsilon^2),$$
$$t\in(-\infty,0],\quad \epsilon<\hat{\epsilon}. \tag{A2}$$

Moreover, choosing $\hat{\epsilon}$ sufficiently small, we can parametrize $q_\epsilon^u(t,\tau)$ by t_0, the first intersection time of $q_\epsilon^u(t,\tau)$ with Σ_t, i.e., $q_\epsilon^u(t_0,t_0)\in\Sigma_{t_0}$. $q_\epsilon^s(t,\tau)$ can be expressed similarly to (A2) for $t\in[0,\infty)$.

Lemma A.1: For all finite t_0, $H(q_\epsilon^s(t,t_0))\to0$ as $t\to\infty$ and $H(q_\epsilon^u(t,t_0))\to0$ as $t\to-\infty$.

Proof. By assumptions **A1** and **A4** $H(0,0)=0$ and $q_\epsilon^{s,u}(t)\to(0,0)$ as $t\to\pm\infty$ (respectively). Hence by continuity $H(q^{s,u}\epsilon(t))\to0$ as $t\to\pm\infty$ as indicated. □

Lemma A.2: Let $q(t)=q(t,\tau_{i-1})$ denote a solution of (2.1) satisfying $|q(\tau_{i-1},\tau_{i-1})-q_0(0)|=O(\sigma)$ for some small parameter σ. Then for σ and ϵ sufficiently small the orbit $q(t)$ is trapped for $t\in(\tau_{i-1},\tau_i)$ if and only if $H_i\equiv H(q(t_i^*,\tau_{i-1}))<0$ and $t_i^*<\infty$, where $t_i^*\in(\tau_{i-1},\tau_i)$ denotes the crossing time of Σ_h by $q(t,\tau_{i-1})$. Similarly, $q(t,\tau_{i-1})$ is untrapped for $t\in(\tau_{i-1},\tau_i)$ iff $H_i>0$ and $t_i^*<\infty$; τ_i may be infinite in this case.

Proof. ⇒: Let N_δ be a neighborhood of the origin for which the $O(3)$ terms of $H(x,y)$ are negligible and in particular $H(x,0)<0$, $H(0,y)>0$, $(x,y)\in N_\delta$. For σ sufficiently small $q(t_i^*)\in N_\delta$, hence, by **A1**, $H_i<0$ and $t_i^*<\infty$ imply $y(t_i^*)=0$, $x(t_i^*)=\sqrt{-2H_i}$, hence, for ϵ sufficiently small $(x(t_i^*),y(t_i^*),t_i^*)\in D$. By the construction of D and by uniqueness of solutions of ODEs, an orbit may enter or exit the interior of D only through the segment $I(s(t),u(t))$. This may occur, by definition, at $t=\tau_{i-1}$ and $t=\tau_i$, respectively. Similarly, if $H_i>0$ and t_i^* is finite, then $(x(t_i^*), y(t_i^*),t_i^*)\notin \bar{D}$. By the same argument $(x(t),y(t),t)$ remains out of $\bar{D}$ until it reaches the cross section Σ_t.

$\Leftarrow$: Assume $q(t)$ is trapped $[q(t)\in D]$ for $t\in(\tau_{i-1},\tau_i)$. Choose $\hat{\epsilon}$ sufficiently small so that Σ_h is crossed by all solutions entering $[N_\delta\backslash(W^s\cap N_\delta)]$ for all $\epsilon<\hat{\epsilon}$. For sufficiently small σ and ϵ, $q(t)$ enters N_δ in finite time (here σ may be ϵ dependent). Since, $q(t)\notin \partial D$, $q(t)$ crosses Σ_h at finite time, t_i^*. By **A1** and **A4**, for ϵ sufficiently small, the left horizontal part of Σ_h is completely contained in D and the perpendicular part of Σ_h is completely contained in $\bar{D}^c$ (the complement of $\bar{D}$). Therefore, for sufficiently small ϵ, the result follows immediately from **A1**. □

Let $\hat{\epsilon}$ be sufficiently small for Lemmas A.1 and A.2 to hold for all $\epsilon<\hat{\epsilon}$. Define the function:

$$H_1(t_0,\epsilon):([0,T),[0,\hat{\epsilon}])\to R,$$

$$H_1(t_0,\epsilon)=\begin{cases}0, & \text{if } q_\epsilon^u(t_0,t_0)=q_\epsilon^s(t_0,t_0);\\ H(q_\epsilon^u(t_1^*,t_0)), & \text{otherwise.}\end{cases} \tag{A3}$$

Lemma A.3: For sufficiently small ϵ, $H_1(t_0,\epsilon)$ is continuous *in* (t_0,ϵ), *differentiable in* t_0 *for all* $\epsilon>0$, *and* $H_1(t_0,\epsilon)=O(\epsilon)$.

Proof: First we show that $H_1(t_0,\epsilon)$ is continuous in the limit $\epsilon\to 0$. From (A2) it follows that $q_\epsilon^u(t_0,t_0)$ is ϵ-close to $q_\epsilon^s(t_0,t_0)$, hence $q_\epsilon^u(t,t_0)$ remains ϵ-close to $q_\epsilon^s(t,t_0)$ for a finite interval of time, in which it enters a neighborhood of the origin where linearized equations govern. It follows from the form of the linearized equations (see proofs of Lemmas A.6–A.8 in Appendix B for more details) that $|q_\epsilon^u(t_1^*,t_0)|=O(\sqrt{\epsilon})$ and therefore that $H_1(t_0,\epsilon)=O(\epsilon)$, and in particular $H_1(t_0,\epsilon)\to 0$ as $\epsilon\to 0$ for all t_0. From (A3) it follows that $H_1(t_0,0)\equiv 0$, hence $H_1(t_0,\epsilon)$ is continuous in this limit. Consider $\epsilon>0$. For (t_0,ϵ) values for which $H_1(t_0,\epsilon)$ is either strictly positive or strictly negative, t_1^* is finite and it depends smoothly on t_0 and analytically on ϵ. Using (A2) and Lemma A.2 we conclude that $H_1(t_0,\epsilon)$ is a C^r function in t_0 and analytic in ϵ for orbits which are either trapped or untrapped for $t\in(t_0,\tau_1)$. Since $q_\epsilon^s(t,t_0)\in\partial D$ for $t\geqslant t_0$, it follows from Lemmas A.1 and A.2 that the definition (A3) of $H_1(t_0,\epsilon)$ on $\partial D\cap W^s$ makes $H_1(t_0,\epsilon)$ continuous across ∂D. Moreover, using the form of H and the form of the solutions in the neighborhood of the fixed point, it follows that $H_1(t_0,\epsilon)$ is C^1 in t_0 for all $\epsilon>0$. □

Similarly, we may define $H_2(t_0,\epsilon)$, or in general $H_n(t_0,\epsilon)$. For open flows, these are defined only on the partial segments $I_n(\epsilon)\subseteq[0,T)$ for which H_{n-1} is negative:

$$I_1(\epsilon)=[0,T),$$

$$I_n(\epsilon)=\{t_0|t_0\in I_{n-1}(\epsilon) \text{ and } H_{n-1}(t_0,\epsilon)<0\},\quad n\geqslant 2,$$

and

$$H_n:(I_n(\epsilon),(0,\hat{\epsilon}))\to R,$$

$$H_n(t_0,\epsilon)=\begin{cases}0, & \text{if } q_\epsilon^u(\tau_{n-1},t_0)=q_\epsilon^s(\tau_{n-1},\tau_{n-1});\\ H(q_\epsilon^u(t_n^*,t_0)), & \text{otherwise.}\end{cases}$$

Lemma A.4: For sufficiently small ϵ, $H_2(t_0,\epsilon)$ is continuous in (t_0,ϵ) and differentiable in t_0 for $\epsilon>0$.

