Exercise 6: Motion, Segmentation, and Illumination

Q1:

Let the spatial derivatives of the image intensity be denoted by I_x, I_y and the temporal derivative by I_t. At each location in the image, we assume the following Brightness Constancy Constraint:

$$I_x u + I_y v + I_t = 0$$

where $[u, v]^t$ is the velocity (or displacement) at the pixel $[x, y]^t$. Lucas and Kanade assumed that within an image patch around each pixel, the displacement vector $[u, v]^t$ is constant. They derived a set of 2 equations in the 2 unknowns $[u, v]^t$ that best fits the measurements of I_x, I_y, I_t within this patch, in the least-squares sense (which we also derived in class). Write this set of two equations, and describe the degenerate cases (cases where the solution is not unique), both algebraically, as well geometrically (i.e., provide an explanation of what geometric settings within an image patch would give rise to such singular cases).

Q2:

Let $P_i = (X_i, Y_i, Z_i)^T$, $i = 1, ..., N$ be a set of 3D points in space with image projections $p_i = (x_i, y_i)^T = f\frac{P_i}{Z_i}$, where f is the focal length of the camera. In the general case of a rigid camera motion

$$P_i' = RP_i + t$$

where R is the rotational component and t is the translational component of camera motion, with projections $p_i' = f\frac{P_i'}{Z_i'}$.

Show that in the case of pure translation (i.e., the camera does not rotate), the induced displacements of all image points form a radial flow field emerging from the focus-of-expansion (FOE) of the camera $e = f\frac{t}{t_Z}$ (this is also the epipole), where $t = (t_X, t_Y, t_Z)^T$.

(*Hint:* A radial displacement field has the form: $p_i' - p_i = \alpha_i(p_i - e)$, where α_i is a point-dependent scalar.)

Q3:

(a) What is the computational complexity of Hough Transform for finding lines in a collection of points?

(b) Suggest a Hough Transform method for finding circles given a collection of points. What would be the complexity of this method?

Q4:

(a) Given a surface $z(x, y)$ (so every point on the surface is given by $(x, y, z(x, y)))$, write an expression for the surface normal.

(b) Suppose following a photometric stereo algorithm we recover the correct surface normals for $z(x, y)$, but only up to the following linear transformation (the ambiguity matrix):

$$\begin{pmatrix} 1 & 0 & -\alpha \\ 0 & 1 & -\beta \\ 0 & 0 & 1 \end{pmatrix}$$

Show that these recovered normals form an integrable surface. (In other words, call the surface determined by the recovered normals $z'(x, y)$; show that $z'_{xy} = z'_{yx}$. Subscripts denote partial differentiation.) In addition, write an expression for z' (by integrating the relevant components of the surface normals).

(c) Repeat this process for normals transformed by:

$$\begin{pmatrix} 2 & 0 & -\alpha \\ 0 & 1 & -\beta \\ 0 & 0 & 1 \end{pmatrix}$$

Do these normals form an integrable surface?

Q5: (Bonus points)

Given three points P_1, P_2, P_3 in 3D ($P_i = (X_i, Y_i, Z_i)$ in world's coordinate system) and three corresponding points p_1, p_2, p_3 in the 2D image ($p_i = (x_i, y_i)$). Assume these points are related to the camera's coordinate system by a rotation and translation in 3D, and projected onto the image by a scaled orthographic projection ($(X_i, Y_i, Z_i) \rightarrow (sX_i, sY_i)$). Show how to derive the parameters of rotation, translation and scale from these correspondences.