Introduction to Computer Vision
Exercise 4: Projective Geometry and Structure from Motion
Due Date : 2.1.02

Question 1:

Given two pencils of lines: $\{x+5=y;\ 2x=y;\ x=5\}$, and $\{x+y=5;\ 2x+y=0;\ x=-5\}$, find the family of homographies $\{H\}$ that rectifies these lines (i.e., that transforms each pencil of lines into a parallel set of lines), and satisfies the following two conditions:
(i) $h_{33}=1$.
(ii) H keeps the point $(x,y)=(0,0)$ fixed.

Question 2:

Fixed points of a homography.

1. What are the fixed points of the following homography?

$$H=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 3 & 1 \end{pmatrix}$$

2. What is the maximal number of linearly-independent fixed points that a homography can have?
3. Can there be a line of fixed points? If so – explain when.

Question 3:

$x^2+y^2=1$ is the canonical circle and $4x^2+y^2=1$ is an ellipse in a Euclidean plane. Write expressions for the same shapes in the Projective plane. Write them also in matrix notation.

Find a simple projective transformation (a homography) that transforms the canonical circle to the ellipse above. Is this transformation Euclidean? Affine? Projective? Explain.

Question 4:

A point $P=(X,Y,Z)^T$ projects to $p=(x,y)^T=s\cdot(X,Y)^T$ with a scalar $s\neq 0$ under an orthographic projection. Suppose a second orthographic image is obtained after 3×3 rotation R and translation $t=(t_x,t_y,t_z)^T$. Derive the epipolar constraint for the two images. (*Hint:* eliminate Z.) Show that the epipolar lines are parallel. Write the epipolar constraint in the form of a Fundamental matrix $p'^TFp=0$ using homogeneous coordinates and derive the coordinates of the two epipoles.

Question 5:

Let $P_i=(X_i,Y_i,Z_i)^T$, $i=1,...,N$ be a set of 3D points in space with image projections $p_i=(x_i,y_i)^T=\frac{1}{Z_i}P_i$. Assuming a rigid camera motion

$$P_i'=RP_i+t$$

where R is the rotational component and t is the translational component of camera motion, with projections $p'_i = \frac{1}{Z'_i} P'_i$.

Show that in the case of pure rotation ($t = 0$), it is not possible to recover the structure of the scene (the depth Z_i) given any number of matching pairs p_i, p'_i.

(*Hint:* show the existence of a mapping from p_i to p'_i that does not include 3D structure (i.e., depth Z_i). Explain why the existence of such a mapping proves the claim.)

Question 6:

Two images of the same scene are related by general rotation and translation. Corresponding points p_i and p'_i in the two images satisfy the fundamental relation $p'^T_i F p_i = 0$. Suppose that all the points P_i lie on some plane, and so corresponding points in the two images are related by a homography $p'_i \cong H p_i$ (i.e., $p'_i = \alpha_i H p_i$ for some scalar $\alpha_i \neq 0$). Show that in this case there exist infinitely many solutions to the fundamental matrix F.

(*Hint:* A matrix M is called skew-symmetric if it has the form:

$$M = \begin{pmatrix} 0 & m_{12} & m_{13} \\ -m_{12} & 0 & m_{23} \\ -m_{13} & -m_{23} & 0 \end{pmatrix}.$$

(a) Show that $H^T F$ is a skew-symmetric matrix. (You can use the fact that a matrix M is skew-symmetric *iff* $\forall x : \; x^T M x = 0$.)
(b) Show that this implies the following constraint on F:

$$H^T F + F^T H = 0.$$

(c) Show that this provides only 6 linearly independent equations for F (instead of 8).
(d) If the points P_i come from more than just one plane in the scene, how many planes are needed in order to uniquely recover F?