Proof: We proved in Lemma A.3 that $H_1(t_0,\epsilon)$ is continuous in its arguments, hence it is negative in a neighborhood of (t_0,ϵ) values for which $H_2(t_0,\epsilon)$ is defined, and $H_2(t_0,\epsilon)$ is defined on such open sets (therefore the case $\epsilon=0$ is excluded). By Lemma A.3 $H_1(t_0,\epsilon)=O(\epsilon)$, hence, $q_\epsilon^u(t_1^*,t_0)$ is $\sqrt{\epsilon}$-close to the origin and therefore $q_\epsilon^u(\tau_1,t_0)$ is ϵ-close to $q_0(t-\tau_1)$. Therefore, by Lemma A.2, the sign of $H_2(t_0,\epsilon)$ *determines whether* $q_\epsilon^u(t,t_0)$ *is trapped/untrapped for* $t\in(\tau_1,\tau_2)$. It follows that t_2^*, the second crossing time of Σ_h, is smooth for the trapped and untrapped orbits, hence $H_2(t_0,\epsilon)$ is continuous and differentiable in t_0 whenever it is strictly negative or strictly positive. If the orbit is neither trapped nor untrapped, then, since $q_\epsilon^u(\tau_1,t_0)\in \mathrm{I}\ (u(\tau_1),s(\tau_1))$, it follows that $q_\epsilon^u(\tau_1,t_0)$ must belong to the stable manifold, namely $q_\epsilon^u(\tau_1,t_0)=q_\epsilon^s(\tau_1,\tau_1)$ and then $H_2(t_0,\epsilon)=0$, as required for continuity. Using the smooth dependence on initial conditions up to τ_1, the form of H, and the form of the solutions in the neighborhood of the fixed point for $t\to t_2^*$, it follows that $H_2(t_0,\epsilon)$ is C^1 in t_0 for all $\epsilon>0$ also when $t_2^*\to\infty$.

Lemma A.5: For sufficiently small $\epsilon>0$ and for all $t_0\in I_2(\epsilon)$, $q_\epsilon^u(t,t_0)$ is a secondary homoclinic orbit if and only if $H_2(t_0,\epsilon)=0$, and is transverse if and only if $\partial H_2(t,\epsilon)/\partial t$ is bounded away from zero at t_0.

Proof: $\Rightarrow$: If the orbit is a secondary homoclinic orbit then there exists a τ_1 such that $q_\epsilon^u(\tau_1,t_0)=q_\epsilon^s(\tau_1,\tau_1)$, hence, by definition, $H_2(t_0,\epsilon)=0$. If the intersection is transverse, then, it remains transverse up to a neighborhood of the origin, where the analysis of the linear equations show that this implies that $\partial H_2/\partial t$ must be nonvanishing.

$\Leftarrow$: Let $H_2(t_0,\epsilon)=0$ and assume $q_\epsilon^u(t,t_0)$ is not a secondary homoclinic orbit. Since $t_0\in I_2(\epsilon)$, it follows that t_1^* is finite and that $q_\epsilon^u(\tau_1,t_0)$ is ϵ-close to $q_\epsilon^s(\tau_1,\tau_1)$. Hence, since $q_\epsilon^u(\tau_1,t_0)$ is not a SIP, for ϵ sufficiently small there exists a finite t_2^* such that $q_\epsilon^u(t_2^*,t_0)\in\Sigma_h\cap N_\delta$. It follows from **A1** that $H_2(t_0,\epsilon)$ must be either positive or negative—contradicting our assumption. Recall that t_0 parametrizes the unstable manifold. Since $\partial H_2/\partial t$ is nonvanishing $H_2(t_0,\epsilon)$ must change sign at t_0. By Lemma A.2, $H_2(t,\epsilon)$ changes sign at t_0 iff orbits with $H_2(t,\epsilon)<0$ are trapped and the ones with $H_2(t,\epsilon)>0$ are untrapped. Hence, by continuity, the intersection is topologically transverse, and by using the linearized structure, it can be proven to be transverse. □

Therefore, there exists a SIP of transition number l, given by $q_\epsilon^u(t,t_0)$ if and only if $H_1(t_0,\epsilon)<0$, $lT\leqslant\tau_1(t_0,\epsilon)<(l+1)T$ and $H_2(t_0,\epsilon)=0$.

Though not necessary for our current work, it is interesting to generalize the above results to $H_n(t_0,\epsilon)$ for any $n\geqslant 2$ (see Sec. VI).

Conjecture A.1: For sufficiently small $\hat{\epsilon}$ and $n=o(1/\hat{\epsilon})$, $H_n(t_0,\epsilon)$ is continuous in (t_0,ϵ) for all $t_0\in I_n(\epsilon)$, $0<\epsilon<\hat{\epsilon}$.

Proof outline: By induction on n. Assume $H_n(t_0,\epsilon)$, $n\geqslant 2$ is continuous in (t_0,ϵ) and $|H_n|=O(n\epsilon)$. From Lemmas A.4 and A.8 these assumptions hold for $n=2$. It implies that $H_n(t_0,\epsilon)=o(1)\equiv\sigma^2$ for all $\epsilon<\hat{\epsilon}$ and we may repeat the same argument as in Lemma A.4 for $q_\epsilon^u(\tau_n,t_0)$ to prove that H_{n+1} is continuous. Using the Melnikov integral along

$q_0(t-\tau_n)$ it follows that (see Lemmas A.6–A.8) $H_{n+1}=H_n+\epsilon M(\tau_n)+O(|H_n|^{3/2})=O[(n+1)\epsilon]$ and the induction assumption is established. □

Conjecture A.2: Let $n=o(1/\hat{\epsilon})$ and $\hat{\epsilon}$ sufficiently small. Then for all $\epsilon\in(0,\hat{\epsilon})$, $i\leqslant n$ and $t_0\in I_i(\epsilon)$ $q_\epsilon^u(t,t_0)$ is an ith homoclinic orbit if and only if $H_i(t_0,\epsilon)=0$ and is topologically transverse if and only if $H_i(t,\epsilon)$ changes sign as a function of t at t_0.

Proof outline: Repeat arguments of Lemma A.5, using Conjecture A.1.

B. $H_2(t_0,\epsilon)$ may be approximated by $\epsilon_0\bar{h}_2(t_0,\epsilon_0)$

We prove consequently that H_1, τ_1, and H_2 are well approximated by h_1, t_1, and h_2 of (3.3). As the proofs of this part are quite technical and long, we quote the lemmas, give the essential ideas of the proofs, and present the details in Appendix B.

In Lemmas A.6–A.8, let (2.1) satisfy assumptions **(A0)**–**(A5)**, and let t_0 satisfy $M(t_0)\leqslant-\eta<0$.

Lemma A.6: $H_1(t_0,\epsilon)=h_1(t_0)+O[|h_1(t_0)|^{3/2},\epsilon|h_1|,\epsilon^2]$, where h_1 is given by Eq. (3.3).

Proof: We approximate the change in energy by the integral of $f\wedge g$ evaluated along $q_0(t-t_0)$. The expansion in ϵ gives an $O(\epsilon^2)$ error. Since $q_\epsilon^u(t_1^*,t_0)\neq(0,0)$ we obtain a boundary error, which is proportional to $(\sqrt{h_1})^p$, where p is the order of $H(x,y)$ for small (x,y) along the stable manifold. By our assumptions, $p\geqslant3$, and it gives the above error estimates. For the forced pendulum, for example, $p=4$, and the error is $O(\epsilon^2)$. □

Lemma A.7: $\tau_1(t_0)=t_1+O(\epsilon,\sqrt{-h_1},\epsilon^2/|h_1|)$, where h_1, $t_1(t_0)$ is given by Eq. (3.3).

Proof: The energy function measured on Σ_h fixes the distance to the fixed point, hence it fixes the time of passage near it (this passage time is the dominant term in $\tau_1-\tau_0$). Choosing to approximate the perturbed orbit by the "correct" unperturbed periodic orbit—the one which has the same passage time, we obtain the above error estimates. □

Lemma A.8: $H_2(t_0,\epsilon)=h_2(t_0)+O(|h_1|^{3/2},\epsilon|h_1|,\epsilon^2,\epsilon|h_2|,|h_2|^{3/2},\epsilon\sqrt{-h_1},\epsilon^3/|h_1|)$, where $h_1(t_0,\epsilon)$, $h_2(t_0,\epsilon)$ are given by Eq. (3.3).

Proof: We approximate the change in energy between H_1 and H_2 by the integral of $f\wedge g$ evaluated along the unperturbed homoclinic orbit $q_0(t-\tau_1)$, using the previous estimates of τ_1 and H_1. As in Lemma A.6, there are contributions to the error from the usual perturbation terms and from the boundary terms, which can be smaller than indicated if H has no $O(3)$ terms in the stable direction near the origin, see Appendix B for details.

Using (3.3) we conclude the following:

Corollary A.1: Let (2.1) satisfy **A0–A5** and let $t_0\in\{t|M(t)\leqslant-\eta<0\}$. Then

$$H_2(t_0,\epsilon)=h_2(t_0,\epsilon)+O(\epsilon^{3/2})=\epsilon[\bar{h}_2(t_0,\epsilon)+O(\epsilon^{1/2})], \tag{A4}$$

and

$$\frac{\partial H_2(t_0,\epsilon)}{\partial t_0}=\epsilon\left(\frac{\partial\bar{h}_2(t_0,\epsilon)}{\partial t_0}+O(\epsilon^{1/2})\right). \tag{A5}$$

The second equation follows from the first as the partial derivatives w.r.t. t_0 of the error terms listed in Lemma A.8 are still of $O(\epsilon^{3/2})$. Therefore, using the linear structure near the origin, as in the proofs of Lemmas A.6–A.8, one may prove that the angle between the manifolds at the intersection point (t_0,ϵ_0) is proportional to $\tilde{\epsilon}[\partial\bar{h}_2(\tilde{t},\tilde{\epsilon})/\partial\tilde{t}]$.

