Introduction to Computer Vision
Exercise 4: Projective Geometry and Structure from Motion
Due Date: 2.1.02

Question 1:
Given two pencils of lines: \{x + 5 = y; 2x = y; x = 5\}, and \{x + y = 5; 2x + y = 0; x = -5\}, find the family of homographies \{H\} that rectifies these lines (i.e., that transforms each pencil of lines into a parallel set of lines), and satisfies the following two conditions:
(i) \(h_{33} = 1\).
(ii) \(H\) keeps the point \((x, y) = (0, 0)\) fixed.

Question 2:
Fixed points of a homography.
1. What are the fixed points of the following homography?
\[
H = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 3 & 1
\end{pmatrix}
\]
2. What is the maximal number of linearly-independent fixed points that a homography can have?
3. Can there be a line of fixed points? If so – explain when.

Question 3:
\(x^2 + y^2 = 1\) is the canonical circle and \(4x^2 + y^2 = 1\) is an ellipse in a Euclidean plane. Write expressions for the same shapes in the Projective plane. Write them also in matrix notation.
Find a simple projective transformation (a homography) that transforms the canonical circle to the ellipse above. Is this transformation Euclidean? Affine? Projective? Explain.

Question 4:
A point \(P = (X, Y, Z)^T\) projects to \(p = (x, y)^T = s \cdot (X, Y)^T\) with a scalar \(s \neq 0\) under an orthographic projection. Suppose a second orthographic image is obtained after \(3 \times 3\) rotation \(R\) and translation \(t = (t_x, t_y, t_z)^T\). Derive the epipolar constraint for the two images. (Hint: eliminate \(Z\).) Show that the epipolar lines are parallel. Write the epipolar constraint in the form of a Fundamental matrix \(p'^T F p = 0\) using homogeneous coordinates and derive the coordinates of the two epipoles.

Question 5:
Let \(P_i = (X_i, Y_i, Z_i)^T, i = 1, ..., N\) be a set of 3D points in space with image projections \(p_i = (x_i, y_i)^T = \frac{1}{Z_i} P_i\). Assuming a rigid camera motion
\[
P_i' = RP_i + t
\]
where \( R \) is the rotational component and \( t \) is the translational component of camera motion, with projections \( p_i' = \frac{1}{Z_i} P_i' \).

Show that in the case of pure rotation \( (t = 0) \), it is not possible to recover the structure of the scene (the depth \( Z_i \)) given any number of matching pairs \( p_i, p_i' \).

(Hint: show the existence of a mapping from \( p_i \) to \( p_i' \) that does not include 3D structure (i.e., depth \( Z_i \)). Explain why the existence of such a mapping proves the claim.)

Question 6:

Two images of the same scene are related by general rotation and translation. Corresponding points \( p_i \) and \( p_i' \) in the two images satisfy the fundamental relation \( p_i'^T F p_i = 0 \). Suppose that all the points \( P_i \) lie on some plane, and so corresponding points in the two images are related by a homography \( p_i' \cong Hp_i \) (i.e., \( p_i' = \alpha_i Hp_i \) for some scalar \( \alpha_i \neq 0 \)). Show that in this case there exist infinitely many solutions to the fundamental matrix \( F \).

(Hint: A matrix \( M \) is called skew-symmetric if it has the form:
\[
M = \begin{pmatrix}
0 & m_{12} & m_{13} \\
-m_{12} & 0 & m_{23} \\
-m_{13} & -m_{23} & 0
\end{pmatrix}.
\]

(a) Show that \( H^T F \) is a skew-symmetric matrix. (You can use the fact that a matrix \( M \) is skew-symmetric \( \iff \forall x : x^T M x = 0 \).)

(b) Show that this implies the following constraint on \( F \):
\[
H^T F + F^T H = 0.
\]

(c) Show that this provides only 6 linearly independent equations for \( F \) (instead of 8).

(d) If the points \( P_i \) come from more than just one plane in the scene, how many planes are needed in order to uniquely recover \( F \)?