C. Simple zeros of $\bar{h}_2(t_0,\epsilon_0)$ imply simple zeros of $H_2(t,\epsilon)$

We prove that if (t_0,ϵ_0) satisfy C1 and C2, then given $0<\beta<\alpha<1/2$, for $\epsilon_0<\hat{\epsilon}(\alpha,\beta)$, $H_2(t_0\pm\Delta t,\tilde{\epsilon})$ changes sign for $\Delta t=O(\epsilon_0^\beta)$ for all $\tilde{\epsilon}\in[\epsilon_0-\Delta\epsilon,\epsilon_0+\Delta\epsilon]$ where $\Delta\epsilon=O(\epsilon_0^{1+\alpha})$. Hence, by continuity of $H_2(t_0,\epsilon)$ (Lemma A.4), there exists $\tilde{t}\in(t_0-\Delta t,t_0+\Delta t)$ such that $H_2(\tilde{t},\tilde{\epsilon})=0$, and the zero is topologically simple. Therefore, by Lemma A.5 the manifolds intersect topologically transversely at a SIP, parametrized by $(\tilde{t},\tilde{\epsilon})$. By Corollary A.1 it follows that $\partial H_2/\partial t_0$ is bounded away from zero, hence the intersection is transverse. Then, we show that for sufficiently small ϵ_0 the transition number is estimated correctly: $l(\tilde{t},\tilde{\epsilon})=l(t_0,\epsilon_0)$.

Note that it is impossible to simply apply the implicit function theorem to H_2/ϵ since $\bar{h}_2(t,\epsilon)$ is singular at $\epsilon=0$. Nonetheless, using (A4), it is easy to prove that H_2 must change sign as indicated; let $\Delta t=K\epsilon_0^\beta$, $K>0$ and $\Delta\epsilon=\epsilon_0^{1+\alpha}$. Then

$$\begin{aligned}H_2(t_0\pm\Delta t,\epsilon_0+\Delta\epsilon)&=h_2(t_0\pm K\epsilon_0^\beta,\epsilon_0+\Delta\epsilon)+O[(\epsilon_0+\Delta\epsilon)^{3/2}]\\&=\pm K\epsilon_0^{1+\beta}\frac{\partial\bar{h}_2}{\partial t_0}(t_0,\epsilon_0)+O(\epsilon_0^{1+\alpha},\epsilon_0^{1+2\beta},\epsilon_0^{3/2}).\end{aligned}$$

In the error estimates above and hereafter, we use the following property of $\bar{h}_2(t_0,\epsilon_0)$, which follows from (1.1) and assumption **A3**: taking derivatives of $\bar{h}_2(t_0,\epsilon_0)$ (or its derivatives) w.r.t. t_0 does not change its order of magnitude, whereas taking a derivative of $\bar{h}_2(t_0,\epsilon_0)$ (or its derivatives) w.r.t. ϵ multiplies the function by a factor of order $1/\epsilon$. Since, by C2, $(\partial\bar{h}_2/\partial t_0)(t_0,\epsilon_0)$ is bounded away from zero, we obtain that the first term is dominant for ϵ_0 sufficiently small, and $H_2(t_0\pm\Delta t,\epsilon_0+\Delta\epsilon)$ has opposites signs. By Corollary A.1 we conclude that $\partial H_2/\partial t$ is bounded away from zero in the indicated intervals hence that there exists a transverse secondary homoclinic intersection at $(t,\epsilon)=(\tilde{t},\tilde{\epsilon})$ for all $\tilde{\epsilon}\in(\epsilon_0-\Delta\epsilon,\epsilon_0+\Delta\epsilon)$ with a $\tilde{t}(\tilde{\epsilon})\in(t_0-\Delta t,t_0+\Delta t)$. We verify that the transition number of this SIP is $l(t_0,\epsilon_0)$; Using Lemma A.7 we find

$$\begin{aligned}\tau_1(\tilde{t},\tilde{\epsilon})&=t_1(\tilde{t},\tilde{\epsilon})+O\left(\tilde{\epsilon},\sqrt{-\tilde{\epsilon}M(\tilde{t})},\frac{\tilde{\epsilon}}{M(\tilde{t})}\right)\\&=t_1(t_0,\epsilon_0)+O(\epsilon_0^\alpha,\epsilon_0^\beta,\sqrt{\epsilon_0}).\end{aligned}$$

In the last equality we used **A3** and the assumption that $M(t_0)$ is bounded away from zero. If $t_1(t_0,\epsilon_0)$ is bounded away from lT and $(l+1)T$ as $\epsilon\to0^+$, then, for ϵ_0 sufficiently small $\tau_1(\tilde{t},\tilde{\epsilon})\in[lT,(l+1)T]$. Indeed $t_1(t_0,\epsilon_0)$ must be bounded away from jT for any integer j: assume it is not, namely, there exists a function $\theta(t_0;\epsilon_0)$ such that

$$t_1(t_0,\epsilon_0)=jT+\theta(t_0;\epsilon_0), \quad \lim_{\epsilon_0\to 0^+}\theta(t_0;\epsilon_0)\to 0.$$

Then, by **A5**,

$$\bar{h}_2(t_0,\epsilon_0)=M(t_0)+M(t_1)=M(t_0)+O[\theta(t_0;\epsilon_0)],$$

but we assumed $M(t_0)\leqslant-\eta<0$ and $\bar{h}_2(t_0,\epsilon_0)=0$ for all (t_0,ϵ_0) satisfying C1 and C2. Hence we conclude that for such (t_0,ϵ_0), $t_1(t_0,\epsilon_0)$ must be bounded away from jT for all j. □□□

2. Proof of Theorem 2

This theorem is composed of statements regarding the behavior of $\bar{h}_2(t_0,\epsilon)$ which are easily verified. By assumption **A5** and the continuity of $M(t)$, for $\eta\in(0,\bar{M}]$ (2.6), the set

$$\bar{I}_\eta=\{t_0|\bar{M}\geqslant -M(t_0)\geqslant\eta\} \tag{A6}$$

is closed and nonempty. For all $t_0\in\bar{I}_\eta$ the equation $M(x)=-M(t_0)$ has $k=k(t_0)\geqslant 1$ solutions $x=M^{-1,i}(-M(t_0))\in(0,T)$. Therefore, given an l and an $\eta\in(0,\bar{M}]$, we can solve $\bar{h}_2(t_0,\epsilon_0)=0$ for all $t_0\in\bar{I}_\eta$ as follows:

$$\begin{aligned} t_1^i(t_0,l)&=M^{-1,i}(-M(t_0))+lT,\\ \epsilon^i(t_0,l)&=\frac{1}{M(t_0)}P^{-1}[-t_0+M^{-1,i}(-M(t_0))+lT]\\ &=\frac{1}{M(t_0)}P^{-1}[-t_0+t_1^i(t_0,l)]. \end{aligned} \tag{A7}$$

By (1.2) $j(t_0,\epsilon^i(t_0,l))=l$, proving the first claim in the theorem.

From (A7) it follows that if $t_0\in\bar{I}_\eta$ then $\epsilon^i(t_0,l)\in[\epsilon^i_{\min}(l),\epsilon^i_{\max}(l,\eta)]$, where

$$\epsilon^i_{\min}(l)=\min_{t_0\in I_0+}\epsilon^i(t_0,l)\propto P^{-1}(lT),$$

$$\epsilon^i_{\max}(l,\eta)=\max_{t_0\in\bar{I}_\eta}\epsilon^i(t_0,l)\propto\frac{1}{\eta}P^{-1}(lT).$$

Hence, for sufficiently large $l=l(\eta,\hat{\epsilon})$ the condition $\epsilon^i(t_0,l)<\hat{\epsilon}$ is satisfied for all $t_0\in\bar{I}_\eta$. Let $l_0=l_0(\hat{\epsilon})$ be the minimal value of l for which $\epsilon^i_{\min}(l)<\hat{\epsilon}$. It follows that there exists an $\hat{\eta}=\eta(l_0,\hat{\epsilon})$ for which $\epsilon^i(t_0,l)<\hat{\epsilon}$ for all $t_0\in\bar{I}_{\hat{\eta}}$ and for all $l\geqslant l_0$, hence condition C1 is satisfied there.

Now, we show that there are closed intervals in $\bar{I}_{\hat{\eta}}$ for which condition C2 is satisfied. It follows from (3.5) that $(\partial\bar{h}_2/\partial t_0)(t_0,\epsilon_0)=0$ in one of two cases;

case 1: $\quad M'(t_0)=M'(t_1)=0,$

case 2: $\quad M'(t_0)\neq 0 \quad$ and $M'(t_1)\neq 0.$

Using Eq. (A7) we find

$$\begin{aligned} M'(t_1^i(t_0,l))&=M'[M^{-1,i}(-M(t_0))+lT]\\ &=M'[M^{-1,i}(-M(t_0))]. \end{aligned} \tag{A8}$$

Namely $M'(t_1^i)$ depends only on t_0 and i (and not on ϵ_0 nor on l). Therefore, by assumption (**A5**) on $M(t)$, it may vanish only on a finite set of values of $t_0\in\bar{I}_{\hat{\eta}}$. Let us define the closed intervals $I_{\eta',\hat{\eta}}\subseteq\bar{I}_{\hat{\eta}}$ on which *additionally* $|M'(t_1^i(t_0,l))|\geqslant\eta'>0$ (so case 1 is excluded on $I_{\eta',\hat{\eta}}$). Consider $t_0\in I_{\eta',\hat{\eta}}$, then $(\partial\bar{h}_2/\partial t_0)(t_0,\epsilon)=0$ implies

$$M'(t_1^i(t_0,l))=\frac{-M'(t_0)}{1+\epsilon M'(t_0)P'(\epsilon M(t_0))}, \tag{A9}$$

which, with (A8) and **A3** gives an equation for t_0:

$$M'[M^{-1,i}(-M(t_0))]=\frac{-M'(t_0)}{1+C(1)[M'(t_0)/M(t_0)]+o(\epsilon)}. \tag{A10}$$

Equation (A10) may have a finite number of solutions, $t_s^0\in I_{\eta'}$ for $\epsilon=0$. Then $(\partial\bar{h}_2/\partial t_0)(t_0,\epsilon)$ may vanish on the (finite) set of points $t_s=t_s^0+o(\epsilon)$. Excluding open intervals of order $o(\hat{\epsilon})$ around each of the solutions t_s^0, we obtain a set of closed intervals $\hat{I}\subseteq I_{\eta',\hat{\eta}}$ such that for all $t_0\in\hat{I}$ and $\epsilon^i(t_0,l)$ conditions C1 and C2 are satisfied and $\epsilon^i(t_0,l)<\hat{\epsilon}$ for all $l\geqslant l_0$, proving the second claim of the theorem.

From the construction of the intervals $\hat{I}$ and $I_{\eta',\hat{\eta}}$ it is clear that C2 may be violated on the countable infinity set of values $[t_0,\epsilon^i(t_0,l)]$, $l=1,\ldots,\infty$ if $M'(t_0)=M'(t_1^i)=0$ [specifically, if $\max M(t)=-\min M(t)$ such a value exists]. Similarly, if there exists a solution t_0 for (A10) with $\epsilon=0$, then, generically, for $\tilde{l}$ sufficiently large, there exist solutions $[\tilde{t}_0(l),\epsilon^i(\tilde{t}_0(l),l)]$ of (A10) for all $l>\tilde{l}$, thus violating C2 on a *countable infinity set* of values. We suspect that for any periodic function $M(t)$ which satisfies **A5** Eq. (A10) must have solutions when $\epsilon=0$. □□□.

APPENDIX B: PROOFS OF LEMMAS A.6–A.8

In Lemmas A.6–A.8, let (2.1) satisfy assumption **A0–A5**, and let t_0 satisfy $M(t_0)\leqslant-\eta<0$.

Lemma A.6: $H_1(t_0,\epsilon)=h_1(t_0)+O[|h_1(t_0)|^{3/2},\epsilon|h_1|,\epsilon^2]$, where h_1 is given by Eq. (3.3).

Proof: Using the same arguments as in the derivation of the Melnikov function (see Wiggins[1] or Guckenheimer and Holmes[2]) we find:

$$\begin{aligned}H_1=\int_{-\infty}^{t_1^*}\frac{dH}{dt}\bigg|_{q_\epsilon^u(t,t_0)}dt&=\int_{-\infty}^{0}\frac{dH}{dt}\bigg|_{q_0(t-t_0)+\epsilon q_1^u(t,t_0)+O(\epsilon^2)}dt+\int_0^{t_1^*}\frac{dH}{dt}\bigg|_{q_0(t-t_0)+\epsilon q_1^s(t,t_0)+O(\epsilon^2)}dt\\&=\int_{-\infty}^{t_1^*}\frac{dH}{dt}\bigg|_{q_0(t-t_0)}+\epsilon\int_{-\infty}^{t_1^*}\frac{d}{dt}(\nabla H\cdot q_1^{u,s})(t,t_0)+O(\epsilon^2)\\&=H(q_0(t-t_0),t)\bigg|_{-\infty}^{t_1^*}+\epsilon\int_{-\infty}^{t_1^*}(f\wedge g)_{(q_0(t-t_0),t)}+O(\epsilon^2)\\&=H(q_0(t_1^*-t_0),t_1^*)+h_1(t_0)-\epsilon\int_{t_1^*}^{\infty}(f\wedge g)\bigg|_{(q_0(t-t_0),t)}dt+O(\epsilon^2)\\&=h_1(t_0)+O(|q_0(t_1^*-t_0)|^3)+O(\epsilon|q_0(t_1^*-t_0)|^2)+O(\epsilon^2).\end{aligned}\tag{B1}$$

In the last equality we used the fact that near the origin $H(x,y)=x^2-y^2+O(3)$ which vanishes on the stable manifold to $O(3)$, that $(f\wedge g)$ is $O(2)$ there and that $q_0(t)$ decays exponentially as $t\to\infty$. Now, we need to show that the error terms are of the indicated order. First, if $t_1^*=\infty$, it follows that $H_1=h_1(t_0)+O(\epsilon^2)$. Since in this case, by definition $H_1=0$, we obtain $h_1=O(\epsilon^2)$ and the error estimates are consistent.

Consider the case of finite t_1^*. Let N_δ be a neighborhood of the origin for which linearization is valid. For $\hat{\epsilon}$ sufficiently small $q_\epsilon^u(t_1^*,t_0)\in N_\delta$ for all $\epsilon<\hat{\epsilon}$, hence we can study the behavior of $q_\epsilon^u(t_1^*,t_0)$ using linearization. We solve the linearized equations in this neighborhood, and in the process we determine $t_1^*-t_0$ and $q_0(t_1^*-t_0)$. Let

$$v=-(x-y),\quad u=-(x+y).\tag{B2}$$

Using **A1** and **A4**, we conclude that in N_δ

$$\begin{aligned}\dot v&=-v(1+\epsilon a(t))+\epsilon b(t)u+O(2),\\\dot u&=u(1+\epsilon c(t))+\epsilon d(t)v+O(2),\end{aligned}\tag{B3}$$

where $a(t)$, $b(t)$, $c(t)$, $d(t)$ are T periodic functions which depend on the form of $g(x,y,t,\epsilon)$ near the origin and depend on ϵ analytically. We choose a solution to Eq. (B3) which hits Σ_h at $(x,y,t)=(-x_{\min},0,t_1^*)$, $x_{\min}>0$ (see Fig. 7). An orbit passing through the neighborhood N_δ enters the neighborhood with

$$(v_{\text{in}},u_{\text{in}},t_{\text{in}})=(\delta,u_{\text{in}},t_{\text{in}}),\tag{B4}$$

passes through Σ_h with

$$(v_{\text{mid}},u_{\text{mid}},t_1^*)=(x_{\min},x_{\min},t_1^*),\tag{B5}$$

and exits N_δ with

$$(v_{\text{out}},u_{\text{out}},t_{\text{out}})=(v_{\text{out}},\delta,t_{\text{out}}).\tag{B6}$$

We now find t_{in}, u_{in}, t_{out}, v_{out} in terms of t_1^*, $x_{\min}$.

We write u, v as

$$\begin{aligned}v&=v_0(t)+\sum_{i=1}^{\infty}\epsilon^i v_i(t),\quad v_0(t)=x_{\min}e^{-(t-t_1^*)},\\u&=u_0(t)+\sum_{i=1}^{\infty}\epsilon^i u_i(t),\quad u_0(t)=x_{\min}e^{t-t_1^*}.\end{aligned}\tag{B7}$$

The equations for u_n, v_n are given by

$$\begin{aligned}\dot v_n&=-v_n+\sum_{i=1}^{n-1}[a_{n-i}(t)v_i(t)+b_{n-i}(t)u_i(t)]+O(2),\\\dot u_n&=u_n+\sum_{i=1}^{n-1}[c_{n-i}(t)u_i(t)+d_{n-i}(t)v_i(t)]+O(2).\end{aligned}\tag{B8}$$

With the initial conditions

$$u_n(t_1^*)=0,\quad v_n(t_1^*)=0,\quad n\geq 1.\tag{B9}$$

Here the periodic coefficients a_i, b_i, c_i, d_i are the ϵ-expansion coefficients of the a, b, c, d coefficients of Eq. (B3). The $O(2)$ terms in (B8) are quadratic terms in (u,v) and *not* in ϵ. Hence,

$$\begin{aligned}v_1(t)&=e^{-(t-t_1^*)}\int_t^{t_1^*}e^{s-t_1^*}[v_0(s)a(s)+u_0(s)b(s)]ds\\&=x_{\min}e^{-(t-t_1^*)}\int_t^{t_1^*}[a(s)+e^{2(s-t_1^*)}b(s)]ds,\end{aligned}\tag{B10}$$

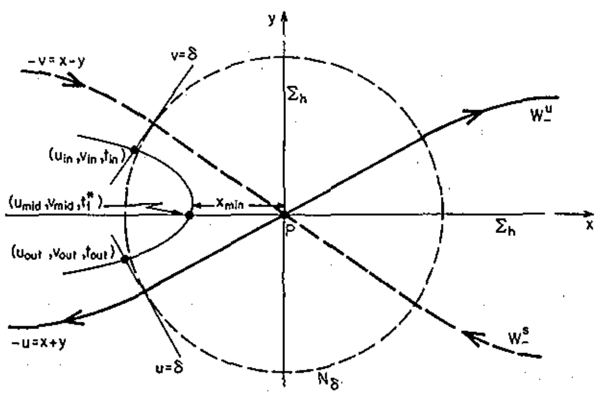

FIG. 7. Behavior near the origin. —— An orbit passing near the fixed point $P=(0,0)$.

$$u_1(t)=e^{t-t_1^*}\int_t^{t_1^*}e^{-(s-t_1^*)}[u_0(s)c(s)+v_0(s)d(s)]ds$$

$$=x_{\min}e^{t-t_1^*}\int_t^{t_1^*}[c(s)+e^{-2(s-t_1^*)}d(s)]ds.$$

Using Eq. (B8), one can show inductively that in general

$$|v_i(t)|\leqslant x_{\min}e^{-(t-t_1^*)}(A_i+e^{2(t-t_1^*)}B_i),$$

$$|u_i(t)|\leqslant x_{\min}e^{t-t_1^*}(C_i+e^{-2(t-t_1^*)}D_i),$$

where the constants A_i, B_i, C_i, D_i are related to each other via a recursion relation involving quadratic terms in these coefficients (here assumption **A4** is crucial). It follows that for ϵ sufficiently small the series of (B7) converges uniformly for $t<t_1^*$ and in particular

$$v_{\text{in}}=v(t_{\text{in}})=x_{\min}e^{-(t_{\text{in}}-t_1^*)}\left(1+\epsilon\int_{t_{\text{in}}}^{t_1^*}[a(s)+e^{2(s-t_1^*)}b(s)]ds+O(\epsilon^2)\right)$$

$$=x_{\min}e^{t_1^*-t_{\text{in}}}[1+O(\epsilon)]. \tag{B11}$$

Since $v_{\text{in}}=\delta$ [see Eq. (B4)] we find

$$e^{-(t_1^*-t_{\text{in}})}=\frac{x_{\min}}{\delta}[1+O(\epsilon)],$$

$$t_1^*-t_{\text{in}}=\log\frac{\delta}{x_{\min}}+O(\epsilon). \tag{B12}$$

Now, since $q_0(t)\approx e^{-t}$ for large t, and since by definition $t_{\text{in}}=t_0+\tau$ where $\tau>0$ is an order one quantity [time of flight of $q_\epsilon^u(t,t_0)$ from Σ_t to the boundary of N_δ, which is bounded away, by a distance δ, from the origin], we find

$$q_0(t_1^*-t_0)=e^{-(t_1^*-t_0)}+O(e^{-2(t_1^*-t_0)})$$

$$=Ke^{-(t_1^*-t_{\text{in}})}[1+O(Ke^{-(t_1^*-t_{\text{in}})})]$$

$$=K\frac{x_{\min}}{\delta}\left[1+O(\epsilon)+O\left(\frac{x_{\min}}{\delta}\right)\right],\quad K<1. \tag{B13}$$

Moreover, by definition,

$$H_1=H(-x_{\min},0,t_1^*)=-\tfrac{1}{2}x_{\min}^2[1+O(x_{\min})], \tag{B14}$$

which implies

$$x_{\min}=\sqrt{-2H_1}[1+O(\sqrt{|H_1|})].$$

Therefore, (recall that δ is fixed, independent of ϵ),

$$|q_0(t_1^*-t_0)|=O(\sqrt{H_1}). \tag{B15}$$

and substituting in Eq. (B1) we obtain

$$H_1(t_0)=h_1(t_0)+O(|H_1|^{3/2},\epsilon|H_1|,\epsilon^2)$$

$$=h_1(t_0)+O(|h_1|^{3/2},\epsilon|h_1|,\epsilon^2). \qquad \square$$

Lemma A.7: $\tau_1(t_0)=t_1+O(\epsilon,\sqrt{-h_1},\epsilon^2/|h_1|)$, where h_1, $t_1(t_0)$ is given by Eq. (3.3).

Proof: Recall that $t_1(t_0)-t_0$ is given by the period of the unperturbed orbit with energy h_1. Namely, we claim that by picking the unperturbed periodic solution $q^{h_1}(t)$ we obtain a good approximation to the perturbed solution $q(t)$ which hits Σ_t at t_0 and Σ_h with energy H_1. We use the construction of the solutions in N_δ, and show that indeed $t_{\text{out}}-t_{\text{in}}|_{q(t)}=t_{\text{out}}-t_{\text{in}}|_{q^{h_1}}+O(\sqrt{-h_1},\epsilon,\epsilon^2/|h_1|)$. Then we show that the entry and exit coordinates to and from N_δ of $q(t)$ and q^{h_1} are close, and since $q(t)$ and $q^{h_1}(t)$ spend a finite amount of time outside of N_δ a standard application of Gronwall's inequality completes the proof of the lemma.

Calculating $t_{\text{out}}-t_1^*$, as we did for $t_1^*-t_{\text{in}}$ in Eq. (B12), we obtain

$$t_{\text{out}}-t_{\text{in}}=2\log\frac{\delta}{x_{\min}}+O(\epsilon). \tag{B16}$$

Now,

$$x_{\min}(q_\epsilon^u(t,t_0))=\sqrt{-2H_1}[1+O(\sqrt{H_1})]$$

$$=\sqrt{-2h_1\left[1+O\left(\sqrt{-h_1},\epsilon,\frac{\epsilon^2}{|h_1|}\right)\right]}\times[1+O(\sqrt{|h_1|})]$$

$$=x_{\min}(q^{h_1}(t_1^*))\left[1+O\left(\sqrt{-h_1},\epsilon,\frac{\epsilon^2}{|h_1|}\right)\right]. \tag{B17}$$

Therefore,

$$t_{\text{out}}-t_{\text{in}}|_{q_\epsilon^u(t,t_0)}=t_{\text{out}}-t_{\text{in}}|_{q^{h_1}(t)}+O\left(\sqrt{-h_1},\epsilon,\frac{\epsilon^2}{|h_1|}\right). \tag{B18}$$

It follows from (B12) that

$$v(q^{h_1}(t_{\text{in}}))=x_{\min}(q^{h_1}(t_1^*))\exp(t_1^*-t_{\text{in}}|_{q_\epsilon^u(t,t_0)})$$

$$=\delta\left[1+O\left(\sqrt{-h_1},\epsilon,\frac{\epsilon^2}{|h_1|}\right)\right]$$

$$=v_{\text{in}}(q_\epsilon^u(t_{\text{in}},t_0))+O\left(\sqrt{-h_1},\epsilon,\frac{\epsilon^2}{|h_1|}\right)$$

and similarly for u_{out}. The u_{in}'s (and similarly the v_{out}'s) must be close to each other as well:

$$u(q_\epsilon^u(t_{\text{in}}))$$

$$=x_{\min}e^{t_{\text{in}}-t_1^*}\left(1+\epsilon\int_{t_{\text{in}}}^{t_1^*}[c(s)+e^{-2(s-t_1^*)}d(s)]ds\right)$$

$$\leqslant\frac{x_{\min}^2}{\delta}\left\{1+\epsilon\left[C+D\left(\frac{\delta}{x_{\min}}\right)^2\right]\right\}$$

$$=\frac{x_{\min}^2}{\delta}[1+O(\epsilon)]+\epsilon\delta D,$$

where C, D are the maximal values of the periodic functions $c(t)$, $d(t)$ defined in the proof of the previous lemma. Hence, using (B17), we find

$$u(q^u_\epsilon(t_{\text{in}},t_0))=u(q^{h_1})+O\left(\epsilon,\epsilon\sqrt{|h_1|},|h_1|^{3/2},\frac{\epsilon^3}{|h_1|}\right).$$

Hence, using the standard application of the Gronwall's inequality, we obtain that the solutions $q^u_\epsilon(t,t_0)$ and $q^{h_1}(t)$ remain $O(\sqrt{|h_1|})$-close on the finite time intervals $t\in[t_0,t_{\text{in}}]$ and $t\in[t_{\text{out}},\tau_1]$. Combining this with the result (B18) we are done. □

Lemma A.8: $H_2(t_0,\epsilon)=h_2(t_0,\epsilon)+O(|h_1|^{3/2},\epsilon|h_1|,\epsilon^2,\epsilon|h_2|,|h_2|^{3/2},\epsilon\sqrt{-h_1},\epsilon^3/|h_1|)$, where $h_1(t_0,\epsilon)$, $h_2(t_0,\epsilon)$ are given by Eq. (3.3).

Proof: If $H_2(t_0)\neq 0$ then, using the same argument as in Lemma A.3, and using Lemmas A.3–A.6, we obtain

$$\begin{aligned}H_2&=H_1+\int_{t_1^*}^{t_2^*}\frac{dH^\epsilon}{dt}\\&=h_1+O(|h_1|^{3/2},\epsilon|h_1|,\epsilon^2)+\epsilon M(\tau_1)\\&\quad+O(|H_1|^{3/2},|H_2|^{3/2},\epsilon|H_1|,\epsilon|H_2|)\\&=h_1+\epsilon M(t_1)\\&\quad+O(\epsilon\sqrt{|h_1|},\epsilon^3/|h_1|,|h_1|^{3/2},|h_2|^{3/2},\epsilon|h_1|,\epsilon|h_2|,\epsilon^2).\end{aligned}\tag{B19}$$

If $H_2(t_0)=0$, then, by definition $t_2^*=\infty$ and (B19) becomes

$$\begin{aligned}H_2&=H_1+\int_{t_1^*}^{\infty}\frac{dH^\epsilon}{dt}\\&=h_1+\epsilon M(t_1)+O\left(\epsilon\sqrt{|h_1|},\frac{\epsilon^3}{|h_1|},|h_1|^{3/2},\epsilon|h_1|,\epsilon^2\right).\end{aligned}\tag{B20}$$

It follows that h_2 is of the same order as the error in (B20), hence the error term in the lemma is consistent in this case too. □

APPENDIX C: SECONDARY HOMOCLINIC TANGENCIES

Theorems 3 and 4 are concerned with the degeneracies of the partial derivatives of $\bar{h}_2(t,\epsilon)$. If one assumes that $H_2(t,\epsilon)$ is sufficiently differentiable near its zeros, then it can be shown as in Theorem 1, that by using the appropriate scales for ϵ and t, the derivatives of H_2 have similar properties and the assertions follow immediately [whether H_2 is sufficiently smooth is still an open question since fractional powers appear in the error estimates of, e.g., (B19); additional restrictions on H, such as evenness and local symmetries may be needed]. Instead, we prove that H_2 undergoes "topologically" a certain bifurcation. For example, we claim that case **S1** of Theorem 3 corresponds to a saddle–node bifurcation of $\bar{h}_2(t_0,\epsilon_0)$. We prove that $H_2(t_0+\Delta t,\epsilon_0+\Delta\epsilon)$ changes signs twice as Δt varies on one side of the bifurcation value (say $\Delta\epsilon>0$) and does not change sign on the other side of it ($\Delta\epsilon<0$) as long as Δt and $\Delta\epsilon$ scale correctly with ϵ_0—small enough to allow Taylor expansion and large enough to dominate the error in approximating H_2 by $\epsilon\bar{h}_2$. The latter restriction excludes the possibility of identifying the exact infinitesimal structure of H_2 near the bifurcation without further information regarding its differentiability. Nonetheless, it follows from the geometry that there must be a tangency of *at least* quadratic order for some $(\tilde{t},\tilde{\epsilon})\in[(t_0+\Delta t,t_0-\Delta t),(\epsilon_0-\Delta\epsilon,\epsilon_0+\Delta\epsilon)]$. It follows that generically if **S1** is satisfied the tangency is indeed quadratic.

Theorem 4 is proved in exactly the same fashion.

1. Proof of Theorem 3

Case S1: Let $t_0\in\bar{I}_\eta$ and let $t_1=t_1^i(t_0,l)$, $\epsilon_0=\epsilon^i(t_0,l)>0$ of (A7) be sufficiently small. Provided the following generic conditions are satisfied (see assumption **A3** for the definition of the C's):

$$M'(t_0)\neq 0,\quad M'(t_1(t_0,\epsilon_0))\neq 0,$$

and

$$M''(t_0)\left(1+C(1)\frac{M'(t_1)}{M(t_0)}\right)+M''(t_1)\left(1+C(1)\frac{M'(t_0)}{M(t_0)}\right)^2-M'(t_1)\left(C(2)\frac{M'(t_0)}{M(t_0)}\right)^2\neq 0,$$

we prove that there is a secondary homoclinic saddle–node bifurcation.

First, notice that $\bar{h}_2(t_0,\epsilon_0)$ satisfies the conditions for a saddle–node bifurcation:[1] by assumption, $\bar{h}_2(t_0,\epsilon_0)=\partial\bar{h}_2/\partial t_0=0$. From (1.1) and **A3**, and since $M(t_0)\neq 0$ by C1 we find:

$$\begin{aligned}\frac{\partial\bar{h}_2}{\partial\epsilon}&=M'(t_1)P'(\epsilon_0M(t_0))M(t_0)\\&=\frac{M'(t_1)}{\epsilon_0}[C(1)+o(\epsilon_0)].\end{aligned}\tag{C1}$$

Hence, for ϵ_0 sufficiently small $\partial\bar{h}_2/\partial\epsilon\neq 0$. Finally, from (3.5) and **A3** we obtain

$$\begin{aligned}\frac{\partial^2\bar{h}_2}{\partial t_0^2}&=M''(t_0)+M''(t_1)[1+\epsilon_0P'(\epsilon_0M(t_0))M'(t_0)]^2+M'(t_1)[\epsilon_0^2P''(\epsilon_0M(t_0))M'(t_0)^2+\epsilon_0P'(\epsilon_0M(t_0))M''(t_0)]\\&=M''(t_0)\left(1+C(1)\frac{M'(t_1)}{M(t_0)}\right)+M''(t_1)\left(1+C(1)\frac{M'(t_0)}{M(t_0)}\right)^2-M'(t_1)\left(C(2)\frac{M'(t_0)}{M(t_0)}\right)^2+o(\epsilon_0).\end{aligned}\tag{C2}$$

Hence for ϵ_0 sufficiently small $\partial^2\bar{h}_2/\partial t_0^2 \neq 0$ as well. We conclude that locally

$$h_2(t_0+\Delta t,\epsilon_0+\Delta\epsilon) = \frac{1}{2}\epsilon_0\Delta t^2 \frac{\partial^2\bar{h}_2}{\partial t_0^2}(t_0,\epsilon_0)+\epsilon_0\frac{\Delta\epsilon}{\epsilon_0}\frac{\partial h_2}{\partial\epsilon}(t_0,\epsilon_0) + O\left[\epsilon_0\Delta t^3,\epsilon_0\frac{\Delta\epsilon}{\epsilon_0}\Delta t,\epsilon_0\left(\frac{\Delta\epsilon}{\epsilon_0}\right)^2\right],$$

provided $\Delta\epsilon$ and Δt are sufficiently small, and in particular $\Delta\epsilon=o(\epsilon_0)$. Using (A4), we find

$$H_2(t_0+\Delta t,\epsilon_0+\Delta\epsilon) = \frac{1}{2}\epsilon_0\Delta t^2 \frac{\partial^2\bar{h}_2}{\partial t_0^2}(t_0,\epsilon_0)+\epsilon_0\frac{\Delta\epsilon}{\epsilon_0}\frac{\partial h_2}{\partial\epsilon}(t_0,\epsilon_0) + O\left[\epsilon_0^{3/2},\epsilon_0\Delta t^3,\epsilon_0\frac{\Delta\epsilon}{\epsilon_0}\Delta t,\epsilon_0\left(\frac{\Delta\epsilon}{\epsilon_0}\right)^2\right]. \tag{C3}$$

Choosing $\Delta\epsilon=O(\epsilon^{1+\alpha})$ and $\Delta t=O(\sqrt{\Delta\epsilon/\epsilon_0})=O(\epsilon_0^{\alpha/2})$, the first two terms above are of the same order and dominate the error provided $0<\alpha<1/2$. It follows that on these scales $H_2(t+\Delta t,\epsilon_0+\Delta\epsilon)$ changes sign twice for

$$\Delta\epsilon\,\frac{\partial\bar{h}_2(t_0,\epsilon_0)}{\partial\epsilon}\Bigg/\frac{\partial^2\bar{h}_2(t_0,\epsilon_0)}{\partial t_0^2}<0$$

and that it does not change sign when $\Delta\epsilon$ has the opposite sign for Δt sufficiently small, provided $\Delta\epsilon=O(\epsilon_0^{1+\alpha})$.

It follows from Lemma A.6 that the manifolds intersect topologically transversely at two homoclinic points on one side of the bifurcation value and have no intersection on the other side of the bifurcation value. Since (2.1) depends continuously on ϵ, and since the manifolds are smooth (C^r), it follows that there must be a value of $\tilde{\epsilon}$ for which the manifolds are tangent, with at least a quadratic degeneracy.

Finally, H_2 is given locally by (C3) on these scales of $(\Delta\epsilon,\Delta t)$, hence $H_2(\tilde{t},\tilde{\epsilon})=0$ for

$$t^{\tilde{}\pm}=t_0\pm\frac{2[\partial\bar{h}_2(t_0,\epsilon_0)/\partial\epsilon]}{[\partial^2\bar{h}_2(t_0,\epsilon_0)/\partial t_0^2]}\sqrt{\frac{\Delta\epsilon}{\epsilon_0}}+O(\epsilon_0^{\alpha},\epsilon_0^{(1-\alpha)/2}),$$

and near the bifurcation the angle between manifolds at the emerging homoclinic points scales like $\tilde{\epsilon}(\partial\bar{h}_2/\partial t_0)\times(\tilde{t}^{\pm},\tilde{\epsilon})\propto\pm\epsilon_0^{1+\alpha/2}$. The $\pm$ signs reflect the opposite orientations of the manifolds at the emerging bifurcation points. We note that $\alpha=1/3$ seems to optimize the error estimates above.

Case S2: Let $M'(t_0)=M'(t_1(t_0,\epsilon_0))=0$ and $M''(t_0)\neq -M''(t_1)\neq 0$ [it follows from (3.5) that if $\partial\bar{h}_2(t_0,\epsilon_0)/\partial t_0=0$, then $M'(t_0)=0$ iff $M'(t_1)=0$, hence we need not consider the cases $M'(t_1)=0$, $M'(t_0)\neq 0$ nor $M'(t_0)=0$, $M'(t_1)\neq 0$]. It follows from (3.5), (C1) that

$$\bar{h}_2(t_0,\epsilon_0)=\frac{\partial\bar{h}_2}{\partial\epsilon}(t_0,\epsilon_0)=\frac{\partial\bar{h}_2}{\partial t_0}(t_0,\epsilon_0)=0,$$

$$\frac{\partial^2\bar{h}_2}{\partial t_0^2}(t_0,\epsilon_0)=M''(t_0)+M''(t_1)=A\neq 0, \tag{C4}$$

$$\frac{\partial^2\bar{h}_2}{\partial\epsilon\partial t_0}(t_0,\epsilon_0)=P'(\epsilon_0M(t_0))M''(t_1)M(t_0)=\frac{1}{\epsilon_0}B\neq 0,$$

$$\frac{\partial^2\bar{h}_2}{\partial\epsilon^2}(t_0,\epsilon_0)=M''(t_1)P'(\epsilon M(t_0))^2M(t_0)^2=\frac{1}{\epsilon_0^2}C\neq 0.$$

To leading order in ϵ_0, A,B,C are order one functions of t_0. Using (C4) we find

$$\bar{h}_2(t_0+\Delta t,\epsilon_0+\Delta\epsilon)=\frac{1}{2}A\Delta t^2+B\Delta t\frac{\Delta\epsilon}{\epsilon_0}+\frac{1}{2}C\left(\frac{\Delta\epsilon}{\epsilon_0}\right)^2+O(3) \tag{C5}$$

where the $O(3)$ terms are third-order terms in Δt and $\Delta\epsilon/\epsilon_0$. There are two cases to consider, depending on the sign of D:

$$D=B^2-AC = -\epsilon_0^2[P'(\epsilon_0M(t_0))M(t_0)]^2M''(t_1)M''(t_0).$$

Case 1, $D>0$: Then, $\bar{h}_2(t,\epsilon)=0$ has two branches of solutions for (t,ϵ) on either side of the bifurcation point (t_0,ϵ_0), and $\bar{h}_2$ undergoes a transcritical bifurcation; The solutions are given by

$$\tilde{t}^{\pm}=t_0+\frac{\Delta\epsilon}{\epsilon_0}\left(\frac{-B\pm\sqrt{D}}{A}\right)+O\left[\left(\frac{\Delta\epsilon}{\epsilon_0}\right)^2\right].$$

And indeed,

$$\bar{h}_2\left(t-\frac{B\Delta\epsilon}{A\epsilon_0}\pm K\frac{\sqrt{D}\Delta\epsilon}{A\epsilon_0},\epsilon_0+\Delta\epsilon\right) =\left(\frac{\Delta\epsilon}{\epsilon_0}\right)^2(K^2-1)\frac{D}{2A}+O\left[\left(\frac{\Delta\epsilon}{\epsilon_0}\right)^3\right]. \tag{C6}$$

Taking $K>1$ and $K<1$ in (C6) gives opposite signs of $\bar{h}_2$, hence $\bar{h}_2$ vanishes on two separate branches of solutions for all $\Delta\epsilon$ sufficiently small $[\Delta\epsilon=o(\epsilon_0)]$. Using Corollary A.1, we conclude that $H_2[t-(B\Delta\epsilon/A\epsilon_0)\pm K(\sqrt{D}\Delta\epsilon/\ A\epsilon_0)$, $\epsilon_0+\Delta\epsilon]$ changes sign with $K-1$, provided the dominant term of $\epsilon_0\bar{h}_2$, of order $\epsilon_0(\Delta\epsilon/\epsilon_0)^2$, dominates the error of H_2-h_2, of order $\epsilon_0^{3/2}$. Choosing $\Delta\epsilon=\epsilon_0^{1+\alpha}$ and $0<\alpha<1/4$ ensures this dominance. The angle between the manifolds at the bifurcating solutions is given by $\tilde{\epsilon}\partial\bar{h}_2/\partial t(\tilde{t}^{\pm},\tilde{\epsilon})\propto\pm\Delta\epsilon\sqrt{D}\propto\pm\epsilon_0^{1+\alpha}$.

Using Lemma A.6 it follows that the manifolds intersect transversely at two homoclinic points on both sides of the bifurcation value; on one side the homoclinic points approach each other as ϵ is increased and on the other side the homoclinic points depart from each other as ϵ is increased. It follows that, generically, three scenarios may occur: (1) each of the homoclinic points may change its direction of propagation before they collide, thus no tangency occurs. (2) The homoclinic points collide, disappear in a saddle–node bifurcation, and then reappear in a second saddle–node bifurcation, hence creating homoclinic tangencies at two ϵ values in the interval $(\epsilon_0-\Delta\epsilon,\epsilon_0+\Delta\epsilon)$. (3) The homoclinic points collide and create a (degenerate) tangency, then continue in reverse directions.

Case 2, $D<0$: Then, excluding the point (t_0,ϵ_0), $\bar{h}_2(t,\epsilon)=0$ has no solutions in a neighborhood of the bifurcation point. Since $H_2=h_2+O(\epsilon_0^{3/2})$, we conclude that

$H_2(t,\epsilon)$ has no zeros for $(t,\epsilon)=[t_0+O(\epsilon_0^\alpha),\epsilon_0+O(\epsilon_0^{1+\alpha})]$ for $\alpha<1/4$. However, we may not conclude on the structure closer to the bifurcation point, and in particular, we cannot conclude whether $H_2(t,\epsilon)$ vanishes for some $(\tilde{t},\tilde{\epsilon})\approx(t_0,\epsilon_0)$. If H_2 vanishes, then necessarily $\Delta t=o(\epsilon_0^\alpha)$, hence the angle of intersection must scale like $o(\epsilon_0^{5/4})$. If H_2 depends on an additional parameter then, generically, we expect this bifurcation to create a quadratic tangency as the other parameter is varied.

Case S3: Let $M'(t_0)=M'(t_1(t_0,\epsilon_0))=0$ and $M''(t_0)=-M''(t_1)$ and assume

$$M''(t_1)\neq 0,$$

$$M'''(t_0)+M'''(t_1)+3C(1)M''(t_0)M''(t_1)/M(t_0)\neq 0.$$

It follows from **A3** that for ϵ_0 sufficiently small:

$$\frac{\partial \bar{h}_2}{\partial \epsilon}(t_0,\epsilon_0)=\frac{\partial \bar{h}_2}{\partial t_0}(t_0,\epsilon_0)=\frac{\partial^2 \bar{h}_2}{\partial t_0^2}(t_0,\epsilon_0)=0,$$

$$\frac{\partial^2 \bar{h}_2}{\partial t_0 \partial \epsilon}(t_0,\epsilon_0)=P'(\epsilon_0 M(t_0))M''(t_1)M(t_0)=\frac{1}{\epsilon_0}B\neq 0,$$

$$\begin{aligned}\frac{\partial^3 \bar{h}_2}{\partial t_0^3}(t_0,\epsilon_0)&=M'''(t_0)+M'''(t_1)\\&\quad+3\epsilon_0 M''(t_0)M''(t_1)P'(\epsilon_0 M(t_0))\\&=A'\neq 0,\end{aligned}$$

where, to leading order in ϵ_0, A', B are order one functions of t_0. Therefore,

$$\bar{h}_2(t_0+\Delta t,\epsilon_0+\Delta\epsilon)=\frac{1}{6}A'\Delta t^3+B\Delta t\,\frac{\Delta\epsilon}{\epsilon_0}+O\left[\left(\frac{\Delta\epsilon}{\epsilon_0}\right)^2\right]+O(3). \tag{C7}$$

Choosing $\Delta t=O(\sqrt{\Delta\epsilon/\epsilon_0})$, we obtain a dominance of the first two terms. Hence, $\bar{h}_2$ undergoes topologically a pitchfork bifurcation (the higher-order terms may break the symmetry), and similar arguments to the previous case show that H_2 undergoes topologically a pitchfork bifurcation as well, provided $(\Delta t,\Delta\epsilon)=[O(\epsilon_0^{\alpha/2}),O(\epsilon_0^{1+\alpha})]$, $\alpha<1/3$. The angle between the manifolds is thus given by $\tilde{\epsilon}[\partial\bar{h}_2(\tilde{t},\tilde{\epsilon})/\partial\tilde{t}]=O(\epsilon_0^{1+\alpha})$. It follows that the manifolds undergo a homoclinic bifurcation. If the system is symmetric a pitchfork bifurcation, an at least a third-order tangency from which three homoclinic points emerges. If not, generically a perturbed pitchfork will occur in which one solution does not bifurcate and two additional solutions are created via a quadratic tangency.

Case **S3** may seem highly nongeneric. However, if $M(t)$ is odd in t then either **S1** or **S3** occur at the bifurcation. Since $M(t)$ is odd and T periodic and assuming, for simplicity, that it vanishes exactly twice in $[0,T)$, we obtain that

$$M'(t_1)=M'[M^{-1,i}(-M(t_0))]=\pm M'(t_0). \tag{C8}$$

Hence (A10) implies that $(\partial\bar{h}_2/\partial t_0)(t_0,\epsilon)=0$ in one of two cases: (1) If $M'(t_1)=M'(t_0)\neq 0$ and $M'(t_0)/M(t_0)=-[2/C(1)]+o(\epsilon)$. Then, using $M''(t_1)=-M''(t_0)$ the conditions of **S1** are satisfied.

(2) If $M'(t_0)=M'(t_1)=0$ and $M''(t_1)\neq 0$, by symmetry $M''(t_0)=-M''(t_1)$ and $M'''(t_0)=M'''(t_1)=0$. Therefore, the conditions of **S3** are satisfied. Note that in particular case **S2** cannot occur if $M(t)$ is odd in t. □□□

2. Proof of Theorem 4

Let $t_0\in\bar{I}_\eta(\mu_0)$ and let $t_1=t_1^i(t_0,l;\mu_0)$, $\epsilon_0=\epsilon^i(t_0,l;\mu_0)$ of (A7). Notice that since μ appears in g and not in f of (2.1), $P(h)$ is independent of μ yet $M(t)=M(t;\mu)$. Provided the following generic conditions are satisfied (see assumption **A3**):

$$\frac{\partial M(t_0;\mu_0)}{\partial t_0}=M'(t_0;\mu_0)\neq 0,$$

$$M'(t_1(t_0,\epsilon_0;\mu_0);\mu_0)\neq 0,$$

$$\frac{\partial M(t_1;\mu_0)}{\partial \mu}\neq 0,$$

and

$$\begin{aligned}&M''(t_0;\mu_0)\left(1+C(1)\,\frac{M'(t_1;\mu_0)}{M(t_0;\mu_0)}\right)\\&\quad+M''(t_1;\mu_0)\left(1+C(1)\,\frac{M'(t_0;\mu_0)}{M(t_0;\mu_0)}\right)^2\\&\quad-M'(t_1;\mu_0)\left(C(2)\,\frac{M'(t_0;\mu_0)}{M(t_0;\mu_0)}\right)^2\neq 0,\end{aligned}$$

we prove that the manifolds undergo a tangent bifurcation along a p-dimensional surface in the ϵ,μ parameter space.

Using $M'(t_0;\mu_0)\neq 0$ and $\partial\bar{h}_2(t_0,\epsilon_0)/\partial t_0=0$ we find from (3.5)

$$\frac{\partial\bar{h}_2(t_0,\epsilon_0;\mu_0)}{\partial\mu}=\frac{\partial M(t_1;\mu_0)}{\partial\mu}\neq 0.$$

Clearly (C1) and (C2) are unchanged, hence, $\bar{h}_2(t,\epsilon,\mu)$ undergoes a saddle–node bifurcation along the p-dimensional surface:

$$\epsilon_b(\mu)=\epsilon_0\left(1-\frac{(\mu-\mu_0)(\partial\bar{h}_2/\partial\mu)(t_0,\epsilon_0,\mu_0)}{(\partial h_2/\partial\epsilon)(t_0,\epsilon_0,\mu_0)}+O[(\mu-\mu_0)^2]\right). \tag{C9}$$

Moreover, provided $\Delta\epsilon$, Δt, and $\Delta\mu$ are sufficiently small, and in particular $\Delta\epsilon=o(\epsilon_0)$, we may write:

$$\begin{aligned}&\bar{h}_2(t_0+\Delta t,\epsilon_b(\mu_0+\Delta\mu)+\Delta\epsilon,\mu_0+\Delta\mu)\\&\quad=\Delta t^2\,\frac{\partial^2\bar{h}_2}{\partial t_0^2}(t_0,\epsilon_0,\mu_0)+\frac{\Delta\epsilon}{\epsilon_0}\,\frac{\partial h_2}{\partial\epsilon}(t_0,\epsilon_0,\mu_0)\\&\qquad+O\left[\Delta t^3,\frac{\Delta\epsilon}{\epsilon_0}\Delta t,\left(\frac{\Delta\epsilon}{\epsilon_0}\right)^2,\epsilon_0\Delta\mu^2,\Delta\mu\Delta t,\Delta\mu\,\frac{\Delta\epsilon}{\epsilon_0}\right].\end{aligned}$$

It follows from (A4) that $H_2[t_0+\Delta t,\ \epsilon_b(\mu_0+\Delta\mu)+\Delta\epsilon,\mu_0+\Delta\mu]$ changes sign like $\bar{h}_2(t_0+\Delta t,\epsilon_b(\mu_0$

$+\Delta\mu)+\Delta\epsilon,\mu_0+\Delta\mu)$ does, provided $(\Delta t,\Delta\epsilon) = [O(\epsilon_0^{\alpha/2}), O(\epsilon_0^{1+\alpha})]$, $\alpha<1/2$ and $\Delta\mu$ is sufficiently small: $\Delta\mu = o(\epsilon_0^{\alpha/2})$. It follows that the manifolds undergo a tangent bifurcation along a p-dimensional surface which is $O(\epsilon_0^{1+\alpha}, \epsilon_0\Delta\mu^2)$ close to the surface $(\epsilon_b(\mu),\mu)$ of (C9). □□□

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